J. Operations Research Soc. of Japan
Vol. 17, No. 1, March 1974

ON INTEGER FRACTIONAL PROGRAMMING

YUICHIRO ANZAI

Keio University

(Received May 18, 1973)

Abstract

An algorithm to solve integer linear fractional programming problems is proposed. The procedure is reduced to the solution of a sequence of integer linear subproblems. The number of subproblems necessary to be solved is expected to be fairly small, and the finite convergence to the global optimum is guaranteed. Some properties of the algorithm including the relations to the generalized Lagrangian method and to continuous fractional programming are discussed. Besides, the application to some goal programming problems is described by an example.

1. Introduction

The problems of the type:

$$(1) \qquad \min_{\boldsymbol{x}} \frac{\boldsymbol{p}^T\boldsymbol{x}+r}{\boldsymbol{q}^T\boldsymbol{x}+s}$$

$$\text{subj. to} \quad \boldsymbol{x} \in \bar{X} = \{\boldsymbol{x} \mid A\boldsymbol{x} \leqslant \boldsymbol{b}, \ \boldsymbol{x} \geqslant \boldsymbol{0}\} \subseteq E^n$$

$$\text{where} \quad \boldsymbol{q}^T\boldsymbol{x}+s>0 \quad \text{for all} \quad \boldsymbol{x} \in \bar{X}$$

are called linear fractional programs. They belong to the class of quasiconvex programs since the objective function is quasiconvex over a convex subset of E^n, and a lot of effective solution procedures have been proposed in [5], [6], [12], [15], [16], [19].

If we restrict the variables in (1) to be integers, the problem becomes an integer linear fractional program. In this paper, an algorithm to solve such programs is proposed, and some properties including the relations to the generalized Lagrangian method and to some (continuous) linear fractional programming algorithms are discussed. Furthermore, a goal programming problem is solved by the algorithm so that its applicability to practical problems is suggested. The procedure is reduced to solving integer linear subproblems in each iteration, and the finite convergence to the global optimum is guaranteed if the feasible region is bounded and the degeneracy is avoided. Besides, the number of integer linear programs necessary to be solved is expected to be fairly small.

2. Development of the Algorithm

We restrict our attention to the problems of the following type in this paper:

(2)
$$\min_{\boldsymbol{x}} \frac{\boldsymbol{p}^T\boldsymbol{x}+r}{\boldsymbol{q}^T\boldsymbol{x}+s}$$

subj. to $\boldsymbol{x} \in X=\{\boldsymbol{x} \mid A\boldsymbol{x} \leqslant \boldsymbol{b},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x}$ is an integer vector$\}$
where $\boldsymbol{q}^T\boldsymbol{x}+s>0$ for all $\boldsymbol{x} \in X$.

To begin with, note that (2) is equivalent to the following problem:

$$\min_{\boldsymbol{x},\, x_{n+1}} \frac{\boldsymbol{p}^T\boldsymbol{x}+rx_{n+1}}{\boldsymbol{q}^T\boldsymbol{x}+sx_{n+1}}$$

subj. to $\boldsymbol{x} \in X,\ x_{n+1}=1$
where $\boldsymbol{q}^T\boldsymbol{x}+sx_{n+1}>0$ for all $\boldsymbol{x} \in X,\ x_{n+1}=1$.

Hence, it is sufficient to develop an algorithm for the problems of the following type without loss of generality:

(3)
$$\min_{\boldsymbol{x}} \frac{\boldsymbol{p}^T\boldsymbol{x}}{\boldsymbol{q}^T\boldsymbol{x}}$$

subj. to $\boldsymbol{x} \in X=\{\boldsymbol{x} \mid A\boldsymbol{x} \leqslant \boldsymbol{b},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x}$ is an integer vector$\}$.

We deal with the problem (3) in this paper, and assume that X

is nonvoid and bounded, and $q^T x > 0$ for all $x \in X$. Let $X = \{x^1, \cdots, x^s\}$.

Lemma 1. There exists a finite optimal solution such that only one component is positive and others all zero, $y^* = (y_1^*, y_2^*, \cdots, y_s^*)^T$, of the problem:

$$
(4) \qquad \begin{aligned} &\min \sum_k p^T x^k y_k \\ &\text{subj. to } \sum_k q^T x^k y_k = 1 \\ &y_k \geqslant 0. \end{aligned}
$$

Proof. Since $X \neq \phi$, bounded and $q^T x > 0$ for all $x \in X$ by the assumption, the linear program (4) has a finite optimal solution. Furthermore, since it has just one equality constraint with positive right-hand side, there exists at least one optimal solution, y^*, such that just one component is strictly positive.

