On Input-to-State Stability for Nonlinear Systems with Delayed Feedbacks

MICHAEL MALISOFF
Department of Mathematics
Louisiana State University

Joint with Frédéric Mazenc (Projet MERE INRIA-INRA)
and Zongli Lin (University of Virginia)

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OUTLINE

- Background and Motivation
- Definitions and Assumptions
- Main Theorem
- Extensions to Cascades
- Identification Theory Example
- Two Other Examples
- Conclusions and Summary

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BACKGROUND

Lyapunov Functions: Useful for building stabilizing feedbacks when given by explicit expressions. For example, Lyapunov functions for

$$\dot{x} = f(x)$$

yield input-to-state stability (ISS) stabilizers $K(x)$ for

$$\dot{x}(t) = f(x(t)) + g(x(t))[K(x(t)) + d(t)]$$

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STATEMENT of PROBLEM

Our Approach: Use a given Lyapunov function V for a UGAS system

$$\dot{x} = f(x,t) + g(x,t)u_s(x,t) \qquad (\Sigma_{\rm nd})$$

to explicitly construct an ISS Lyapunov-Krassovski functional for

$$\dot{x}(t) = f(x(t),t) + g(x(t),t)[u_s(\xi_\tau(t),t) + d(t)], \qquad (\Sigma_{\rm d})$$

where $\xi_\tau(t) = (x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n))$ and $0 \leq \tau_i \leq \bar{\tau}$.

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Motivation: General delayed systems of this kind are useful in networks; see e.g. Nesic-Teel (2004).

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ASSUMPTIONS and DEFINITIONS

Assumption A: f, g, and u_s are locally Lipschitz. $\exists$ constant $\bar{L} > 0$ s.t. $\forall x \in \mathbb{R}^n, t \geq 0$, (A1) $|f(x,t)| \leq \bar{L}|x|$, (A2) $|g(x,t)| \leq \bar{L}(|x| + 1)$, and (A3) $|\partial u_s / \partial x(x,t)| \leq \bar{L}$.

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Consequences: (I) For all choices of $t_o \geq 0$, $\tau > 0$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$, and $d \in \mathcal{L}_m^\infty([0,\infty))$, the initial value problem

$$\begin{aligned} &\dot{x}(t) = f(x(t),t) + g(x(t),t)[u(x(t-\tau),t) + d(t)] \\ &\forall t \geq t_o \ \text{ a.e. and } \ x(r) = x_o(r) \ \ \forall r \in [t_o - \tau, t_o]\,. \end{aligned} \qquad \text{(IP)}$$

has a unique solution $t \mapsto x(t; t_o, x_o, d, \tau)$ defined on $[t_o - \tau, +\infty)$.

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(II) For each $\kappa \in \mathbb{N}$ and $\tau > 0$, we can construct $\bar{\gamma}_{\kappa,\tau} \in \mathcal{K}_\infty$ s.t.

$|x(t; t_o, x_o, d, \tau)| \leq \bar{\gamma}_{\kappa,\tau}(|x_o|_{[t_o-\tau,t_o]}) + \bar{\gamma}_{\kappa,\tau}(|d|_{[t_o,t]})\ \forall t \in [t_o, t_o + \kappa\tau]$

for all $t_o \geq 0$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$, and $d \in \mathcal{L}_m^\infty([0,\infty))$.

INPUT-TO-STATE STABILITY

We say $\beta \in \mathcal{KL}$ provided $\beta(\cdot, t) \in \mathcal{K}_\infty \; \forall t \geq 0$, $\beta(s, \cdot)$ non-increasing $\forall s \geq 0$, and $\beta(s, t) \to 0$ as $t \to +\infty \; \forall s \geq 0$. Set $x_t(\theta) = x(t + \theta)$.

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ISS: We say that (Σ_d) is input-to-state stable (ISS) provided there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t; t_o, x_o, d, \tau)| \ \leq \ \beta(|x_o|_{[t_o-\tau, t_o]}, t - t_o) + \gamma(|d|_{[t_o, t]})$$

for all $t_o \geq 0$, $\tau \in (0, \bar{\tau}]$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$, and $d \in \mathcal{L}_m^\infty([0, \infty))$.

