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ON INFORMATION BASED MINIMAL REPAIR AND THE REDUCTION IN REMAINING SYSTEM LIFETIME DUE TO THE FAILURE OF A SPECIFIC MODULE

of

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1. Introduction

An important discussion of minimal repair models is given in the review paper Bergman ((1985),p.24). Here the time τ to failure of a device under study is assumed to have an absolutely continuous cumulative distribution function F and failure rate function r. In the terminology of Bergman (1985) a statistical minimal repair of the device means that if a failure occurs at time t then, after the repair, the survival probability to time $t+s$ equals $(1-F(t+s))/(1-F(t))$ and the failure rate function equals $r(t+s)$, $s>0$. However, the author points out that we have to distinguish between statistical minimal repair, and physical minimal repair in which case the failed unit is restored to the exact physical condition as it had just before the failure. He argues that if the population, from which the device is taken, is not homogeneous, then each failure gives us some information on the subpopulation to which the device belongs.

This difference is made clearer by Arjas and Norros (1987) who simply use the term "black box" minimal repair for statistical minimal repair. To them (and to me) this seems to be a rather abstract notion for a device consisting of several components by simply asking: How does one repair a black box without knowing what is inside? A main point of these authors is that the notion of minimal repair must be related to the information at hand. The paper Aven (1983) on optimal replacement under a minimal repair strategy is in this spirit and generalizes earlier work by Barlow and Hunter (1960). Some further references to papers in this area are given in Bergman (1985) and Arjas and Norros (1987).

In the latter paper minimal repair transformations are shown to be special cases of a general transformation of hazard rates. Furthermore, it is shown that the "black box" minimal repair modeling leads to a stochastically longer total life length than the socalled $\mathbb{F}$-minimal repair, where $\mathbb{F}$ stands for the information which identifies the state of the considered device. In the first part of the present paper we question the fruitfulness of the $\mathbb{F}$-minimal repair concept of Arjas and Norros (1987). For instance it is indicated that for a system of components this does not incorporate the natural minimal repair based on information on the component level. For the case of independent components, some results are also given comparing "black box" minimal repair of a system with the natural minimal repair based on information on the component level.

The socalled Natvig measure of the importance of a component in a coherent system was introduced in Natvig (1979). This measure is for the case of components not undergoing repair proportional to the expected reduction in remaining system lifetime due to the failure of the component. In Natvig (1982) this reduction was interpreted as the increase in remaining system lifetime due to a "black box" minimal repair of the component at its time of failure. Note that a "black box" minimal repair of a <u>single</u> component device is not an abstract notion. A further treatment of this measure is given in Natvig (1985) and in Norros (1986). The latter cleverly applies a martingale approach to treat the case of dependent components.

In the second part of this paper we consider the reduction in remaining system lifetime due to the failure of a specific module of several components and explore the relation to the reduction in remaining system lifetime due to the failure of a component inside the module. Again this former reduction also equals the increase in remaining system lifetime due to a minimal repair of the module at its time of failure. The expected value of this reduction/increase is proportional to the Natvig measure of the importance of the module as treated in our papers mentioned above. In his Ph.D. thesis Xie (1987) considers this measure and treats, for the case of independent components, without any reflection this minimal repair of the module as a "black box" minimal repair. On the basis of this some results are derived. Here we argue, as implicit in our earlier papers and as Arjas and Norros (1987), that the minimal repair of the module often more reasonably should be based on information on the module's components. (These arguments were in fact presented by this author as a faculty opponent in the disputation of Xie's thesis before seeing the ideas of Arjas and Norros (1987).) This leads to results different to the ones derived by Xie (1987).

2. Objections to the $\mathbb{F}$-minimal repair concept of Arjas and Norros (1987)

We start by reproducing the main steps of Arjas and Norros (1987) leading to the $\mathbb{F}$-minimal repair concept. Consider a probability space $(\Omega,\mathcal{F},P)$, and an increasing family of sub-σ algebras $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$. Let S be a totally unpredictable finite $\mathbb{F}$-stopping time and denote by $N = (N_t)_{t\geq 0}$ the corresponding single point counting process $N_t = 1_{\{t\geq S\}}$, $t\geq 0$. Let $A^{\mathbb{F}} = (A_t^{\mathbb{F}})_{t\geq 0}$ be the $\mathbb{F}$-compensator of N.

S is viewed as the life length of a device and $\mathcal{F}_t$ as the available information at time t. While $\mathbb{F}$ is completely general, as a special case, the history generated by N is considered. This is denoted by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$. $\mathbb{G}$ is minimal in the sense that for any other history $\mathbb{F}$ such that S is an $\mathbb{F}$-stopping time, $\mathcal{G}_t \subset \mathcal{F}_t$ must hold for all t. The $\mathbb{G}$-compensator of N, $A^{\mathbb{G}}$, now satisfies

$$A_t^{\mathbb{G}} = R(t\wedge S), \qquad t\geq 0, \tag{2.1}$$

where $R(t) = -\ln \bar{F}(t)$ $(\bar{F}(t) = P(S>t))$ is the cumulative hazard function corresponding to S.

