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On infinitesimal projective transformations of a Riemannian manifold with constant scalar curvature

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§ 1. Introduction

Let M be a connected differentiable Riemannian manifold of dimension n and g_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K, respectively, the components of the metric tensor, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature, here and hereafter the indices $a, b, c, \cdots, i, j, k, \cdots$, run over the range $1, 2, 3, \cdots, n$. We shall denote $g^{ja}\nabla_a$ by ∇^j. An infinitesimal transformation v^h on M is said to be projective if it satisfies,

$$\mathcal{L}\left\{ {h \atop j\ i} \right\} = \nabla_j \nabla_i v^h + K_{aji}{}^h v^a = \delta_j^h \varphi_i + \delta_i^h \varphi_j\,, \tag{1-1}$$

where $\mathcal{L}$ denotes the operator of Lie derivative with respect to v^h $\left\{ {h \atop j\ i} \right\}$ the Christoffel's symbol, δ_j^h the Kronecker's delta and φ_i the associated gradient vector, respectively. In (1-1), if we put $\nabla_i v^i = (n+1)f$, then we have $f_i = \varphi_i$, where f_i means $\nabla_i f$, thus in the following discussions, we use f_i instead of φ_i.

For the infinitesimal projective transformations, the following results are known.

THEOREM A. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits nonaffin infinitesimal projective transformations, then M is a space of positive constant curvature.* [6], [9], [11].

THEOREM B. *Let M be a compact Riemannian manifold with constant scalar curvature. If the scalar curvature is nonpositive, then an infinitesimal projective transformation is a motion.* [14].

The purpose of this paper is to prove the following theorem.

THEOREM. *Let M be a compact, connected and simply connected n-dimensional, $(n>2)$, Riemannian manifold with constant scalar curvature K. If M admits nonisometric infinitesimal projective transformations, then*

M is isometric to a sphere of radius $\sqrt{n(n-1)/K}$.

The following theorem is well known.

THEOREM C. *Let M be a complete, connected and simply connected Riemannian manifold of dimension n. In order that M admits a non-trivial solution* ϕ *for the system of differential equations,*

$$\nabla_k \nabla_j \phi_i + k(2\phi_k g_{ji} + \phi_j g_{ik} + \phi_i g_{kj}) = 0\,, \quad k>0\,, \tag{1-2}$$

where $\phi_i = \nabla_i \phi$, *it is necessary and sufficient that M be isometric with a sphere of radius* $\sqrt{1/k}$. [7], [10].

We have proved the following theorems.

THEOREM D. *If the projective Killing vector* v^h *can be decomposed as follows,*

$$v^h = w^h - \frac{n(n-1)}{2K} f^h\,,$$

then f^h *satisfies the differential equation of* (1-2), *where* w^h *is the Killing vector and the scalar curvature K is positive constant* [15].

THEOREM E. *In a compact Riemannian manifold with positive constant scalar curvature, the projective Killing vector* v^h *is decomposable as follows,*

$$v^h = w^h - \frac{n(n-1)}{2K} f^h$$

if and only if $G_{ai} f^a = 0$, *where* w^h *is the Killing vector and* $G_{ji} = K_{ji} - \frac{K}{n} g_{ji}$. [15].

First of all we shall prove the following propositions.

PORPOSITION 1. *In a Riemannian manifold with constant scalar curvature, there is the following equation.*

$$2G_{ai} f^a + (n-1) F_i = 0\,,$$

where $F = \Delta f + \frac{2(n+1)}{n(n-1)} Kf$, F_i *means* $\nabla_i F$ *and* Δf *denotes* $\nabla_i f^i$.

PROPOSITION 2. *In a compact Riemannian manifold with constant scalar curvature, we have the following equation,*

$$\int G_{ab} f^a f^b \, d\sigma = \frac{n-1}{2} \int F^2 d\sigma\,,$$

where $d\sigma$ *denotes the volume element.*

Thus from Proposition 2, Proposition 1, Theorem E, Theorem D and Theorem C, to prove the Theorem, it is sufficient to show that $\int G_{ab} f^a f^b d\sigma$ is nonpositive.