Theorem 1. Let $y^* = (y_1^*, y_2^*, \cdots, y_s^*)^T$ be an optimal solution of (4) such that $y_r^* > 0$ and that $y_k^* = 0$ for all $k \neq r$. Then, x^r is an optimal solution of (3).

Proof. Since $y_k = 1/q^T x^k$, $y_i = 0$ $(i \neq k)$ is a feasible solution of (4) for all k,

$$\frac{p^T x^r}{q^T x^r} = \min_k \frac{p^T x^k}{q^T x^k}$$

which implies that x^r is an optimal solution of (3).

Suppose that a feasible basic solution of (4) is given by the primal revised simplex method, and let π be the current value of the simplex multiplier. Then, the cost coefficient of y_k is $p^T x^k - \pi q^T x^k$ for $k = 1, \cdots, s$. Let

$$z = p^T x^r - \pi q^T x^r = \min_k p^T x^k - \pi q^T x^k.$$

If $z \geqslant 0$, the current feasible basis is optimal. Otherwise, y_r should be brought into the basis. As $X = \{x^1, \cdots, x^s\}$, x^r can be found by solving the following integer linear program:

$$
(5) \qquad \begin{aligned} &\min_x p^T x - \pi q^T x \\ &\text{subj. to } x \in X. \end{aligned}
$$

If the optimal objective value of (5) is nonnegative, an optimal solution of (4) has been obtained.

Barring the degeneracy, the optimum of (4) is attained by solving the finite number of the problems (5) by the assumption that X is bounded. Furthermore, if $\boldsymbol{x}^*$ is the optimal solution of (5) at some iteration in the solution of (4), the basis inverse and the cost coefficient of the basic variable are $1/\boldsymbol{q}^T\boldsymbol{x}^*$ and $\boldsymbol{p}^T\boldsymbol{x}^*$, respectively. Thus, the current value of the simplex multiplier is $\pi=\boldsymbol{q}^T\boldsymbol{x}^*/\boldsymbol{q}^T\boldsymbol{x}^*$.

Combining the above discussions with Theorem 1 provides the finite algorithm to solve (3) as follows.

Step 1. Set $\pi=M$, where M is a sufficiently large number.

Step 2. Find an optimal solution of (5), $\boldsymbol{x}^*$. If the optimal objective value is nonnegative, stop. $\boldsymbol{x}^*$ is an optimal solution of (3). Otherwise, go to Step 3.

Step 3. Set $\pi=\boldsymbol{p}^T\boldsymbol{x}^*/\boldsymbol{q}^T\boldsymbol{x}^*$ and go to Step 2.

Note that

(a) the primal feasibility of (4) is retained after the first feasible solution is found, which implies that a value of the objective function of (4) at any iteration in Phase 2 provides an upper bound of the objective value of the original problem (3), and that (b) if (4) is infeasible, then (3) is infeasible.

Practical interest for the algorithm consists in how many integer linear programs (5) should be solved to obtain an optimal solution of (4). But the number of iterations necessary in most linear programming problems is empirically around $3m$ where m is the number of constraints. Since $m=1$ in (4), the number of problems necessary to be solved is fairly small, perhaps around 3 after the first feasible solution is found.

Illustrative Example

Let us solve the following problem by the algorithm:

$$\min_{x_1, x_2} \frac{x_1-3x_2}{3x_1+2x_2}$$

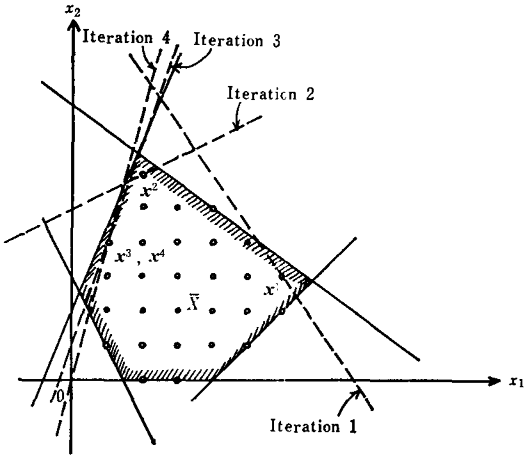

Fig. 1. Feasible region and optimal contours of subproblems of the example in Section 2.

$$\text{subj. to } \boldsymbol{x}=(x_1, x_2)^T \in X=\{\boldsymbol{x} \mid -5x_1+2x_2 \leqslant 4, \; 2x_1+x_2 \geqslant 3, \; x_1-x_2 \leqslant 4, \; x_1, x_2 \geqslant 0, \text{ integer}\}.$$

Let the optimal solution and the corresponding objective value of (5) at the ith iteration be $\boldsymbol{x}^i$ and z^i, respectively. The optimal contour of the objective function of (5) in each iteration is illustrated in Fig. 1.