Consider now the change of distributions which arises from "exactly one minimal repair, taking place at the first failure. Under the $\mathbb{G}$-history, where this corresponds to the "black box" minimal repair, the transformed survival function is given by

$$\begin{aligned} Q^{\mathbb{G}}(S>t) &= \bar{F}(t) - \int_0^t (\bar{F}(t)/\bar{F}(s))d\bar{F}(s) \\ &= \bar{F}(t)\ (1+R(t)) \end{aligned} \tag{2.2}$$

The corresponding $\mathbb{G}$-compensator, $B^{\mathbb{G}}$, becomes from (2.1) and (2.2)

$$\begin{aligned} B_t^{\mathbb{G}} &= \ln[\bar{F}(t\wedge S)(1+R(t\wedge S))] \\ &= A_t^{\mathbb{G}} - \ln(1+A_t^{\mathbb{G}}) \end{aligned} \tag{2.3}$$

For a general history $\mathbb{F}$ the $\mathbb{F}$-minimal repair is defined through the $\mathbb{F}$-compensator, $B^{\mathbb{F}}$, given by

$$B_t^{\mathbb{F}} = A_t^{\mathbb{F}} - \ln(1+A_t^{\mathbb{F}}) \qquad (2.4)$$

As we see this definition it is a pure mathematical generalization of (2.3) just replacing the compensator $A^{\mathbb{G}}$ by $A^{\mathbb{F}}$. Furthermore, for a device of components there seems to be no reason that the above defined $\mathbb{F}$-minimal repair concept should incorporate the natural minimal repair based on information on the component level. A somewhat surprising result in Arjas and Norros (1987) is that the transformed life length corresponding to the $\mathbb{F}$-minimal repair is stochastically shorter than the one corresponding to the "black box" minimal repair. However, one should have in mind that the above defined $\mathbb{F}$-minimal repair concept is not as general as one might wish.

To clarify this, consider the simple binary system of three independent binary components depicted in Figure 2.1.

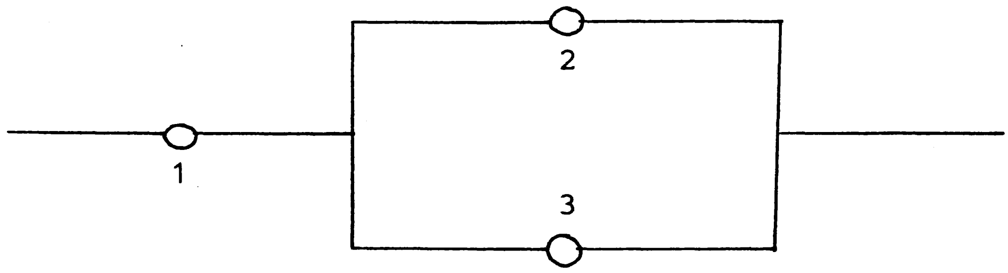

Figure 2.1. A simple binary system.

Let the i th component have an exponential life length distribution with failure rate λ_i, i=1,2,3. Define $\mathbb{G}$ and $\mathbb{F}$ as the histories generated respectively by observing when the system fails and when each component fails. If S is the life length of the system, the corresponding $\mathbb{G}$-compensator follows from (2.1)

$$A_t^{\mathbb{G}} = \lambda_1(t\wedge S) - \ln(e^{-\lambda_2(t\wedge S)} + e^{-\lambda_3(t\wedge S)} - e^{-(\lambda_2+\lambda_3)(t\wedge S)}) \qquad (2.5)$$

Let now S_i be the life length of the i th component, i=1,2,3 and consider the following event

$$S_2 < S_3 = S < S_1 \tag{2.6}$$

Then parallel to (2.1) the $\mathbb{F}$-compensator, evaluated on this event, takes the form

$$A_t^{\mathbb{F}} = \begin{cases} \lambda_1 t - \ln(e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_2+\lambda_3)t}) & 0<t<S_2 \\ \lambda_1 (t\wedge S) - \ln(e^{-\lambda_2 S_2} + e^{-\lambda_3 S_2} - e^{-(\lambda_2+\lambda_3)S_2}) \\ \quad + \lambda_3[(t\wedge S)-S_2] & S_2<t \end{cases}$$

