120. On Infinitesimal Linear Isotropy Group of an Affinely Connected Manifold

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 13, 1965)

Introduction. Let M be a differentiable manifold with an affine connection of class C^∞. For each point p in M we denote by L_p the group of all linear transformations of the tangent space M_p at p. The *infinitesimal linear isotropy group* K_p is the subgroup of L_p consisting of all linear transformations of M_p which leave invariant the torsion tensor $(T)_p$, the curvature tensor $(R)_p$, and all their succesive covariant differentials $(\nabla T)_p$, $(\nabla^2 T)_p, \cdots, (\nabla R)_p$, $(\nabla^2 R)_p, \cdots$ [3]. Let $A(M)$ be the group of all affine automorphisms of M, H_p the subgroup of $A(M)$ consisting of all elements of $A(M)$ which fix the point p, and dH_p the linear isotropy group determined by H_p. In § 2, we shall investigate sufficient conditions that $dH_p=K_p$ at each p in M, and treat some applications. We discussed similar problems in a Riemannian manifold [6], [7]. Throughout this note we make use of the summation convention.

§ 1. **Preliminaries.** *Lemma* 1. *Let M be a differentiable manifold with an affine connection of class C^∞. If $f\in H_p$, then $(df)_p\in K_p$ at each p in M.*

Proof. Let B be the frame bundle of M, and let the structural equations be

$$d\tilde{\theta}^j=\tilde{\theta}^k\wedge\tilde{\theta}^j_k+\frac{1}{2}\tilde{P}^j_{km}\tilde{\theta}^k\wedge\tilde{\theta}^m,$$

$$d\tilde{\theta}^j_i=\tilde{\theta}^l_i\wedge\tilde{\theta}^j_l+\frac{1}{2}\tilde{S}^j_{ikm}\tilde{\theta}^k\wedge\tilde{\theta}^m.$$

f induces on B a transformation $\tilde{f}$ in the natural way. Taking a coordinate system $\{x^1,\cdots,x^n\}$ around p in M, we introduce a coordinate system $\{x^1,\cdots,x^n,X^1_1,\cdots,X^n_n\}$ in B. Then we have

$$\tilde{P}^j_{km}=\tilde{Y}^j_i\tilde{X}^p_k\tilde{X}^q_m T^i_{pq},$$

$$\tilde{P}^j_{km,m_t,\cdots,m_1}=\tilde{X}^{p_1}_{m_1}\cdots\tilde{X}^{p_t}_{m_t}\tilde{Y}^j_i\tilde{X}^p_k\tilde{X}^q_m\nabla_{p_1}\cdots\nabla_{p_t}T^i_{pq},$$

where the matrix $||\tilde{Y}^i_j||$ is the inverse matrix of $||\tilde{X}^i_j||$ and T^i_{pq} are the components of the torsion tensor T with respect to the coordinate system. Since f is an affine automorphism of M, we have

$$(1)\qquad \delta\tilde{f}\tilde{P}^j_{km}=\tilde{P}^j_{km},\quad \delta\tilde{f}\tilde{P}^j_{km,m_t,\cdots,m_1}=\tilde{P}^j_{km,m_t,\cdots,m_1}.$$

Denoting by $||a^i_j||$ the matrix defined by $(df)_p(\partial/\partial x^j)_p=a^i_j(\partial/\partial x^i)_p$, and by $||b^i_j||$ the inverse matrix of $||a^i_j||$, we get from (1) that

$$b^j_i a^p_l a^q_m T^i_{pq}(x(p)) = T^j_{lm}(x(p)),$$
$$b^j_i a^p_l a^q_m a^{p_t}_{m_t} \cdots a^{p_1}_{m_1} \nabla_{p_1} \cdots \nabla_{p_t} T^i_{pq}(x(p)) = \nabla_{m_1} \cdots \nabla_{m_t} T^j_{lm}(x(p)).$$

These mean that $(df)_p$ leaves invariant the torsion tensor at p and all their succesive covariant differentials. Similarly we can show that $(df)_p$ also leaves invariant the curvature tensor at p and all their succesive covariant differentials.