Iteration 1. Let π be a sufficiently large number, say, 100, then the subproblem (5) becomes:

$$\min_{\boldsymbol{x} \in X} (x_1-3x_2)-100(3x_1+2x_2)$$
$$=-299x_1-203x_2.$$

Then, $\boldsymbol{x}^1=(6, 3)^T$, $z^1=-2403$. So $\pi=\left.\dfrac{x_1-3x_2}{3x_1+2x_2}\right|_{\boldsymbol{x}^1}=-1/8$.

Iteration 2: the subproblem (5) is:

$$\min_{\boldsymbol{x} \in X} (x_1-3x_2)+(1/8)(3x_1+2x_2)$$

$$=(11/8)x_1-(11/4)x_2\,.$$

Then, $\boldsymbol{x}^2=(2, 6)^T$, $z^2=-55/4$, and $\pi=\left.\dfrac{x_1-3x_2}{3x_1+2x_2}\right|_{\boldsymbol{x}^2}=-8/9$.

Iteration 3. the subproblem (5) is:

$$\min_{\boldsymbol{x}\in X}(x_1-3x_2)+(8/9)(3x_1+2x_2)$$
$$=(11/3)x_1-(11/9)x_2\,.$$

Then, $\boldsymbol{x}^3=(1, 4)^T$, $z^3=-11/9$, and $\pi=-1$.

Iteration 4: the subproblem (5) is:

$$\min_{\boldsymbol{x}\in X}(x_1-3x_2)+(3x_1+2x_2)$$
$$=4x_1-x_2\,.$$

Then, $\boldsymbol{x}^4=(1, 4)^T$, $z^4=0$. Since the optimal objective value of (5) is nonnegative, an optimal solution is obtained, which is $x_1=1$, $x_2=4$, and the optimal objective value is -1.

Extension to multisector problems. Consider the following large-scale nonlinear integer programming problem, in which p subsystems are coupled in the fractional objective function:

$$\text{(6)}\qquad \min_{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_p}\frac{\boldsymbol{p}_1^T\boldsymbol{x}_1+\cdots+\boldsymbol{p}_p^T\boldsymbol{x}_p}{\boldsymbol{q}_1^T\boldsymbol{x}_1+\cdots+\boldsymbol{q}_p^T\boldsymbol{x}_p}$$

subj. to $\boldsymbol{x}_i\in X_i=\{\boldsymbol{x}_i\,|\,A_i\boldsymbol{x}_i\leqslant\boldsymbol{b}_i,\ \boldsymbol{x}_i\geqslant\boldsymbol{0},\text{ integer vector}\}$,
$i=1,\cdots,p$.

We assume that X_i is nonvoid and bounded for all $i=1,\cdots,p$, and $\sum_{i=1}^{p}\boldsymbol{q}_i^T\boldsymbol{x}_i>0$ for $\boldsymbol{x}_i\in X_i$, $i=1,\cdots,p$.

Applying the above algorithm to the problem (6), subproblems (5) become:

$$\min_{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_p}\sum_{i=1}^{p}\boldsymbol{p}_i^T\boldsymbol{x}_i-\pi\sum_{i=1}^{p}\boldsymbol{q}_i^T\boldsymbol{x}_i$$

subj. to $\boldsymbol{x}_i\in X_i$, $i=1,\cdots,p$,

or

$$\text{(7)}\qquad \begin{array}{l}\min_{\boldsymbol{x}_i}\boldsymbol{p}_i^T\boldsymbol{x}_i-\pi\boldsymbol{q}_i^T\boldsymbol{x}_i\\ \text{subj. to }\boldsymbol{x}_i\in X_i\end{array}\qquad i=1,\cdots,p\,.$$

Though (6) is the nonlinearly coupled system, its optimal solution can be obtained by the iterative solution of p independent linear subproblems (7). Since the efficiency of integer programming algorithms generally decreases nonlinearly with the increase of the number of variables, the above machinary may improve the computational efficiency. Actually, the problem (3) is the special case of (6) for $p=1$.