From (2.4) we now get on the same event

$$B_t^{\mathbb{F}} = \begin{cases} \lambda_1 t - \ln(e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_2+\lambda_3)t}) \\ - \ln[1+\lambda_1 t - \ln(e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_2+\lambda_3)t})] & 0<t<S_2 \\ \lambda_1 (t\wedge S) - \ln(e^{-\lambda_2 S_2} + e^{-\lambda_3 S_2} - e^{-(\lambda_2+\lambda_3)S_2}) + \lambda_3[(t\wedge S)-S_2] \\ - \ln\{1+\lambda_1 (t\wedge S) - \ln(e^{-\lambda_2 S_2} + e^{-\lambda_3 S_2} - e^{-(\lambda_2+\lambda_3)S_2}) \\ \quad + \lambda_3[(t\wedge S)-S_2]\} & S_2<t \end{cases} \tag{2.7}$$

The natural minimal repair based on the information on the component level given by (2.6), is a "black box" minimal repair of the third component at S_3. The corresponding compensator is

$$B_t^{*\mathbb{F}} = \begin{cases} \lambda_1 t - \ln(e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_2+\lambda_3)t}) & 0<t<S_2 \\ \lambda_1 (t\wedge S') - \ln(e^{-\lambda_2 S_2} + e^{-\lambda_3 S_2} - e^{-(\lambda_2+\lambda_3)S_2}) \\ \quad + \lambda_3[(t\wedge S')-S_2] & S_2<t \end{cases} \tag{2.8}$$

Here S' is the time of the final breakdown of the system given by

$$S' = \min(S_3+S_3', S_1),$$

where S_3' is exponentially distributed with failure rate λ_3 and independent of S_1 and S_3.

Noting the difference between (2.7) and (2.8) it is clear that in general the $\mathbb{F}$-minimal repair concept does not incorporate the natural minimal repair based on information on the component level.

3. Comparison of "black box" minimal repair of a system with the natural minimal repair based on information on the component level

Consider a system consisting of n components. Let $(i=1,..,n)$

$$X_i(t) = \begin{cases} 1 & \text{if the } i \text{ th component functions at time } t \\ 0 & \text{if the } i \text{ th component is failed at time } t \end{cases}$$

Assume also that the stochastic processes $\{X_i(t),\ t>0\}$ $i=1,\ldots,n$ are mutually independent. Introduce

$$\underline{X}(t) = (X_1(t),\ldots,X_n(t))$$

and let

$$\phi(\underline{X}(t)) = \begin{cases} 1 & \text{if the system functions at time } t \\ 0 & \text{if the system is failed at time } t \end{cases}$$

Now let the i th component have an absolutely continuous life distribution $F_i(t)$ with density $f_i(t)$. Then the reliability of this component at time t is given by

$$P(X_i(t)=1) = 1 - F_i(t) \overset{\text{def}}{=} \bar{F}_i(t)$$

Introduce

$$\underline{\bar{F}}(t) = (\bar{F}_1(t),\ldots,\bar{F}_n(t))$$

Then the reliability of the system at time t is given by

$$P(\phi(\underline{X}(t)=1) = h(\underline{\bar{F}}(t)),$$

where h is the system's reliability function. The following notation will be used

$$(\cdot_i,\underline{x}) = (x_1\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n)$$

We also assume the structure function ϕ to be coherent. For an excellent introduction to coherent structure theory, we refer to Barlow and Proschan (1975a).

The following random variables are of key interest when concentrating on system behaviour after a minimal repair.

X = remaining system lifetime just after the failure of the system, which, however, is immediately "black box" minimally repaired.

Y_i = remaining system lifetime just after the simultaneous failure of the i th component and the system. This component is, however, immediately "black box" minimally repaired.

As in Natvig (1979,1982) let

$$\bar{H}^1_{i,t}(u) = \frac{\bar{F}_i(t+u)}{\bar{F}_i(t)}, \quad \bar{H}^0_{i,t}(u) = 0$$

$$\underline{\bar{H}}^{\underline{x}}_t(u) = (\bar{H}^{x_1}_{1,t}(u),\ldots,\bar{H}^{x_n}_{n,t}(u))$$

Note that the vector $\underline{\bar{H}}^{\underline{x}}_t(u)$ gives the conditional reliabilities of the components at time $t+u$, given the state vector of the components, $\underline{x}$, at time t. Furthermore, let

$$I^{(i)}_B(t) = h(1_i,\underline{\bar{F}}(t)) - h(0_i,\underline{\bar{F}}(t))$$

be the Birnbaum (1969) measure of the importance of the i th component at time t, which is obviously the probability of the component being critical for system functioning at t. Finally let

$$I^{(i)}_{B-P} = \int_0^\infty f_i(t)\; I^{(i)}_B\; dt$$

be the Barlow and Proschan (1975b) measure of the importance of the i th component. This is the probability of a simultaneous failure of the i th component and the system or the probability of the i th component "causing" system failure.