From (1-1), we have the following equations,

$$\mathcal{L}K_{kji}{}^h = -\delta_k^h \nabla_j f_i + \delta_j^h \nabla_k f_i \,, \tag{1-3}$$

$$\mathcal{L}K_{ji} = -(n-1) \nabla_j f_i \,, \tag{1-4}$$

$$\mathcal{L}K = -2\nabla^a v^b K_{ab} - (n-1) \Delta f \,, \tag{1-5}$$

$$\nabla^a \nabla_a v_i + K_{ai} v_i = 2f_i \,, \tag{1-6}$$

$$\nabla^a \nabla_i v_a - K_{ai} v^a = (n+1) f_i \,. \tag{1-7}$$

§ 2. Proof of Proposition 1.

LEMMA 1. *There exists the following equation,*

$$\mathcal{L}g^{ab} \nabla_i K_{ab} = 0 \,.$$

PROOF. From (1-1) and (1-6), we have

$$\begin{aligned}
0 &= \nabla^j (\nabla_j \nabla_k v_i + K_{ajki} v^a - g_{ji} f_k - g_{ik} f_j) \\
&= \nabla^j \nabla_j \nabla_k v_i + \nabla_j K_{kia}{}^j v^a + K_{ajki} \nabla^j v^a - \nabla_i f_k - g_{ik} \Delta f \\
&= \nabla^j (\nabla_k \nabla_j v_i - K_{jki}{}^a v^a) + (\nabla_k K_{ia} - \nabla_i K_{ka}) v^a + K_{ajki} \nabla^j v^a \\
&\quad - \nabla_i f_k - g_{ik} \nabla f \\
&= \nabla^j \nabla_k \nabla_j v_i - \nabla_j K_{aik}{}^j v^a - K_{jkia} \nabla^j v^a + (\nabla_k K_{ia} - \nabla_i K_{ka}) v^a \\
&\quad + K_{ajki} \nabla^j v^a - \nabla_i f_k - g_{ik} \Delta f \\
&= \nabla_k \nabla^j \nabla_j v_i - K^j{}_{kj}{}^a \nabla_a v_i - K^j{}_{ki}{}^a \nabla_j v_a - (\nabla_a K_{ik} - \nabla_i K_{ak}) v^a \\
&\quad + (K_{ajki} - K_{jkia}) \nabla^j v^a + (\nabla_k K_{ia} - \nabla_i K_{ka}) v^a - \nabla_i f_k \\
&\quad - g_{ik} \Delta f \\
&= \nabla_k (2f_i - K_{ia} v^a) + K_k^a \nabla_a v_i + (K_{ajki} - 2K_{jkia}) \nabla^j v^a \\
&\quad + (\nabla_k K_{ia} - \nabla_a K_{ik}) v^a - \nabla_i f_k - g_{ik} \Delta f \\
&= \nabla_k f_i - K_{ia} \nabla_k v^a + K_k^a \nabla_a v_i + (K_{ajki} - 2K_{jkia}) \nabla^j v^a \\
&\quad - \nabla_a K_{ik} v^a - g_{ik} \Delta f \,.
\end{aligned}$$

Operate ∇^k on the above equation, we obtain the following equation by means of (1-1), (1-6), $\nabla^k \nabla_k f_i = \nabla_i (\Delta f) + K_{ia} f^a$, *and* $\nabla^k K_{ik} = \frac{1}{2} \nabla_i K = 0$,