Illustrative Example

$$\min \frac{-x_1-3x_2-x_3+x_4-x_5}{2x_1+x_2+3x_3+5x_4+x_5}$$

$$\begin{aligned} \text{subj. to } & \boldsymbol{x}_1=(x_1, x_2)^T \in X_1=\{\boldsymbol{x} \mid -x_1+2x_2 \leqslant 9, \\ & 4x_1+x_2 \leqslant 21,\ x_1-3x_2 \leqslant 0,\ 4x_1+5x_2 \geqslant 12, \\ & x_1, x_2 \geqslant 0, \text{ integer}\} \\ & \boldsymbol{x}_2=(x_3, x_4, x_5)^T \in X_2=\{\boldsymbol{x} \mid -2x_3+6x_4 \leqslant 21, \\ & 7x_3+4x_4 \leqslant 39,\ x_5=1,\ x_3, x_4, x_5 \geqslant 0, \text{ integer}\}. \end{aligned}$$

Iteration 1: let $\pi=100$, then the two subproblems (7) become:

subproblem 1: $\min_{\boldsymbol{x}_1 \in X_1} (-1, -3)\boldsymbol{x}_1-100(2, 1)\boldsymbol{x}_1$

$$=-201x_1-103x_2.$$

subproblem 2: $\min_{\boldsymbol{x}_2 \in X_2} (-1, 1, -1)\boldsymbol{x}_2-100(3, 5, 1)\boldsymbol{x}_2$

$$=-301x_3-499x_4-101x_5.$$

Optimal solutions are $\boldsymbol{x}_1^1=(4, 5)^T$, $\boldsymbol{x}_2^1=(3, 4, 1)^T$. The sum of the optimal objective values is $z^1=z_1^1+z_2^1=-4319.$ Then,

$$\pi=\frac{(-1, -3)\boldsymbol{x}_1^1+(-1, 1, -1)\boldsymbol{x}_2^1}{(2, 1)\boldsymbol{x}_1^1+(3, 5, 1)\boldsymbol{x}_2^1}=-19/43.$$

Iteration 2: the subproblems (7) are:

subproblem 1: $\min_{\boldsymbol{x}_1 \in X_1} (-1, -3)\boldsymbol{x}_1+(19/43)(2, 1)\boldsymbol{x}_1$

$$=-(5/43)x_1-(110/43)x_2,$$

subproblem 2: $\min_{\boldsymbol{x}_2 \in X_2} (-1, 1, -1)\boldsymbol{x}_2+(19/43)(3, 5, 1)\boldsymbol{x}_2$

$$=(14/43)x_3+(138/43)x_4-(24/43)x_5.$$

Then, $\boldsymbol{x}_1^2=(3, 6)^T$, $\boldsymbol{x}_2^2=(0, 0, 1)^T$, and $z^2=z_1^2+z_2^2=-675/43-24/43=-699/43$. And substituting $\boldsymbol{x}_1^2$ and $\boldsymbol{x}_2^2$ into the objective function, we obtain

$\pi=-22/13$.

Iteration 3: the subproblems (7) are:

subproblem 1: $\min_{\boldsymbol{x}_1 \in X_1} (-1, -3)\boldsymbol{x}_1+(22/13)(2, 1)\boldsymbol{x}_1$

$=(31/13)x_1-(17/13)x_2$,

subproblem 2: $\min_{\boldsymbol{x}_2 \in X_2} (-1, 1, -1)\boldsymbol{x}_2+(22/13)(3, 5, 1)\boldsymbol{x}_2$

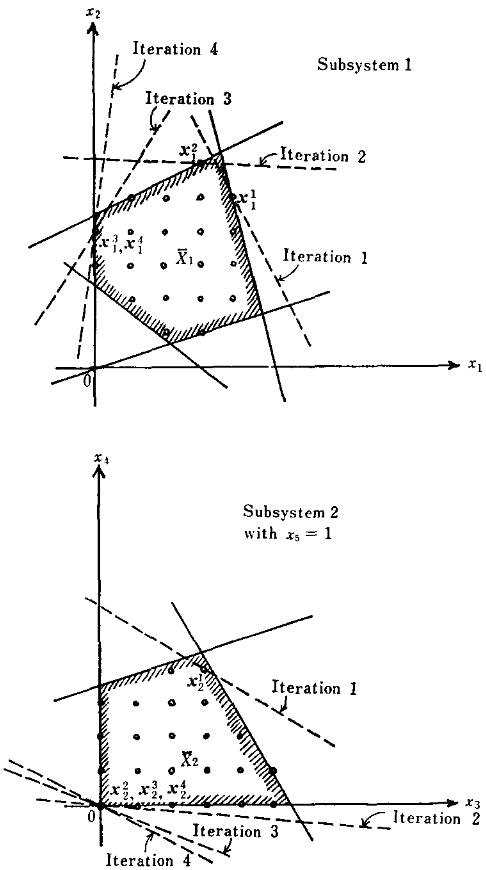

Fig. 2. Feasible region and optimal contours of subproblems of the multi-sector example in Section 2.

$$=(53/13)x_3+(123/13)x_4+(9/13)x_5.$$

Then, $\boldsymbol{x}_1^3=(0,\ 4)^T$, $\boldsymbol{x}_2^3=(0,\ 0,\ 1)^T$, and $z^3=z_1^3+z_2^3=-98/13+9/13=-89/13$. Substituting $\boldsymbol{x}_1^3$ and $\boldsymbol{x}_2^3$ into the objective function, we obtain $\pi=-13/5$.