Let M be an analytic manifold with an analytic affine connection. Take a normal coordinate system $\{x^1, \cdots, x^n\}$ at o in M, whose coordinate neighborhood is U. Let k be an element of K_o, and $||a^i_j||$ the matrix defined by $k(\partial/\partial x^j)_p = a^i_j(\partial/\partial x^i)_p$. Then we can choose V, a connected open neighborhood of o in U, such that the transformation f defined by

$$y^i = a^i_j x^j (i=1, 2, \cdots, n)$$

is a diffeomorphism from V into U. Let F_o be the frame $\{o, (\partial/\partial x^1)_o, \cdots, (\partial/\partial x^n)_o\}$. For each $p \in U (p \neq o)$ we put $F_p = \tau_{po} F_o$ where τ_{po} is the parallel translation along the unique geodesic from o to p. Thus we have an analytic local cross section I from U into B. By putting $P^i_{jk} = \delta I \tilde{P}^i_{jk}$, $S^i_{jkl} = \delta I \tilde{S}^i_{jkl}$, we obtain analytic functions $P^i_{jk}(x)$, $S^i_{jkl}(x)$ on U. Then we have for $tx = (tx^1, \cdots, tx^n)$

$$(\partial^n/\partial t^n)_{t=0} P^i_{jk}(tx) = x^{q_1} \cdots x^{q_n} \nabla_{q_1} \cdots \nabla_{q_n} T^i_{jk}(0) [2].$$

Remarking that k belongs to K_o, for $y^i = a^i_j x^j$ we have the following.

$$\begin{aligned}(\partial^n/\partial t^n)_{t=0} P^i_{jk}(ty) &= y^{q_1} \cdots y^{q_n} \nabla_{q_1} \cdots \nabla_{q_n} T^i_{jk}(0) \\ &= a^{q_1}_{p_1} \cdots a^{q_n}_{p_n} x^{p_1} \cdots x^{p_n} \nabla_{q_1} \cdots \nabla_{q_n} T^i_{jk}(0) \\ &= a^i_\alpha b^\beta_j b^\gamma_k x^{p_1} \cdots x^{p_n} \nabla_{p_1} \cdots \nabla_{p_n} T^\alpha_{\beta\gamma}(0) \\ &= a^i_\alpha b^\beta_j b^\gamma_k (\partial^n/\partial t^n)_{t=0} P^\alpha_{\beta\gamma}(tx).\end{aligned}$$

On the other hand, $P^i_{jk}(0) = T^i_{jk}(0) = a^i_\alpha b^\beta_j b^\gamma_k T^\alpha_{\beta\gamma}(0) = a^i_\alpha b^\beta_j b^\gamma_k P^\alpha_{\beta\gamma}(0)$. $P^i_{jk}(x)$ being analytic on U, we may assume that

$$P^i_{jk}(tx) = \sum_{n=0}^{\infty} (t^n/n!) (\partial^n/\partial t^n)_{t=0} P^i_{jk}(tx), \text{ for } p \in U, \ 0 \leqq t \leqq 1.$$

Consider the transformation $f: y^i = a^i_j x^j$. Then we have

$$\begin{aligned}P^i_{jk}(ty) &= \sum_{n=0}^{\infty} (t^n/n!) (\partial^n/\partial t^n)_{t=0} P^i_{jk}(ty) \\ &= \sum_{n=0}^{\infty} (t^n/n!) a^i_\alpha b^\beta_j b^\gamma_k (\partial^n/\partial t^n)_{t=0} P^\alpha_{\beta\gamma}(tx).\end{aligned}$$

Therefore we have

(2) $$P^i_{jk}(ty) = a^i_\alpha b^\beta_j b^\gamma_k P^\alpha_{\beta\gamma}(tx).$$

Similarly we can prove that

(3) $$S^i_{jkl}(ty) = a^i_\alpha b^\beta_j b^\gamma_k b^\delta_l S^\alpha_{\beta\gamma\delta}(tx).$$

Consider the forms $\theta^i(x, dx)$, $\theta^i_j(x, dx)$ defined by $\theta^i = \delta I \tilde{\theta}^i$, $\theta^i_j = \delta I \tilde{\theta}^i_j$ on U. We substitute tx^i for x^i, then the following (4) and (5) hold as is well known.