$$
\begin{aligned}
0 &= \nabla^k \nabla_k f_i - \nabla_k K_{ia} \nabla^k v^a - K_{ia} \nabla^k \nabla_k v^a + K^a_k \nabla^k \nabla_a v^i \\
&\quad + (\nabla_k K_{jai}{}^k - 2\nabla_k K_{iaj}{}^k) \nabla^j v^a + (K_{ajki} - 2K_{jkia}) \nabla^k \nabla^j v^a \\
&\quad - \nabla^k \nabla_a K_{ik} v^a - \nabla_a K_{ik} \nabla^k v^a - \nabla_i (\Delta f) \\
&= \nabla_i (\Delta f) + K_{ia} f^a - \nabla_k K_{ia} \nabla^k v^a - K_{ia} (2f^a - K^a_b v^b) \\
&\quad + K^a_k (-K_b{}^k{}_{ai} v^b + \delta^k_i f_a + g_{ai} f^k) + (\nabla_j K_{ai} + \nabla_a K_{ij} \\
&\quad - 2\nabla_i K_{aj}) \nabla^j v^a + (K_{ajki} - 2K_{jkia}) (-K_b{}^{kja} v^b + g^{ka} f^j + g^{ja} f^k) \\
&\quad - (\nabla_a \nabla^k K_{ik} - K^k{}_{ai}{}^b K_{bk} - K^k{}_{ak}{}^b K_{ib}) v^a - \nabla_a K_{ik} \nabla^k v^a - \nabla_i (\Delta f) \\
&= \nabla_i K_{ab} \mathcal{L} g_{ab} - (K_{ajki} - 2K_{jkia}) K^{bkja} v_b .
\end{aligned}
$$

On the orther hand, since $K_{jkia} + K_{jiak} + K_{jaki} = 0$, we have

$$
\begin{aligned}
(K_{ajki} - 2K_{jkia}) K^{bkja} &= (K_{jiak} - K_{jkia}) K^{bkja} \\
&= K_{jiak} K^{bkja} - K_{aikj} K^{bkja} \\
&= K_{jiak} K^{bkja} - K_{jiak} K^{bakj} \\
&= K_{jiak} K^{bkja} + K_{jika} K^{bakj} \\
&= K_{jiak} K^{bkja} + K_{jiak} K^{bkaj} \\
&= K_{jiak} (K^{bkja} + K^{bkaj}) \\
&= 0 .
\end{aligned}
$$

Therefore Lemma 1 is proved.
Since the scalar curvature is constant and from Lemma 1, we have

$$
\begin{aligned}
0 &= \mathcal{L}(g^{ab} \nabla_i K_{ab}) \\
&= g^{ab} \mathcal{L} \nabla_i K_{ab} + \nabla_i K_{ab} \mathcal{L} g^{ab} \\
&= g^{ab} \mathcal{L} \nabla_i K_{ab} \\
&= g^{ab} \left\{ \nabla_i \mathcal{L} K_{ab} - \mathcal{L} \left\{ {c \atop i\ a} \right\} K_{cb} - \mathcal{L} \left\{ {c \atop i\ b} \right\} K_{ac} \right\} \\
&= g^{ab} \left\{ -(n-1) \nabla_i \nabla_a f_b - (\delta^c_i f_a + \delta^c_a f_i) K_{cb} - (\delta^c_i f_b + \delta^c_b f_i) K_{ac} \right\} \\
&= -2G_{ia} f^a - (n-1) F_i .
\end{aligned}
$$

Thus Proposition 1 is proved.

§ 3. Proof of Proposition 2.

LEMMA 2. *We have the following equation,*

$$
\nabla^a \nabla_a f_i = \frac{n-3}{n-1} G_{ia} f^a - \frac{n+3}{n(n-1)} K f_i .
$$

PROOF. From Proposition 1, we obtain

$$\nabla_i(\Delta f) = -\frac{2}{n-1} G_{ia} f^a - \frac{2(n+1)}{n(n-1)} K f_i .$$

Therefore we have

$$\begin{aligned}\nabla^a \nabla_a f_i &= \nabla^a \nabla_i f_a \\ &= \nabla_i(\Delta f) - K^a{}_{ia}{}^b f_b \\ &= \frac{n-3}{n-1} G_{ia} f^a - \frac{n+3}{n(n-1)} K f_i .\end{aligned}$$

Thus Lemma 2 is proved.