If we denote the life distribution of the system by $F(t)$ with density $f(t)$, we have

$$F(t) \overset{\text{def}}{=} 1-\bar{F}(t) = 1-h(\underline{\bar{F}}(t))$$

$$f(t) = \sum_{i=1}^{n} (-1)\frac{\partial}{\partial \bar{F}_i(t)}[\bar{F}_i(t)h(1_i,\underline{\bar{F}}(t))+(1-\bar{F}_i(t))h(0_i,\underline{\bar{F}}(t))]\frac{\partial \bar{F}_i(t)}{\partial t}$$

$$= \sum_{i=1}^{n} I_B^{(i)}(t)f_i(t)$$

The following theorem is now more or less straightforward under the stated assumptions.

Theorem 3.1

$$P(X>u) = \int_0^\infty f(t)\frac{\bar{F}(t+u)}{\bar{F}(t)}\,dt, \qquad u \geq 0$$

$$P(Y_i>u) = \int_0^\infty f_i(t) \sum_{(\cdot_i,\underline{x})} [\phi(1_i,\underline{x})-\phi(0_i,\underline{x})] \prod_{j\neq i} F_j(t)^{1-x_j}\,\bar{F}_j(t)^{x_j}$$

$$\times\, h(\underline{\bar{H}}_t^{(1_i,\underline{x})}(u))dt$$

$$= \int_0^\infty f_i(t) \sum_{(\cdot_i,\underline{x})} [\phi(1_i,\underline{x})-\phi(0_i,\underline{x})] \prod_{j\neq i} F_j(t)^{1-x_j}\,\bar{F}_j(t)^{x_j}$$

$$\times\, \frac{\bar{F}_i(t+u)}{\bar{F}_i(t)}\, h(1_i,\underline{\bar{H}}_t^{(\cdot_i,\underline{x})}(u))dt, \qquad u \geq 0$$

Especially

$$P(Y_i>0) = I_{B-P}^{(i)}$$

Also the following random variable is of interest.

Y = remaining system lifetime just <u>after</u> the simultaneous failure of a component and the system. The component is, however, immediately "black box" minimally repaired.

Then obviously

$$P(Y>u) = \sum_{i=1}^{n} P(Y_i>u)I_{B-P}^{(i)}$$

The only stochastic comparison we have arrived at is the following.

Theorem 3.2

Let the i th component be in series with the rest of the system. Then

$$P(Y_i>u) < P(X>u), \qquad u > 0$$

Proof.

In this case

$$\bar{F}(t) = \bar{F}_i(t)h(1_i,\underline{\bar{F}}(t))$$

$$f(t) = f_i(t)h(1_i,\underline{\bar{F}}(t)) - \bar{F}_i(t)\frac{\partial}{\partial t} h(1_i,\underline{\bar{F}}(t))$$

$$> f_i(t)h(1_i,\underline{\bar{F}}(t)) = f_i(t)\bar{F}(t)/\bar{F}_i(t)$$

Hence

$$P(Y_i>u) = \int_0^\infty f_i(t)\frac{\bar{F}_i(t+u)}{\bar{F}_i(t)} h(1_i,\underline{\bar{F}}(t+u))dt$$

$$< \int_0^\infty \frac{f(t)}{\bar{F}(t)} \bar{F}(t+u)dt = P(X>u)$$

However, what we are really interested in comparing is $P(X>u)$ and the conditional survival distribution

$$P(Y_i>u|Y_i>0) = \frac{P(Y_i>u)}{P(Y_i>0)}, \qquad u > 0$$

$P(X>u)$ is the survival distribution of the system after a system failure and an immediate "black box" minimal repair of the system. $P(Y_i>u|Y_i>0)$ on the other hand is the conditional survival distribution of the system based on the information that the i th component has "caused" system failure followed by an immediate, natural "black box" minimal repair of this component. In Egeland (1988), using numerical integration techniques, it is shown for a parallel system of two independent components having exponentially distributed life lengths with failure rate λ_1 and λ_2, that

$$P(X>u) > P(Y_1>u|Y_1>0) = e^{-\lambda_1 u}$$

for all $u > 0$ if $\lambda_1 = \lambda_2 = 1$, whereas the strict inequality is reversed for all $u > 0$ if $\lambda_1 = 1$ and $\lambda_2 = 3$. This again indicates that the stochastic comparison result of Arjas and Norros (1987) is mostly of mathematical interest.

4. On the reduction in remaining system lifetime due to the failure of a specific module

As in Natvig (1979,1982) introduce the random variable

Z_i = reduction in remaining system lifetime due to the failure of the i th component.

In the latter paper this reduction was interpreted as the increase in remaining system lifetime due to a "black box" minimal repair of the i th component at its time of failure.

Let the coherent system ϕ have the modular decomposition $\{M_k, \chi_k\}_{k=1}^{a}$ and introduce the random variable

Z_{M_k} = reduction in remaining system lifetime due to the failure of the k th module.