Iteration 4: the subproblems (7) are:

subproblem 1: $\min_{\boldsymbol{x}_1\in X_1}(-1,\ -3)\boldsymbol{x}_1+(13/5)(2,\ 1)\boldsymbol{x}_1$

$$=(21/5)x_1-(2/5)x_2,$$

subproblem 2: $\min_{\boldsymbol{x}_2\in X_2}(-1,\ 1,\ -1)\boldsymbol{x}_2+(13/5)(3,\ 5,\ 1)\boldsymbol{x}_2$

$$=(34/5)x_3+14x_4+(8/5)x_5.$$

Then, $\boldsymbol{x}_1^4=(0,\ 4)^T$, $\boldsymbol{x}_2^4=(0,\ 0,\ 1)^T$, and $z^4=z_1^4+z_2^4=-8/5+8/5=0$. Since z^4 is nonnegative, an optimal solution is obtained, which is $x_1=0$, $x_2=4$, $x_3=0$, $x_4=0$, $x_5=1$. The optimal objective value is $-13/5$.

The optimal contours of the subproblems in each iteration are shown in Fig. 2.

3. Some Properties and Applications

3.1 Relation to the generalized Lagrangian method

Here, we consider some relations of the algorithm presented above to the generalized Lagrangian method [4], [9], [14], [17], [18], which is among the common methods to attack the nonlinear and/or integer programs.

If we put $1/\boldsymbol{q}^T\boldsymbol{x}=t$ in (3), (3) is equivalent to the parametric program:

$$(8)\qquad \min_{\boldsymbol{x},\,t} t\boldsymbol{p}^T\boldsymbol{x} \quad \text{subj. to } t\boldsymbol{q}^T\boldsymbol{x}=1,\quad \boldsymbol{x}\in X.$$

Define a constrained Lagrangian function corresponding to (8) to be

$$\tilde{L}(\boldsymbol{x},\ t,\ \pi)=t\boldsymbol{p}^T\boldsymbol{x}-\pi(t\boldsymbol{q}^T\boldsymbol{x}-1)$$

where $t>0$ and $\boldsymbol{x}\in X$. If we fix the value of the paramater t, then we can define a generalized Lagrangian problem as follows:

$$(9)\qquad \min_{\boldsymbol{x}} L(\boldsymbol{x})=\boldsymbol{p}^T\boldsymbol{x}-\pi\boldsymbol{q}^T\boldsymbol{x}$$

subj. to $\boldsymbol{x} \in X$.

Since X is a nonvoid, finite set, a finite optimal solution of (9) exists for any π.

In general, the procedures to change the values of multipliers in generalized Lagrangian methods are essentially trial and error, and solving Lagrangian problems may not provide the solution of the original problem whatever the values of multipliers are changed.

However, note that (9) is just the subproblem (5) solved in our algorithm. Furthermore, it can be easily verified that an optimal solution of (9), $\boldsymbol{x}(\pi)$, is an optimal solution of (8) where the parameter t is fixed to be $1/\boldsymbol{q}^T\boldsymbol{x}(\pi)$. Thus, the algorithm presented in Section 2 may be considered as the finite procedure to solve the Lagrangian problem (9) by suitably changing the value of π. In other words, our algorithm can also be classified as the generalized Lagrangian method with the finite convergence.

3.2 Relation to the continuous fractional programming algorithms

The proposed algorithm can be modified to solve the continuous linear fractional programs (1) if $\bar{X}$ is bounded as will be shown below. We assume that $r=s=0$ in (1) without loss of generality and $\bar{X}$ is bounded in this section.