LEMMA 3. *There exists the following equation,*

$$\int \left\{ -\frac{n-2}{n-1} G_{ji} v^i f^i + \frac{2K}{n(n-1)} f^i v_i + f^i f_i \right\} d\sigma = 0 .$$

PROOF. From Lemma 2 and (1-6), we have

$$\begin{aligned}\nabla^j \nabla_j (f^i v_i) &= \nabla^j (\nabla_j f^i v_i + f^i \nabla_j v_i) \\ &= \nabla^j \nabla_j f^i v_i + 2\nabla_j f_i \nabla^i v^i + f^i \nabla^j \nabla_j v_i \\ &= \frac{n-3}{n-1} G_{ji} f^j v^i - \frac{n+3}{n(n-1)} K f^i v_i + 2\nabla_j f_i \nabla^j v^i \\ &\qquad + f^i (2 f_i - K_{ji} v^j) \\ &= -\frac{2}{n-1} G_{ji} f^j v^i - \frac{2(n+1)}{n(n-1)} K f^i v_i + 2 f^i f_i \\ &\qquad + 2\left\{ \nabla^j (v^i \nabla_j f_i) - v^i \nabla^j \nabla_j f_i \right\} \\ &= -\frac{2(n-2)}{n-1} G_{ji} f^j v^i + \frac{4}{n(n-1)} K f^i v_i + 2 f^i f_i + 2\nabla^j (v^i \nabla_j f_i) .\end{aligned}$$

Thus we have Lemma 3 by means of Green's Lemma.

LEMMA 4. *There is the following equation,*

$$\int \left\{ \frac{2}{n-1} G_{ji} f^j v^i + \frac{2(n+1)}{n(n-1)} K f^i v_i + (n+1) f^i f_i \right\} d\sigma = 0 .$$

PROOF. From Lemma 3 and (1-7), we have

$$\begin{aligned}\nabla_j (f^i \nabla_j v^j) &= \nabla_j f^i \nabla_i v^j + f^i \nabla_j \nabla_i v^j \\ &= \nabla_i (\nabla_j f^i v^j) - v^j \nabla_i \nabla_j f^i + f^i \left\{ (n+1) f_i + K_{ij} v^i \right\} \\ &= \nabla_i (\nabla_j f^i v^i) - v^i \left\{ \frac{n-3}{n-1} G_{ji} f^i - \frac{n+3}{n(n-1)} K f_j \right\} \\ &\qquad + (n+1) f^i f_i + K_{ji} v^j f^i\end{aligned}$$

$$=\nabla_i(\nabla_j f^i v^j)+\frac{2}{n-1}G_{ji}v^j f^i+\frac{2(n+1)}{n(n-1)}Kf^i v_i$$
$$+(n+1)f^i f_i .$$

Thus Lemma 4 is proved.

LEMMA 5. *We have the following equation,*

$$\int\left\{\frac{2}{n(n-1)}Kf^i v_i+f^i f_i\right\}d\sigma=0 .$$

PROOF. From Lemma 3 and 4, proof is obvious.

LEMMA 6. *There exists the following equation,*

$$\int\left\{f^i v_i+(n+1)f^2\right\}d\sigma=0 .$$

PROOF. $\nabla_i(fv^i)=f^i v_i+f\nabla_i v^i$
$=f^i v_i+(n+1)f^2 .$

Thus Lemma 6 is proved.

LEMMA 7. *There is the following equation,*

$$2\int G_{ji}f^j f^i d\sigma=(n-1)\int \Delta f F\,d\sigma .$$

PROOF. From Proposition 1, we have

$$2G_{ji}f^j f^i=-(n-1)f_i F^i$$
$$=-(n-1)\left\{\nabla_i(f^i F)-\Delta f F\right\} .$$

Thus Lemma 7 is proved.

LEMMA 8. *We have the following equation,*

$$\int fF\,d\sigma=0 .$$

PROOF. From Lemma 6, we obtain

$$\int f_i v^i d\sigma=-(n+1)\int f^2 d\sigma .$$

Substituting this equation into the equation of Lemma 5, we have

$$0=\int\left\{\frac{2K}{n(n-1)}f_i v^i+f_i f^i\right\}d\sigma$$
$$=-\int\left\{\frac{2(n+1)}{n(n-1)}Kf^2-\nabla_i(ff^i)+f\Delta f\right\}d\sigma$$
$$=-\int fF\,d\sigma .$$

Thus Lemma 8 is proved.

From Lemma 7 and Lemma 8, we have

$$\int F^2 d\sigma = \int F\left(\Delta f + \frac{2(n+1)}{n(n-1)} Kf\right) d\sigma$$
$$= \frac{2}{n-1}\int G_{ji} f^j f^i d\sigma .$$

Therefore Proposition 2 is proved.