Again this reduction also equals the increase in remaining system lifetime due to a minimal repair of the module at its time of failure. Since a module consists of more than one component, we feel that this minimal repair should not be of the "black box" type as in Xie (1987). Having in mind what is going on physically the minimal repair of the module should rather be interpreted as a "black box" minimal repair of the component in the module that "caused" its failure. This was done in Natvig (1979,1982) and will also be our approach in this paper. What we will explore in this section is the relation between Z_i for $i \in M_k$ and Z_{M_k}.

Let now $\{K_r^i\}_{r=1}^{m_i}$ be the set of minimal cut sets containing the i th component. Introduce the following events

$$A_i = \bigcup_{r=1}^{m_i} \bigcup_{t\in[0,\infty)} \bigcup_{\ell\in K_r^i-\{i\}} \bigcup_{u\in[0,\infty)} \bigcup_{\substack{(\cdot_{K_r^i},\underline{x}) \\ \phi(1_\ell,\underline{0}^{K_r^i-\{\ell\}},\underline{x})\phi(1_i,\underline{0}^{K_r^i-\{i\}},\underline{x})=1 \\ \prod_{s<r} \max_{j\in K_s^i-K_r^i}(x_j)=1}} A_{r,t,\ell,u,\underline{x}}$$

where

$$A_{r,t,\ell,u,\underline{x}} = \{X_i(t-)=1, X_i(t)=0\}$$
$$\cap \{X_\ell(t+u-)=1, X_\ell(t+u)=0\} \cap \{\bigcap_{j\in K_r^i-\{i,\ell\}} \{X_j(t+u)=0\}\}$$
$$\cap \{(\cdot_{K_r^i},\underline{X}(t+u)=(\cdot_{K_r^i},\underline{x})\} \cap \{X_{i(\text{min.rep.})}(t+u)=1\}$$

$$B_i = \bigcup_{r=1}^{m_i} \bigcup_{t\in[0,\infty]} \bigcup_{\substack{(\cdot_{K_r^i},\underline{x}) \\ \phi(1_i,\underline{0}^{K_r^i-\{i\}},\underline{x})=1 \\ \prod_{s<r} \max_{j\in K_s^i-K_r^i}(x_j)=1}} B_{r,t,\underline{x}} ,$$

where

$$B_{r,t,\underline{x}} = \{X_i(t-)=1, X_i(t)=0\}$$
$$\cap \{(\cdot_{K_r^i},\underline{X}(t)) = (\cdot_{K_r^i},\underline{x})\} \cap \{\bigcap_{j\in K_r^i-\{i\}} \{X_j(t)=0\}\}$$

We then have:

$$\{Z_i>0\} = A_i \cup B_i \qquad (4.1)$$

Note that in the A_i (B_i) event the i th component is an element of a minimal cut set K_r^i that would have caused system failure by the failure of the ℓ th (i th) component if the i th component had not been minimally repaired. Since several minimal cut sets containing the i th component can fail simultaneously, we have chosen $(\cdot_{K_r^i}, \underline{x})$ such that $\prod_{s<r} \max_{j \in K_s^i - K_r^i} (x_j) = 1$. This ensures the $A_{r,t,\ell,u,\underline{x}}$ and the $B_{r,t,\underline{x}}$ events to be disjoint. We must have $\phi(1_i, \underline{0}^{K_r^i - \{i\}}, \underline{x}) = 1$ to ensure that no minimal cut set, not containing the i th component, fails by the failure of the ℓ th component.

Especially we get for the case of independent components

$$P(Z_i=0) = 1 - \sum_{r=1}^{m_i} \int_0^\infty \sum_{(\cdot_{K_r^i}, \underline{x})} \prod_{s<r} \max_{j \in K_s^i - K_r^i} (x_j)$$

$$\{ \sum_{\ell \in K_r^i - \{i\}} \int_0^\infty \prod_{j \notin K_r^i} (F_j(t+u)^{1-x_j} \bar{F}_j(t+u)^{x_j}) \times \prod_{j \in K_r^i - \{i,\ell\}} F_j(t+u)$$

$$\phi(1_\ell, \underline{0}^{K_r^i - \{\ell\}}, \underline{x}) \phi(1_i, \underline{0}^{K_r^i - \{i\}}, \underline{x}) f_\ell(t+u) \frac{\bar{F}_i(t+u)}{\bar{F}_i(t)} du$$

$$+ \prod_{j \notin K_r^i} (F_j(t)^{1-x_j} \bar{F}_j(t)^{x_j}) \prod_{j \in K_r^i - \{i\}} F_j(t) \phi(1_i, \underline{0}^{K_r^i - \{i\}}, \underline{x}) \} f_i(t) dt$$

as in the correction to Natvig (1982) except for missing the term $\phi(1_i, \underline{0}^{K_r^i - \{i\}}, \underline{x})$ there. The absolutely continuous part of the distribution of Z_i is given in Lemma 2.1 and Theorem 2.3 of the latter paper.