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ISS-LKF: A continuous $U : \mathcal{C}_n(\mathbb{R}) \times [0, \infty) \to [0, \infty)$ is called an *ISS Lyapunov-Krasovskii functional (ISS-LKF)* for (Σ_d) provided $\forall \tau \in (0, \bar{\tau}]$ and $x(t) := x(t; t_o, x_o, d, \tau)$, the function $t \mapsto U(x_t, t)$ is locally AC & $\exists \alpha_i \in \mathcal{K}_\infty$ and $\kappa \in \mathbb{N}$ s.t. $\forall x(t)$, $\phi \in \mathcal{C}_n([-\kappa\bar{\tau}, 0])$, and $t \geq t_o + \kappa\bar{\tau}$,

(i) $\alpha_1(|\phi(0)|) \leq U(\phi, t) \leq \alpha_2(|\phi|_{[-\kappa\bar{\tau}, 0]})$ and

(ii) $D_t U(x_t, t) \leq -\alpha_3(U(x_t, t)) + \alpha_4(|d|_{[t_o, t]})$ a.e..

MORE DEFINITIONS and ASSUMPTIONS

Assumption H: The feedback $u_s \in C^1$. Also, there are $\sigma \in \mathcal{K}_\infty$ such that $\sigma(r) \leq r$ for all $r \geq 0$; constants $K_1 \geq 1$ and $K_i \geq 0$ ($i = 2, 3, 4$); and a C^1 uniformly proper and positive definite $V : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ such that for all $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $l \geq 0$, and $t \geq 0$, we have

H1 $V_t(x,t) + V_x(x,t)[f(x,t) + g(x,t)u_s(x,t)] \leq -\sigma(|x|)^2$;

H2 $|V_x(x,t)g(x,t)| \leq K_1\sigma(|x|)$, $\left|\frac{\partial u_s}{\partial x}(x,t)f(x,l)\right|^2 \leq K_2\sigma(|x|)^2$;

H3 $\left|\frac{\partial u_s}{\partial x}(x,t)g(x,l)\right|^2 \leq K_3(\sigma(|x|) + 1)$; and

H4 $\left[\left|\frac{\partial u_s}{\partial x}(x,t)g(x,l)\right| |u_s(q,l)|\right]^2 \leq K_4[\sigma^2(|x|) + \sigma^2(|q|)]$.

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Allows stable linear system with bounded g and quadratic V, plus cases where the system is not exponentially stable or g is unbounded. Set

$$\bar{\tau} := \frac{1}{4K_1\sqrt{3K_2 + 3K_4 + 1}}$$

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THEOREM 1

Theorem 1: Under the above assumptions, the feedback delayed system

$$\dot{x}(t) = f(x(t), t) + g(x(t), t)[u_s(x(t-\tau), t) + d(t)] \qquad (\Sigma_{\mathrm{d}})$$

with any constant feedback delay satisfying

$$0 < \tau \le \bar{\tau} := \frac{1}{4K_1\sqrt{3K_2 + 3K_4 + 1}}$$

admits the ISS-LKF

$$U(x_t, t) = V(x(t), t) + \frac{1}{4\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \left(\int_{r}^{t} \sigma^2(|x(l)|)\mathrm{d}l \right) \mathrm{d}r$$

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Remark: When $V_t \equiv 0$ and the drift $f \equiv 0$, we can make the delay bound $\bar{\tau}$ arbitrarily large by taking $K_2 = 0$ and scaling u_s:

$$\sigma \to \sqrt{\eta}\sigma, \quad u_s \to \eta u_s, \quad K_1 \to K_1/\sqrt{\eta}, \quad K_4 \to \eta^3 K_4, \quad \eta \downarrow 0.$$

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EXTENSION to CASCADES

Cascades: Under the above assumptions on the subsystem

$$\dot{x}(t) = f(x(t)) + g(x(t))z(t)$$

with fictitious input z and time-invariant functions, we design u to render

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))z(t), \\ \dot{z}(t) = u(x(t-2\tau), x(t-\tau), z(t-\tau)) + d(t) \end{cases} \qquad (\Sigma_{\text{cas}})$$

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ISS Estimate: We use the generalization

$$|(x,z)(t)| \leq \beta(|(x,z)|_{[t_o-\tau,t_o]}, t-t_o) + \gamma(|d|_{[t_o,t]})$$

of our earlier ISS estimate. Initial functions constant on $(-\infty, t_o - \tau]$.