It is well known that the optimum of (1) occurs at an extreme point of $\bar{X}$ [19]. Thus, defining $X=\{\boldsymbol{x} \mid \boldsymbol{x} \in \bar{X},\ \boldsymbol{x}$ is a basic feasible solution$\}=\{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^s\}$, an optimal solution of (4) provides an optimal solution of (1). The column to enter the basis in each simplex iteration of the solution of (4) is determined by $\min_k \boldsymbol{p}^T\boldsymbol{x}^k-\pi\boldsymbol{q}^T\boldsymbol{x}^k$, however, it can be found by solving the linear subproblem:

$$(10) \qquad \min_{\boldsymbol{x}} \boldsymbol{p}^T\boldsymbol{x}-\pi\boldsymbol{q}^T\boldsymbol{x} \quad \text{subj. to } \boldsymbol{x} \in \bar{X},$$

since $\boldsymbol{x}^k$ is an extreme point of $\bar{X}$. Hence, the global optimum of (1) can be obtained directly by the proposed algorithm.

Since the optimum can be obtained at an extreme point of the feasible region in continuous fractional programs, some pivoting algorithms to search neighboring extreme points have been proposed in [12], [19]. Let us compare those with our method.

Adding to constraints the equations related to the denominator and the numerator:

$$\sum_j q_j x_j - z_1 = 0$$

$$\sum_j p_j x_j - z_2 = 0$$

and describing them in the canonical form related to some feasible basis, the equations can be written as follows:

$$\begin{array}{lllll}
x_1 & & +\bar{a}_{1,m+1}x_{m+1} & +\cdots+\bar{a}_{1n}x_n & =\bar{b}_1 \\
\quad\cdot & & + \quad \cdot & & \\
\quad\quad\cdot & & + \quad \cdot & & \\
\quad\quad\quad\cdot & & + \quad \cdot & & \\
\quad\quad\quad x_m & & +\bar{a}_{m,m+1}x_{m+1} & +\cdots+\bar{a}_{mn}x_n & =\bar{b}_m \\
 & -z_1 & +\bar{q}_{m+1}x_{m+1} & +\cdots+\bar{q}_n x_n & =-\bar{z}_1 \\
 & \quad -z_2 & +\bar{p}_{m+1}x_{m+1} & +\cdots+\bar{p}_n x_n & =-\bar{z}_2 .
\end{array}$$

Pivoting at $\bar{a}_{rs}$, new values of z_1, z_2 are

$$-\hat{z}_1 = -\bar{z}_1 - \theta\bar{q}_s$$

$$-\hat{z}_2 = -\bar{z}_2 - \theta\bar{p}_s$$

where $\theta = \bar{b}_r/\bar{a}_{rs} > 0$. Then the value of the fractional objective becomes

$$\hat{z} = \hat{z}_2/\hat{z}_1 = \frac{\bar{z}_2 + \theta\bar{p}_s}{\bar{z}_1 + \theta\bar{q}_s}.$$

The objective value is improved when $\hat{z} < \bar{z}_2/\bar{z}_1$, *i.e.*,

$$\frac{\bar{z}_2 + \theta\bar{p}_s}{\bar{z}_1 + \theta\bar{q}_s} - \frac{\bar{z}_2}{\bar{z}_1} < 0,$$

or, since $\bar{z}_1 + \theta\bar{q}_s > 0$, $\theta > 0$,

(11) $$\bar{c}_s = \bar{p}_s - (\bar{z}_2/\bar{z}_1)\bar{q}_s < 0 .$$

The column to enter the basis is determined by $\min_j \bar{c}_j$. If $\bar{c}_j \geqslant 0$ for all j, then the optimal solution with $\bar{z}_1$, $\bar{z}_2$ provides the optimum of (1).

Note that $\bar{c}_j$ is just the cost coefficient of x_j in our subproblem (10). So the above procedure searches locally a good extreme point of $\bar{X}$, but our method searches globally by resolving the linear program (10), both using the relative cost factor (11).

The relative cost factor (11) is slightly different from ones in Gilmore and Gomory [12] or Swarup [19]. Since $\theta/(\bar{z}_1+\theta\bar{q}_s)$ is eliminated in (11), $\min_j \bar{c}_j$ need not provide the best neighboring extreme point. This situation occurs in the same way as in [19], but, in [12], the best direction is searched by using directional derivatives.

By the above reason, comparing to the other methods, ours is not the locally steepest descent method. Furthermore, different from [12] or [19], several linear programs should be solved in our method, though the primal simplex method can be effectively used by considering π as the cost parameter.

3.3 Relation to the vector optimum problems

As fractional programming is a relative optimization problem, it is naturally related to the optimization problems with multiple objectives as follows [11]:

(12) $$\max_{\boldsymbol{x}} (f_1(\boldsymbol{x}), \cdots, f_m(\boldsymbol{x}))$$
$$\text{subj. to } \boldsymbol{x} \in X \subseteq E^n$$
where $f_i(\boldsymbol{x})$ is an arbitrary function of $\boldsymbol{x}$.

