ON INFINITESIMAL ISOMETRIC DEFORMATIONS

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ABSTRACT. We consider an analytic n-dimensional submanifold M of the Euclidean space E^N, where $N = n(n + 1)/2$, and we prove the existence of analytic, nontrivial, infinitesimal isometric deformations, in a neighborhood of any point of M, which admits a nonasymptotic tangent hyperplane.

1. **Introduction.** Rigidity and isometric deformation problems have been extensively investigated for surfaces in the three dimensional Euclidean space. The theory for submanifolds of the Euclidean space, with codimension one, was developed by E. Cartan [2]. As for submanifolds with higher codimension, very little is known besides the classical rigidity theorem of Allendoerfer [1], which imposes an extremely strong hypothesis on the submanifold. In this paper we consider analytic, n-dimensional submanifolds M of the Euclidean space E^N, where the dimension $N = n(n + 1)/2$ and we investigate the infinitesimal isometric deformation problem.

In §2, we obtain the system of partial differential equations, which determines an infinitesimal isometric deformation. This is the linearized version of an isometric deformation.

In §3, we introduce the concept of asymptotic tangent hyperplane at a point of M and we characterize the set of all such hyperplanes.

Finally in §4, we prove the following

THEOREM. *Let M be an n-dimensional, analytic submanifold of E^N, $N = n(n + 1)/2$, with the induced metric and such that the inclusion $i: M \to E^N$ is nondegenerate. If $x \in M$ is such that there exists a nonasymptotic tangent hyperplane at x, then there is a neighborhood of x in M, which admits an analytic nontrivial infinitesimal isometric deformation.*

Other results, involving the notion of asymptotic submanifolds, can be found in [3], [4] and [5].

I am grateful to Professor S. S. Chern for suggesting this problem and my colleague C. E. Harle for helpful comments.

2. **Infinitesimal isometric deformations.** In this paper, all manifolds and maps are assumed to be analytic. Consider an immersion $f: M \to E^N$, where M is an n-dimensional manifold and E^N is the Euclidean N-dimensional

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space. We define a *deformation* of the immersion f, to be a map $F: I \times M \to E^N$, where $I = [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, such that for each $t \in I$, the map F_t: $M \to E^N$ given by $F_t(x) = F(t, x)$ is an immersion and $F_0 = f$. Each immersion F_t induces a Riemannian metric g_t on M. A deformation F is said to be an *isometric deformation* if $g_t = g_0$ for each $t \in I$. Now we consider the linearized version of isometric deformation. A deformation is said to be an *infinitesimal isometric deformation* if $(d/dt)g_t|_{t=0} = 0$, i.e. for all vector fields X, Y on M

$$\frac{d}{dt} \left\langle F_{t_*}(X), F_{t_*}(Y) \right\rangle \Big|_{t=0} = 0$$

where $\langle \, , \, \rangle$ denotes the Euclidean metric.

Let F be a deformation of an immersion $f: M \to E^N$. For each $x \in M$, let $Z(x)$ be the tangent vector to the curve $t \mapsto F(t, x)$ at $t = 0$. We call Z the *deformation vector field* of F. We say that F is a *trivial deformation* if there exist a skew-symmetric matrix A and a vector v such that for all $x \in M$, $Z(x) = Af(x) + v$.

It is not difficult to see that a deformation F is an infinitesimal isometric deformation if and only if for all tangent vector fields X, Y on M

$$\langle D_X Z, Y \rangle + \langle X, D_Y Z \rangle = 0 \tag{1}$$

where D denotes the usual connection in E^N (we have identified M with $f(M)$). If we denote $Z = T + B$, where T and B are respectively the tangential and normal components of Z, then equation (1) is equivalent to

$$\langle \nabla_X T, Y \rangle + \langle \nabla_Y T, X \rangle + 2\langle s(X, Y), B \rangle = 0 \tag{2}$$

where ∇ denotes the Riemannian connection on M and s the second fundamental form.