EXTENSION to CASCADES (cont'd)

Theorem 2: Set

$$\bar{\tau}_c = \min\{1/\sqrt{8}, \bar{\tau}\} \text{ and } Z(t) = z(t) - u_s(x(t-\tau)).$$

Then for each constant $\tau \in (0, \bar{\tau}_c]$, the dynamics ($\Sigma_{\text{cas}}$) is ISS when

$$u(x(t-2\tau), x(t-\tau), z(t-\tau)) :=$$
$$-Z(t-\tau) + \frac{\partial u_s}{\partial x}(x(t-\tau))\left[f(x(t-\tau)) + g(x(t-\tau))z(t-\tau)\right].$$

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Strategy of Proof: First build an ISS-LKF for the $q = (x, Z)$ dynamics $\dot{q}(t) = F(q(t)) + G(q(t))[U_s(q(t-\tau)) + D(t)]$ with

$$F(q) = \begin{pmatrix} f(x) + g(x)Z \\ 0 \end{pmatrix}, \quad G(q) = \begin{pmatrix} g(x) & 0 \\ 0 & -1 \end{pmatrix},$$

the feedback $U_s(q) = (u_s(x), Z)^T$ and $D(t) = (0, -d(t))^T$.

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EXAMPLE from IDENTIFICATION THEORY

When $m : \mathbb{R} \to \mathbb{R}^n$ is continuous, we build an ISS-LKF for

$$\dot{x}(t) = -m(t)m^T(t)[x(t-\tau) + d(t)]. \qquad (\Sigma_{\mathrm{id}})$$

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Assume $|m(t)| = 1 \ \forall t \in \mathbb{R}$ and $\exists \alpha' \in (0,1)$ and $\beta', \tilde{c} > 0$ such that

$$\alpha' I_{n\times n} \ \leq \ \int_t^{t+\tilde{c}} m(\tau)m^T(\tau)\mathrm{d}\tau \ \leq \ \beta' I_{n\times n} \quad \forall t \in \mathbb{R}.$$

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Lemma: The function $V(x,t) := x^T P(t) x$ satisfies $\dot{V} \leq -\alpha' |x|^2/2$ along all trajectories of $\dot{x}(t) = -m(t)m^T(t)x(t)$ when

$$P(t) = \kappa I + \int_{t-\tilde{c}}^t \int_s^t m(l)m^T(l)\,\mathrm{d}l\,\mathrm{d}s$$

and $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$. Moreover, $|P(t)| \leq \kappa + \tilde{c}^2$ everywhere.

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and $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$. Moreover, $|P(t)| \leq \kappa + \tilde{c}^2$ everywhere.

Corollary: Let $\tau \in (0, \bar{\tau}]$. Then $(\Sigma_{\rm id})$ has this ISS-LKF and so is ISS:

$$U(x_t,t) \;\; = \;\; x^T(t)P(t)x(t) \; + \; \frac{\alpha'}{16\bar{\tau}} \int_{t-2\tau}^t \left(\int_r^t |x(l)|^2 {\rm d}l \right) {\rm d}r.$$

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$$\dot{x}(t) \quad = \quad \sqrt{1+x^2(t)}\left[-\int_0^{x(t-\tau)} \tfrac{1}{\sqrt{1+l^2}}\mathrm{d}l + d(t)\right]$$

is ISS when $0 < \tau \leq 1/8$ and we can build corresponding ISS-LKFs.

OUTLINE

- Background and Motivation
- Definitions and Assumptions
- Main Theorem
- Extensions to Cascades
- Identification Theory Example
- Two Other Examples
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