A common concept of optimality for such a problem is Pareto optimality.

Definition. $\hat{\boldsymbol{x}} \in X$ is said to be a Pareto optimal solution of (12) if there exists no $\boldsymbol{x} \in X$ such that $f_i(\boldsymbol{x}) \geqslant f_i(\hat{\boldsymbol{x}})$ for all $i=1, \cdots, m$ and $f_h(\boldsymbol{x}) > f_h(\hat{\boldsymbol{x}})$ for some h such that $1 \leqslant h \leqslant m$.

The relation between our problem (3) and the problems of the type (12) is specified as follows.

Theorem 2. Consider the vector optimum problem:

$$\text{(13)} \qquad \max_{\boldsymbol{x}} \left(f_1(\boldsymbol{x}) = -\boldsymbol{p}^T\boldsymbol{x},\ f_2(\boldsymbol{x}) = \boldsymbol{q}^T\boldsymbol{x}\right)$$
$$\text{subj. to } \boldsymbol{x} \in X = \{\boldsymbol{x} \mid A\boldsymbol{x} \leqslant \boldsymbol{b},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \text{integer vector}\}$$
$$\text{where } f_1(\boldsymbol{x}) \geqslant 0,\ f_2(\boldsymbol{x}) > 0 \text{ for all } \boldsymbol{x} \in X.$$

Then an optimal solution of (3), $\boldsymbol{x}^*$, is a Pareto optimal solution of (13).

Proof. Suppose that $\boldsymbol{x}^*$ is not a Pareto optimal solution of (13). Then there exists $\tilde{\boldsymbol{x}} \in X$ such that $f_1(\tilde{\boldsymbol{x}}) > f_1(\boldsymbol{x}^*)$ and $f_2(\tilde{\boldsymbol{x}}) \geqslant f_2(\boldsymbol{x}^*)$, or $f_1(\tilde{\boldsymbol{x}}) \geqslant f_1(\boldsymbol{x}^*)$ and $f_2(\tilde{\boldsymbol{x}}) > f_2(\boldsymbol{x}^*)$. For both cases,

$$-\frac{f_1(\boldsymbol{x}^*)}{f_2(\boldsymbol{x}^*)} > -\frac{f_1(\tilde{\boldsymbol{x}})}{f_2(\tilde{\boldsymbol{x}})} \quad \text{for } \boldsymbol{x}^*, \tilde{\boldsymbol{x}} \in X.$$

That is,

$$\frac{\boldsymbol{p}^T\boldsymbol{x}^*}{\boldsymbol{q}^T\boldsymbol{x}^*} > \frac{\boldsymbol{p}^T\tilde{\boldsymbol{x}}}{\boldsymbol{q}^T\tilde{\boldsymbol{x}}} \quad \text{for } \boldsymbol{x}^*, \tilde{\boldsymbol{x}} \in X.$$

It contradicts the assumption that $\boldsymbol{x}^*$ is an optimal solution of (3).

Note that the above result holds also if the feasible region is $\bar{X}$, for the definition of Pareto optimality does not depend on topological properties of X or $f_i(\boldsymbol{x})$.

3.4 Application to goal programming

Fractional programming is essentially a relative optimization of the numerator and denominator function. Hence, as an important field to which it is applicable is the optimization of efficiency. For example, a multi-item production scheduling to maximize the rate of return under the resource and demand constraints. Another example is referred to by Gilmore and Gomory in cutting stock problems in the paper industries [12]. However, in fact, the maximization or minimization of "efficiency" is sometimes meaningless. On the contrary, the goal attainment of efficiency is not only meaningful, but it is sometimes the more practical objective in actual social systems than ones such as the maximization of total profits or the minimization of total costs.

Here, a goal attainment problem of the total rate of return in a capital budgeting problem for the project selection is formulated as the 0–1 integer linear fractional program, and solved by the proposed algorithm.

The goal programming problem to select projects from n condidate projects such that expected profit/total capital investment is closest to the given goal value of the rate of return under the constraints of upper and lower bounds of the total capital investment and the variance of profit can be formulated as follows:

$$\text{(14)} \qquad \min_{x_1,\cdots,x_n} \left| \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} c_i x_i} - \lambda \right|$$

subj. to $G_L \leqslant \sum_{i=1}^{n} c_i x_i \leqslant G_U$

$\sum_{i=1}^{n} w_i x_i \leqslant V$

$x_i=0$ or 1, $i=1, \cdots, n$,

where $w_i = v_i c_i / \sum_{i=1}^{n} c_i x_i$, $p_i = r_i c_i$,

and c_i: capital investment for the ith project

r_i: expected rate of return of the ith project

v_i: variance of rate of return of the ith project

G_U: upper bound of the available total capital

G_L: lower bound of the available total capital

V: upper bound of the sum of weighted variance of profit

λ: goal value of rate of return.