$$(4)\quad \begin{cases} \theta^i = x^i dt + \bar{\theta}^i(t, x, dx), \ \theta^i_j = \bar{\theta}^i_j(t, x, dx) \\ \bar{\theta}^i(0, x, dx) = 0, \ \bar{\theta}^i_j(0, x, dx) = 0. \end{cases}$$

$$(5)\quad \begin{cases} \partial\bar{\theta}^i/\partial t = dx^i + x^j\bar{\theta}^i_j + P^i_{jk}(tx)x^j\bar{\theta}^k \\ \partial\bar{\theta}^i_j/\partial t = S^i_{jkl}(tx)x^k\bar{\theta}^l. \end{cases}$$

We substitute in (5) $y^i = a^i_j x^j$, $\varphi^i = a^i_j\bar{\theta}^j$, $\varphi^i_j = a^i_k b^l_j \bar{\theta}^k_l$. Then we have from (2) and (3),

$$\partial\varphi^i/\partial t = dy^i + y^j\varphi^i_j + P^i_{jk}(ty)y^j\varphi^k,$$
$$\partial\varphi^i_j/\partial t = S^i_{jkl}(ty)y^k\varphi^l.$$

Since $\varphi^i(0, y, dy) = 0$, $\varphi^i_j(0, y, dy) = 0$, according to the uniqueness theorem of differential equations, we have

$$\bar{\theta}^i(t, y, dy) = \varphi^i(t, x, dx), \ \bar{\theta}^i_j(t, y, dy) = \varphi^i_j(t, x, dx),$$

which are equivalent to

$$\theta^i(y, dy) = a^i_j\theta^j(x, dx), \ \theta^i_j(y, dy) = a^i_k b^l_j \theta^k_l(x, dx).$$

Thus we get the following.

Lemma 2. *Let M be an analytic manifold with an analytic affine connection, o a point in M, and k an element of K_o. Then there are a connected open neighborhood V of o and a diffeomorphism f from V into M, such that*

$$\delta f\theta^i = a^i_j\theta^j, \ \delta f\theta^i_j = a^i_k b^l_j \theta^k_l,$$

where $k(\partial/\partial x^j)_o = a^i_j(\partial/\partial x^i)_o$ and $\|b^i_j\|$ is the inverse matrix of $\|a^i_j\|$.

Any open subset of M has an affine connection induced from that of M.

Lemma 3. *Let M be an analytic manifold with an analytic affine connection, o a point in M, and k an element of K_o. Then there are a connected open neighborhood V of o, and an affine isomorphism f from V into M such that $f(o) = o$.*

Proof. Let U be the normal neighborhood of o in Lemma 2, and I the local cross section from U into B. Let f be the diffeomorphism from V into M induced by k in Lemma 2. It is clear that $f(o) = o$. Denoting by ω^k_l the components of the affine connection in the coordinate system, we have

$$(1)\quad \tilde{X}^i_j\tilde{\theta}^j = dx^i,$$
$$(2)\quad \tilde{X}^k_j\tilde{\theta}^j_i = d\tilde{X}^k_i + \omega^k_l\tilde{X}^l_i.$$

Consider the family of frames $\{F_p\}$ on U which we considered in Lemma 2. Denoting each vector of F_p by $L_i = l^k_i(\partial/\partial x^k)_p$, we define X^k_j by

$$(3)\quad \tilde{X}^i_j = X^k_j l^i_k.$$

Applying $\delta f\circ\delta I$ on (1), we get from Lemma 2,

$$(4)\quad (\delta f l^i_j)a^j_k = a^i_j l^j_k.$$

Let $\tilde{f}$ be the transformation induced in B by f. Then it is clear that

$$\delta\tilde{f}\tilde{X}^i_j = a^i_k\tilde{X}^k_j.$$

Applying $\delta\tilde{f}$ on (3) we get from (3) and (4),

(5) $$\delta\tilde{f}X^k_j = a^i_k X^k_j.$$

Applying δI on (2)

(6) $$l^k_j\theta^j_i = dl^k_i + \omega^k_j l^j_i.$$

From (3) and (6) we get

(7) $$X^k_j\tilde{\theta}^j_i = dX^k_i + \theta^k_l X^l_i.$$

Applying $\delta\tilde{f}$ on (7), we get from Lemma 2 and (5) $\delta\tilde{f}\tilde{\theta}^j_i = \tilde{\theta}^j_i$. It is clear that $\delta\tilde{f}\tilde{\theta}^i = \tilde{\theta}^i$. Therefore f is an affine isomorphism from V into M.