§ 4. Proof of Theorem.

LEMMA 9. *There is the following equation,*

$$\int \nabla_j f_i \nabla^j f^i d\sigma = \int\left\{-\frac{n-3}{n-1} G_{ji} f^j f^i + \frac{n+3}{n(n-1)} K f_i f^i\right\} d\sigma .$$

PROOF. From Lemma 2, we have

$$\nabla_j f_i \nabla^j f^i = \nabla_j (f_i \nabla^j f^i) - f_i \nabla_j \nabla^j f^i$$
$$= \nabla_j (f_i \nabla^j f^i) - \frac{n-3}{n-1} G_{ji} f^j f^i + \frac{n+3}{n(n-1)} K f_i f^i .$$

Thus Lemma 9 is proved.

LEMMA 10. *There exists the following equation,*

$$-2\int \nabla_j f_i \nabla^j v^i d\sigma = (n+3)\int f_i f^i d\bar{\sigma} .$$

PROOF. $-2\nabla_j f_i \nabla^j v^i = \nabla_j f_i \mathcal{L} g^{ji}$

$$= \nabla_j (f_i \mathcal{L} g^{ji}) - f_i \nabla_j \mathcal{L} g^{ji}$$
$$= \nabla_j (f_i \mathcal{L} g^{ji}) - f_i \left\{\mathcal{L} \nabla_j g^{ji} - \mathcal{L}\left\{ {j \atop j\ a} \right\} g^{ai} - \mathcal{L}\left\{ {i \atop j\ a} \right\} g^{ja}\right\}$$
$$= \nabla_j (f_i \mathcal{L} g^{ji}) + (n+3) f_i f^i .$$

Thus Lemma 10 is proved.

LEMMA 11. *We have the following equation,*

$$\int (\nabla_j v_i \nabla^j v^i + \nabla_j v_i \nabla^i v^j)\, d\sigma = -(n+3)\int f_i v^i d\sigma$$

PROOF. $\nabla_j v_i \nabla^j v^i + \nabla_j v_i \nabla^i v^j$

$$= -\nabla_j v_i \mathcal{L} g_{ji}$$

$$= -\nabla_j\left\{v_i \mathcal{L} g^{ji}\right\} + v_i \nabla_j \mathcal{L} g^{ji}$$

$$= -\nabla^j (v_i \mathcal{L} g^{ji}) + v_i\left\{\mathcal{L}\nabla_j g^{ji} - \mathcal{L}\left\{ {j \atop j\ a} \right\} g^{ai} - \mathcal{L}\left\{ {i \atop j\ a} \right\} g^{ja}\right\}$$

$$= -\nabla_j (v_i \mathcal{L} g^{ji}) - (n+3) f_i v^i .$$

Thus Lemma 11 is proved.
If we put

$$P_{ji} = \nabla_j f_i + \frac{K}{n(n-1)}(\nabla_j v_i + \nabla_i v_j) ,$$

then we have

$$P_{ji}P^{ji} = \nabla_j f_i \nabla^j f^i + \frac{4K}{n(n-1)}\nabla_j f_i \nabla^j v^i + \frac{2K^2}{n^2(n-1)^2}(\nabla_j v_i \nabla^j v^i + \nabla_j v_i \nabla^i v^j) .$$

Thus by means of Lemma 9, 10, 11, and 5, we have

$$\int P_{ji}P^{ji} d\sigma = -\frac{n-3}{n-1}\int G_{ji} f^j f^i d\sigma .$$

Therefore if dim $M=n\geqq 4$, then $\int G_{ji} f^j f^i d\sigma$ is nonpositive, and Theorem is proved. If dim $M=n=3$, then we have

$$P_{ji} = \nabla_j f_i + \frac{K}{n(n-1)}(\nabla_j v_i + \nabla_j v_i) = 0 .$$

Thus we have

$$0 = g^{ji} P_{ji}$$

$$= \Delta f + \frac{2(n+1)}{n(n-1)} Kf$$

$$= F .$$

In this case, from Proposition 1, Theorem E, Theorem D and Theorem C, Theorem is proved. Therefore Theorem is completely proved.

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