We now turn to the more difficult Z_{M_k} and introduce the events

$$C_i = \bigcup_{r=1}^{m_i} \bigcup_{t\in[0,\infty]} \bigcup_{\ell\in K_r^i - M_k} \bigcup_{u\in[0,\infty]} \bigcup_{\substack{(\cdot_{(M_k-K_r^i)^c},\underline{y}) \\ \chi_k(1_i,\underline{0}^{(K_r^i-\{i\})\cap M_k},\underline{y})-\chi_k(\underline{0}^{K_r^i\cap M_k},\underline{y})=1}} \bigcup_{\substack{(\cdot_{K_r^i},\underline{x}) \\ x_j<y_j \text{ for } j\in M_k-K_r^i \\ \phi(1_\ell,\underline{0}^{K_r^i-\{\ell\}},\underline{x})\phi(1_i,\underline{0}^{K_r^i-\{i\}},\underline{x})=1 \\ \prod_{s<r} \max_{j\in K_s^i-K_r^i}(x_j)=1}} C_{r,t,\ell,u,\underline{y},\underline{x}} ,$$

where

$$C_{r,t,\ell,u,\underline{y},\underline{x}} = \{X_i(t-)=1, X_i(t)=0\}$$

$$\cap\ \{X_\ell((t+u)-)=1, X_\ell(t+u)=0\} \cap \{\bigcap_{j\in K_r^i-\{i,\ell\}-M_k} \{X_j(t+u)=0\}\}$$

$$\cap\ \{(\cdot_{(M_k-K_r^i)^c},\underline{X}(t))=(\cdot_{(M_k-K_r^i)^c},\underline{y})\} \cap \{\bigcap_{j\in(K_r^i-\{i\})\cap M_k} \{X_j(t)=0\}\}$$

$$\cap\ \{(\cdot_{K_r^i},\underline{X}(t+u)) = (\cdot_{K_r^i},\underline{x})\} \cap \{X_{i(\text{min.rep.})}(t+u)=1\}$$

We then have

$$\{Z_{M_k}>0\} = \bigcup_{i\in M_k}\{C_i \cup B_i\} \tag{4.2}$$

Note that in the C_i event the ℓ th component must lie outside the module M_k since the i th component is the one that would have caused module failure if it had not been minimally repaired.

Especially we get for the case of independent components

$$P(Z_{M_k}=0) = 1 - \sum_{i\in M_k} \sum_{r=1}^{m_i} \int_0^\infty \{ \sum_{\ell\in K_r^i - M_k} \sum_{(\cdot_{(M_k-K_r^i)^c},\underline{y})} [\chi_k(1_i,\underline{0}^{(K_r^i-\{i\})\cap M_k},\underline{y})-\chi_k(\underline{0}^{K_r^i\cap M_k},\underline{y})] \sum_{\substack{(\cdot_{K_r^i},\underline{x})\\ x_j\leq y_j \text{ for } j\in M_k-K_r^i}} \prod_{s<r} \max_{j\in K_s^i-K_r^i}(x_j)$$

$$\int_0^\infty \prod_{j\in(K_r^i-\{i\})\cap M_k} F_j(t) \prod_{j\in M_k-K_r^i} F_j(t)^{1-y_j}(\bar{F}_j(t)-\bar{F}_j(t+u))^{y_j-x_j}\bar{F}_j(t+u)^{x_j}]$$

$$\prod_{j\in M_k^c-K_r^i}[F_j(t+u)^{1-x_j}\bar{F}_j(t+u)^{x_j}]$$

$$\prod_{j\in K_r^i-\{i,\ell\}-M_k}(F_j(t+u))\phi(1_\ell,\underline{0}^{K_r^i-\{\ell\}},\underline{x})\phi(1_i,\underline{0}^{K_r^i-\{i\}},\underline{x})f_\ell(t+u)\frac{\bar{F}_i(t+u)}{\bar{F}_i(t)}du$$

$$+\sum_{(\cdot_{K_r^i},\underline{x})}\prod_{s<r}\max_{j\in K_s^i-K_r^i}(x_j)\prod_{j\notin K_r^i}(F_j(t)^{1-x_j}\bar{F}_j(t)^{x_j})\prod_{j\in K_r^i-\{i\}}F_j(t)$$

$$\phi(1_i,\underline{0}^{K_r^i-\{i\}},\underline{x})\}f_i(t)dt$$

as is a somewhat simplified and again slightly corrected version of the expression given in the correction to Natvig (1982). Again the absolutely continuous part of the distribution of Z_{M_k} is given in Theorem 2.6 of the latter paper.

It is now not hard to realize that $Z_{M_k} = Z_i$ for $i \in M_k$ iff the event $C_i \cup B_i$ occurs. Hence

$$Z_{M_k} = \sum_{i\in M_k} I_{C_i\cup B_i} Z_i \leq \sum_{i\in M_k} Z_i \qquad (4.3)$$

Especially

$$EZ_{M_k} \leq \sum_{i \in M_k} EZ_i \tag{4.4}$$

which was shown for the case of independent components in Natvig (1979).