§ 2. Main theorem and its applications. In this section we denote by G^0 the identity component of a Lie group G.

Theorem 1. *If M is a connected, simply connected, analytic manifold with a complete analytic affine connection, then $dH_p = K_p$ at each p in M.*

Proof. We have proved in Lemma 1 that $dH_p \subset K_p$ for each p in M. Let k be an element of K_p. By Lemma 3, k induces an affine isomorphism f from V into M, where V is a connected open neighborhood of p. Since M is a connected, simply connected analytic manifold and the connection is complete analytic, this affine isomorphism f can be uniquely extended to an affine automorphism f'([5] p. 255). Clearly $f'(p) = p$, and $(df')_p = k$. Therefore we have $K_p \subset dH_p$.

Corollary. *Let M be the manifold in Theorem 1. Then each element $k \in K_p$ induces a unique affine automorphism f on M such that $f(p) = p$ and $(df)_p = k$.*

In fact, let g be an affine automorphism of M such that $g(p) = p$, and $(dg)_p = k$. Then from $f(p) = g(p)$ and $(df)_p = (dg)_p$, we get $f = g$ on M([5] p. 254).

In 1927, E. Cartan proved the following theorem ([1] p. 84).

Let M be an affine locally symmetric space. If a linear transformation of M_p leaves invariant the curvature tensor R at p, then this induces a local affine isomorphism in M.

We shall treat this problem globally by imposing some conditions on M.

Theorem 2. *Let M be a connected, simply connected, complete affine locally symmetric space, then $K_p = dH_p$ at each p in M.*

Proof. Since $\nabla R = 0$ and $T = 0$, M is considered to be an analytic manifold with an analytic affine connection ([5] p. 263). By Theorem 1, the conclusion follows.

Denoting by $h(p)$ the linear holonomy group of M at p, $h(p)^0$ is the restricted linear holonomy group.

Lemma 4. Let M be an affine locally symmetric space, then $h(p)^0$ is contained in K^0_p at each p in M.

Proof. Take a local coordinate system $\{x^1, \cdots, x^n\}$ around p. Since M is considered to be an analytic manifold with an analytic affine connection, the Lie algebra of $h(p)$ consists of the following matrices [5].

$$\sum_{r,s} \lambda_{rs} R_{rs} \quad \text{where} \quad (R_{rs})^i_j = (R^i_{jrs})_p.$$

We express each element of K_p by a matrix with respect to the base $\{(\partial/\partial x^1)_p, \cdots, (\partial/\partial x^n)_p\}$. Since $T=0$ and $\nabla R=0$, K_p consists of all matrices $||a^i_j||$ which satisfy $b^i_\alpha a^\beta_j a^\gamma_k a^\delta_l (R^\alpha_{\beta\gamma\delta})_p = (R^i_{jkl})_p$, where $||b^i_j||$ is the inverse matrix of $||a^i_j||$. Therefore the Lie algebra of K_p consists of all matrices $||\mu^i_j||$ which satisfy

$$-\mu^i_h (R^h_{jkl})_p + \mu^h_j (R^i_{hkl})_p + \mu^h_k (R^i_{jhl})_p + \mu^h_l (R^i_{jkh})_p = 0.$$

Since M is locally symmetric, from the Ricci identities,

$$\nabla_s \nabla_r R^i_{jkl} - \nabla_r \nabla_s R^i_{jkl} = R^i_{hrs} R^h_{jkl} - R^h_{jrs} R^i_{hkl} - R^h_{krs} R^i_{jhl} - R^h_{lrs} R^i_{jkh} = 0,$$

$$-(R^i_{hrs})_p (R^h_{jkl})_p + (R^h_{jrs})_p (R^i_{hkl})_p + (R^h_{krs})_p (R^i_{jhl})_p + (R^h_{lrs})_p (R^i_{jkh})_p = 0.$$

These mean that the Lie algebra of $h(p)$ is contained in the Lie algebra of K_p. Therefore we have $h(p)^0 \subset K^0_p$.

Theorem 3. *Let M be a connected, simply connected, complete affine locally symmetric space, then the linear holonomy group $h(p)$ is contained in dH_p at each p in M.*

Proof. Since M is connected and simply connected, $h(p)=h(p)^0$. By Theorem 2, $K_p = dH_p$ at each p in M. Therefore the conclusion follows from Lemma 4.

References

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