Let $x_1, \ldots, x_n$ be a coordinate system defined in a neighborhood V of a point $x \in M$. We denote by $\partial/\partial x_i$ the coordinate vectors and $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$, $1 \leqslant i, j \leqslant n$. We choose an orthonormal frame ξ_α on V, $\alpha = n + 1, \ldots, N$, normal to M. We adopt the following indices convention

$$1 \leqslant i, j, k \leqslant n, \qquad n + 1 \leqslant \alpha, \beta, \gamma \leqslant N,$$

and the summation convention with regard to repeated indices. If we consider $T = t^i \partial/\partial x_i$ and $B = b^\alpha \xi_\alpha$, then equation (2) is equivalent to the system

$$g_{ik} \frac{\partial t^k}{\partial x_i} + g_{ik} \frac{\partial t^k}{\partial x_j}$$

$$+ \left(\left\langle \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right\rangle + \left\langle \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i} \right\rangle \right) t^k + 2b^\alpha h_{ij}^\alpha = 0 \tag{3}$$

where $1 \leqslant i \leqslant j \leqslant n$ and $h_{ij}^\alpha = \langle s(\partial/\partial x_i, \partial/\partial x_j), \xi_\alpha \rangle$.

3. Asymptotic hypersurfaces. Let M be an n-dimensional submanifold of E^N, $N = n(n + 1)/2$, with the induced metric and such that the inclusion i: $M \to E^N$ is nondegenerate. For a fixed point $x \in M$, an l-dimensional,

$0 < l < n$, linear subspace L of the tangent space $T_x M$ is called *asymptotic* if there exists a vector ξ normal to $T_x M$, such that for any $X, Y \in L$, $\langle s(X, Y), \xi \rangle = 0$. If $l = n - 1$ we have an *asymptotic hyperplane* at x. An l-dimensional submanifold V of M, $l < n$, is called *asymptotic* at $x \in V$ if $T_x V$ is asymptotic and asymptotic if this is true for each $x \in V$. The notion of asymptotic submanifolds in a more general context can be found in [3].

The asymptotic hyperplanes at $x \in M$ can be characterized as follows:choose an orthonormal frame $e_1, \ldots, e_N$ defined on a neighborhood of x, such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_N$ are normal to M. Let $\omega^1, \ldots, \omega^N$ be the dual frame. We denote by H^α the second fundamental forms with respect to this frame, i.e. $s(X, Y) = H^\alpha(X, Y)e_\alpha$. From the theory of a submanifold of an Euclidean space, we know that $\omega^\alpha = 0$ on M and $H^\alpha = h_{ij}^\alpha \omega^i \otimes \omega^j$, $h_{ij}^\alpha = h_{ji}^\alpha$. It follows from the definition, that an $(n - 1)$-dimensional linear subspace of $T_x M$, given by $u_i \omega^i = 0$ is asymptotic if and only if there exists $a_\alpha \in \mathbf{R}$, not all zero, such that $a_\alpha H^\alpha = 0$ when restricted to the hyperplane $u_i \omega^i = 0$. It is not difficult to prove that this is equivalent to saying that there exist $a_\alpha, c_i \in \mathbf{R}$ not all zero, such that

$$a_\alpha H^\alpha \equiv u_i \omega^i \otimes c_j \omega^j.$$

This reduces to a homogeneous equation in u_i of degree n, $P(u_1, \ldots, u_n) = 0$, which can be described as follows: we consider the matrices

$$U_0 = \begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix}, \quad U_p = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ u_{p+1} & u_{p+2} & \cdots & u_n \\ u_p & & & 0 \\ & u_p & & \\ & & \ddots & \\ 0 & & & u_p \end{bmatrix}, \tag{4}$$

where $1 < p < n - 1$ and U_p has the first $(p - 1)$ rows equal to zero.

$$A_0 = \begin{bmatrix} h_{11}^{n+1} & h_{22}^{n+1} & \cdots & h_{nn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{11}^N & h_{22}^N & \cdots & h_{nn}^N \end{bmatrix},$$

$$A_p = 2 \begin{bmatrix} h_{pp+1}^{n+1} & h_{pp+2}^{n+1} & \cdots & h_{pn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{pp+1}^N & h_{pp+2}^N & \cdots & h_{pn}^N \end{bmatrix}, \tag{5}$$

$1 \leqslant p \leqslant n - 1$. Then

$$P(u_1, u_2, \ldots, u_n) = \det \begin{bmatrix} U_0 & U_1 & \cdots & U_{n-1} \\ A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix} = 0. \qquad (6)$$

Hence the asymptotic hypersurfaces of M are the solutions of the first order partial differential equation defined by $P(u_1, \ldots, u_n) = 0$.