From (4.3) it also follows that $Z_{M_k} > 0$ implies the existence of $i \in M_k$ such that $Z_i > 0$. The reverse implication is on the other hand not true. Assume for $i \in M_k$ that A_i occurs with $\ell \in K_r^i - M_k$, and hence from (4.1) that $\{Z_i > 0\}$. We can, however, not guarantee that the failure of the i th component causes the failure of the module M_k. Since $\phi(1_i, \underline{0}^{K_r^i - \{i\}}, \underline{x}) = 1$ and $i \in K_r^i$, the module cannot fail before the i th component at time t. It can, however, fail after t along with say $m \in M_k \cap K_r^i$ at time $t+v < t+u$ (assume for instance that the module is a parallel system). Since we can not guarantee that $X_{m(\text{min.rep.})}(t+u) = 1$, the event C_m is not necessarily occuring. In addition B_m is not occuring since we do not have $\bigcap_{j \in K_r^i - \{m\}} (X_j(t+v) = 0)$ due to the fact that $X_\ell(t+u) = 1$.

Finally let M_k be a series system. Then we must have

$(\cdot_{(M_k - K_r^i)^c}, \underline{x}) = (\cdot_{(M_k - K_r^i)^c}, \underline{y}) = (\cdot_{(M_k - \{i\})^c}, \underline{1})$. It is now not hard

to see that $C_i = A_i$. Hence from (4.3) and (4.1)

$$Z_{M_k} = \sum_{i \in M_k} I_{A_i \cup B_i} Z_i = \sum_{i \in M_k} I_{\{Z_i > 0\}} Z_i$$

Since $\{Z_i > 0\}_{i \in M_k}$ are disjoint, we get

$$Z_{M_k} = \sum_{i \in M_k} Z_i \tag{4.5}$$

Especially in this case

$$EZ_{M_k} = \sum_{i \in M_k} EZ_i \ , \tag{4.6}$$

which follows from Theorem 3.7 in Natvig (1979) for the case of independent components. In this case also the existence of $i \in M_k$ such that $Z_i > 0$ implies $Z_{M_k} > 0$.

5. Some comments on the Natvig importance measure of modules as treated in Xie (1987)

In Natvig (1979) the socalled Natvig measure of the importance of the i th component of a coherent system was introduced as

$$I_N^{(i)} = EZ_i / \sum_{j=1}^{n} EZ_j \tag{5.1}$$

Correspondingly the Natvig measure of the importance of the k th module was introduced as

$$I_N^{(M_k)} = EZ_{M_k} / \sum_{j=1}^{a} EZ_{M_j} \tag{5.2}$$

Note that the importance of a module is relative to the specific modular decomposition it is a member of and depends hence totally on the whole modular decomposition.

In Natvig (1985) the following simplified expression for EZ_i was arrived at for the case of independent components

$$EZ_i = \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t))I_B^{(i)}(t)dt \tag{5.3}$$

Xie (1987) suggests the following alternative to (5.2)

$$I_{N'}^{(M_k)} = EZ_{M_k} / \sum_{j=1}^{n} EZ_j \tag{5.4}$$

He claims it to be more reasonable, which we certainly doubt. It is, however, more convenient mathematically when comparing for instance $I_N^{(i)}$ for $i \in M_k$ and $I_{N'}^{(M_k)}$. In fact this boils down to comparing EZ_i for $i \in M_k$ and EZ_{M_k}, as in (4.4) and (4.6), so why bother with the normalization at all in this approach?

When computing (5.4) Xie (1987) is using as numerator

$$EZ_{M_k}^{*} = \int_0^\infty \bar{F}_{M_k}(t)(-\ln \bar{F}_{M_k}(t))I_B^{(M_k)}(t)dt, \tag{5.5}$$

where F_{M_k} is the life distribution of the module. (5.5) is just a copy of (5.3) treating the module M_k as a component. Hence (5.5) gives the expected increase in remaining system lifetime due to a "black box" minimal repair of M_k at its time of failure. We have already questioned whether this is the right approach. A result of Xie (1987), when comparing (5.3) and (5.5), is that

$$EZ_i < EZ^*_{M_k} \qquad \text{for} \quad i \in M_k$$

Then the following question arises. Is this also true in general for our definition of EZ_{M_k}? The answer is no as we shall see.