In our theorem, we prove that there exist nontrivial, infinitesimal, isometric deformations in a neighborhood of any point of M, for which the polynomial $P(u_1, \ldots, u_n) = 0$ is not an identity.

4. Proof of theorem. The proof is essentially based on the Cauchy-Kowalewsky theorem.

(a) Let $x_1, \ldots, x_n$ be a coordinate system defined in a neighborhood V of x in M, such that the coordinate vectors $\partial/\partial x_i$ are orthonormal at x, i.e. $g_{ij}(x) = \delta_{ij}$. Moreover we choose an orthonormal frame ξ_α in V normal to M. We have to prove the local existence of a vector field $Z = t^i \partial/\partial x_i + b^\alpha \xi_\alpha$, which satisfies the system of differential equations (3). This system can be written in matrix notation as

$$\sum_{i=1}^{n} Q^i \frac{\partial t}{\partial x_i} + Rt + 2Db = 0 \qquad (7)$$

where

$$t = \begin{bmatrix} t^1 \\ \vdots \\ t^n \end{bmatrix}, \quad b = \begin{bmatrix} b^{n+1} \\ \vdots \\ b^N \end{bmatrix},$$

Q^i and R are $N \times n$ matrices and D is an $N \times (N - n)$ matrix.

By assumption the inclusion of M in E^N is nondegenerate, hence the matrix D is of maximal rank $N - n$. Therefore we can consider (7) as

$$\sum_{i=1}^{n} \begin{bmatrix} q^i \\ \overline{Q}^i \end{bmatrix} \frac{\partial t}{\partial x_i} + \begin{bmatrix} r \\ R \end{bmatrix} t + 2 \begin{bmatrix} d \\ D \end{bmatrix} b = 0 \qquad (8)$$

where $\overline{D}$ is a nonsingular square matrix in D. It follows that

$$-2 = \overline{D}^{-1} \left(\sum_{i=1}^{n} \overline{Q}^i \frac{\partial t}{\partial x_i} + \overline{R}t \right). \qquad (9)$$

Substituting this relation in the first set of equations in (8) we get

$$\sum_{i=1}^{n} \left(q^i - d\overline{D}^{-1}\overline{Q}^i \right) \frac{\partial t}{\partial x_i} + \left(r - d\overline{D}^{-1}\overline{R} \right) t = 0. \qquad (10)$$

This is a system of n equations with n unknowns to which we apply the Cauchy-Kowalewsky theorem.

The characteristic equation for (10) is given by

$$\det\left[\sum_{i=1}^{n}\left(q^i - d\overline{D}^{-1}\overline{Q}^i\right)u_i\right] = 0. \tag{11}$$

In (b) we prove that at the point x this equation is equivalent to $P(u_1, \ldots, u_n) = 0$ given in (6). Therefore the characteristic hypersurfaces of the system (10) are the asymptotic hypersurfaces of the manifold M.

We consider a hypersurface S of M, which is not asymptotic and contains x (exists by assumption). We observe that if a deformation vector field Z is trivial, then the tangential component of Z is of the form

$$\sum_{j,k} \left\langle Ai + v, \frac{\partial}{\partial x_k} \right\rangle g^{kj} \frac{\partial}{\partial x_j} \tag{12}$$

where $i: M \to E^N$ is the inclusion map, A is a fixed skew-symmetric matrix, v is a vector in E^N and g^{kj} is the inverse matrix of g_{kj}. Since the space of vector fields of the form (12), along the submanifold S is finite dimensional, we can always choose an analytic tangential vector field $t^j \partial/\partial x_j$ in a neighborhood of x along S which is not of the form (12). With this initial condition fixed, it follows from the Cauchy-Kowalewsky theorem, that locally there exists a unique analytic tangential vector field satisfying (10) and the initial condition. Next using (9) we determine the normal component and hence we get a nontrivial deformation vector field Z. Therefore for $\varepsilon > 0$ sufficiently small and $\overline{V}$ a neighborhood of x in M, the map $F(r, y) = i(y) + rZ(y)$, $r \in (-\varepsilon, \varepsilon)$ and $y \in \overline{V}$ is a nontrivial infinitesimal isometric deformation of the inclusion $i: M \to E^N$.

(b) It remains to prove that at x (11) is equivalent to (6). To fix ideas we assume the following order for the indices i, j of the system of equations (3):

$$i = j = 1, \ldots, n; \quad i = 1, j = 2, \ldots, n;$$
$$i = 2, j = 3, \ldots, n, \ldots, i = n - 1, j = n.$$

With this order, we can describe the matrices D and Q^i of equation (7) as

$$2D = \begin{bmatrix} 2 & {}^tA_0 \\ & {}^tA_1 \\ & \vdots \\ & {}^tA_{n-1} \end{bmatrix}, \quad Q^i = \begin{bmatrix} G_0^i \\ G_1^i \\ \vdots \\ G_{n-1}^i \end{bmatrix},$$

where ${}^tA_0, \ldots, {}^tA_{n-1}$ are the transpose matrices of (5); G_0^i is an $n \times n$ matrix defined by

$$G_0^i = \begin{bmatrix} 0 & 0 & \cdots & 0 & & \\ \vdots & \vdots & & \vdots & & \\ 0 & 0 & & 0 & & \\ 2g_{1i} & 2g_{2i} & \cdots & 2g_{ni} & & \\ 0 & 0 & & 0 & & 0 \\ \vdots & \vdots & & \vdots & & \\ 0 & 0 & \cdots & 0 & & \end{bmatrix},$$

where the nonzero elements are on the ith row; and G_p^i, $1 \leqslant p \leqslant n - 1$, is an $(n - p) \times n$ matrix defined by

$$\text{if } p < i \quad G_p^i = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ g_{1p} & g_{2p} & \cdots & g_{np} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \rightsquigarrow (i - p)\text{th row,}$$

$$\text{if } p = i \quad G_p^i = \begin{bmatrix} g_{1i+1} & \cdots & g_{ni+1} \\ g_{1i+2} & \cdots & g_{ni+2} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ g_{1n} & \cdots & g_{nn} \end{bmatrix},$$

$$\text{if } p > i \quad G_p^i = 0.$$

With the same notation as in part (a) we consider the product of the matrices

$$\begin{bmatrix} I_n & -d\overline{D}^{-1} \\ 0 & \overline{D}^{-1} \end{bmatrix} \begin{bmatrix} \Sigma_i q^i u_i & 2d \\ \Sigma_i \overline{Q}^i u_i & 2\overline{D} \end{bmatrix} = \begin{bmatrix} \Sigma_i (q^i - d\overline{D}^{-1}\overline{Q}^i) u_i & 0 \\ * & 2I_{N-n} \end{bmatrix}$$

where I_n denotes the $n \times n$ identity matrix. Therefore u_i satisfy (11) if and only if

$$\det \left[\sum_i Q^i u_i \quad 2D \right] = 0. \tag{13}$$

Now we remark that the coordinate system was chosen, such that at x, $g_{ij}(x) = \delta_{ij}$ and it can easily be verified that

$$\sum_i Q^i(x) u_i = \begin{bmatrix} 2^t U_0 \\ {}^t U_1 \\ \cdot \\ \vdots \\ U_{n-1} \end{bmatrix},$$

where ${}^t U_0 \cdots {}^t U_{n-1}$ are the transpose matrices of (4). Therefore it follows from (13), that at x (11) is equivalent to (6).   Q.E.D.

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