For the case of independent components the following expression is given for EZ_{M_k} in Theorem 3.7 of Natvig (1979)

$$EZ_{M_k} = \sum_{i \in M_k} \int_0^\infty \sum_{(\cdot_i, \underline{x})} [\chi_k(1_i, \underline{x}^{M_k}) - \chi_k(0_i, \underline{x}^{M_k})]$$

$$\prod_{j \neq i} F_j(t)^{1-x_j} \bar{F}_j(t)^{x_j} \int_0^\infty [h(\underline{\bar{H}}_t^{(1_i,\underline{x})}(u)) - h(\underline{\bar{H}}_t^{(0_i,\underline{x})}(u))] du \; f_i(t) dt$$

As remarked in Natvig ((1985),p.47) conditioning on the states of the components outside the module at the time of failure of the module, is unnecessary. Hence we get

$$EZ_{M_k} = \sum_{i \in M_k} \int_0^\infty \sum_{(\cdot_i, \underline{x}^{M_k})} [\chi_k(1_i, \underline{x}^{M_k}) - \chi_k(0_i, \underline{x}^{M_k})]$$

$$\times \prod_{j \in M_k - \{i\}} F_j(t)^{1-x_j} \bar{F}_j(t)^{x_j} \int_0^\infty [h(\underline{\bar{H}}_t^{(1_i,\underline{x}^{M_k})}(u)_{M_k}, \underline{\bar{F}}(t+u))$$

$$- h(\underline{\bar{H}}_t^{(0_i,\underline{x}^{M_k})}(u)_{M_k}, \underline{\bar{F}}(t+u))] du \; f_i(t) dt \qquad (5.6)$$

As a little digression the same reduction technique can be applied to the expressions for $\bar{G}_{M_k}(u,v)$ and $\bar{G}^1_{M_k}(u)$ in Theorem 2.6 of Natvig (1982) giving the absolutely continuous part of the distribution of Z_{M_k}. Note that this is also done in the event C_i in Section 4. Similarly there is no need for conditioning on the component states at all in the expressions for $\bar{G}_i^1(u)$ and $\bar{G}_i^0(v)$ in Lemma 2.1 of Natvig (1982) giving part of the absolutely continuous distribution of Z_i.

Consider now the simple system of Figure 2.1 and let the two components in parallel constitute the module M_k. From (5.3) and (5.6) we easily get

$$EZ_2 = \int_0^\infty \bar{F}_2(t)\bar{F}_1(t)F_3(t)(-\ln \bar{F}_2(t))dt \tag{5.7}$$

$$EZ_{M_k} = \int_0^\infty F_3(t)\int_0^\infty \frac{\bar{F}_2(t+u)}{\bar{F}_2(t)} \bar{F}_1(t+u)du\ f_2(t)dt$$

$$+ \int_0^\infty F_2(t)\int_0^\infty \frac{\bar{F}_3(t+u)}{\bar{F}_3(t)} \bar{F}_1(t+u)du\ f_3(t)dt$$

By now letting F_1 and F_3 be the exponential distribution with failure rate λ and F_2 the exponential distribution with failure rate $\lambda/2$, we get after some straightforward calculations

$$EZ_{M_k} = \frac{125}{900\lambda} < \frac{128}{900\lambda} = EZ_2$$

Hence we have given a reason for our negative answer.

A final question, having (4.4) and (4.6) in mind, is whether the following relations are true or not for the case of independent components

$$EZ^*_{M_k} < \sum_{i \in M_k} EZ_i\ , \tag{5.8}$$

whereas for M_k as a series module

$$EZ^*_{M_k} = \sum_{i \in M_k} EZ_i \tag{5.9}$$

Both relations are claimed to be true in Xie (1987). The proof for (5.9) is correct whereas (5.8) is mixed up with (4.4) as proved in Natvig (1979).

To establish a counterexample to (5.8) consider the same example as above with an exponential life length distribution of the i th component with failure rate λ_i i=1,2,3. By applying (5.5) and (5.7) we want to find λ_1, λ_2, λ_3 such that

$$\int_0^\infty \Big[\frac{1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t} + \frac{1}{\lambda_1+\lambda_3} e^{-(\lambda_1+\lambda_3)t} - \frac{1}{\lambda_1+\lambda_2+\lambda_3} e^{-(\lambda_1+\lambda_2+\lambda_3)t}\Big]$$

$$\times \big[\lambda_2 e^{-\lambda_2 t} + \lambda_3 e^{-\lambda_3 t} - (\lambda_2+\lambda_3) e^{-(\lambda_2+\lambda_3)t}\big] / \big[e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_2+\lambda_3)t}\big] dt$$

$$> \lambda_2\big[(\lambda_1+\lambda_2)^{-2} - (\lambda_1+\lambda_2+\lambda_3)^{-2}\big] + \lambda_3\big[(\lambda_1+\lambda_3)^{-2} - (\lambda_1+\lambda_2+\lambda_3)^{-2}\big]$$

This inequality is established in Egeland (1988) for $\lambda_1 = \lambda_2 = 0.5$, $\lambda_3 = 8$ by using numerical integration techniques. Similarly the reversed inequality is established for $\lambda_1 = \lambda_2 = 0.5$, $\lambda_3 = 1$.

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