Note that $x_i=1$ if the ith project is selected and $x_i=0$ if not.

An optimal solution of (14) can be obtained by solving the following two 0–1 fractional programs:

$$\text{(15)} \qquad \min_{x_1,\cdots,x_n} \frac{\pm\{\sum_{i=1}^{n} (r_i - \lambda c_i) x_i\}}{\sum_{i=1}^{n} c_i x_i}$$

$$\text{subj. to } G_L \leqslant \sum_{i=1}^{n} c_i x_i \leqslant G_U$$

$$\sum_{i=1}^{n} (v_i - V) c_i x_i \leqslant 0$$

$$\pm\{\sum_{i=1}^{n} (r_i - \lambda c_i) x_i\} \geqslant 0$$

$$x_i = 0 \text{ or } 1,\ i = 1, \cdots, n.$$

The above problems can be solved by the proposed algorithm, where the subproblems (5) become 0–1 linear programs.

The computational result of an example problem with 12 projects

Table 1. Data for the project selection problem.

Project No.	Capital investment	Expected rate of return	Variance of rate of return
1	47.0	0.17	0.0060
2	55.0	0.25	0.0045
3	86.0	0.20	0.0058
4	23.0	0.41	0.0033
5	98.0	0.59	0.0125
6	25.0	0.33	0.0050
7	74.0	0.48	0.0100
8	74.0	0.35	0.0088
9	70.0	0.08	0.0015
10	45.0	0.27	0.0045
11	93.0	0.15	0.0075
12	55.0	0.56	0.0090

Parameters: $G=200.0$, $G=125.0$, $V=0.007$.

Table 2. Computational results for the project selection problem.

Goal value of rate of return (A)	Projects selected	Total capital investment	Expected profit	Expected rate of return (B)	$\|(A)-(B)\|$
0.1	9, 11	163.0	19.6	0.120	0.020
0.2	4, 9, 10	138.0	27.2	0.197	0.003
0.3	4, 9, 10, 12	193.0	58.0	0.300	0.000
0.4	2, 4, 12	133.0	54.0	0.406	0.006
0.5	4, 6, 10, 12	148.0	60.6	0.410	0.090
0.6	4, 6, 10, 12	148.0	60.6	0.410	0.190

is tabulated in Tables 1 and 2. Subproblems are solved by Balas' additive algorithm [1] and the average execution time for one case is about 30–40 secs (IBM 7040, FORTRAN IV).

Note that (14) is essentially different from the problem with the objective function, minimize $\left|\sum_{i=1}^{n} p_i x_i - \lambda \sum_{i=1}^{n} c_i x_i\right|$. They have been sometimes confounded with in practical systems. 0–1 fractional programming algorithms were treated with by Ivanescu and Rudeaunu [13], but they considered basically the problems such that every term in the denominator has the plus sign and there are no constraints except 0–1 conditions.

Comments

1. As in nonlinear programming, it would be more popular to solve integer nonlinear programs by the iterative solution of integer linear subproblems (Benders' partitioning method [3] is a typical example). Thus, the presented algorithm, which provides the solution of integer nonlinear programs, also suggests one of the main directions to attack formidable integer nonlinear programming, though it does not conquer the difficulties of handling with linear integer programs.

2. Note that only the finiteness property of X is used in the construction of subproblems (5). This implies that, even if X in (3) is characterized as a finite set of another type, it is sufficient that practically solvable subproblems are constructed. It makes very wide the applicability of the algorithm.

Acknowledgement

The author thanks Associate Professors K. Shimizu and H. Yanai of Keio University for their helpful suggestions.

References

[1] Balas, E., "An Additive Algorithm for Solving Linear Programs with Zero-

one Variables," *Opns. Res.*, **13** (1965), 517-544.
[2] Bector, C.R., "Programming Problems with Convex Fractional Functions," *Opns. Res.*, **16** (1968), 383-391.
[3] Benders, J.F., "Partitioning Procedures for Solving Mixed-Variables Programming Problems," *Numerische Mathematik*, **4** (1962), 238-252.
[4] Brooks, R. and A. Geoffrion, "Finding Everett's Lagrange Multipliers by Linear Programming," *Opns. Res.*, **14** (1966), 1149-1153.
[5] Charnes, A. and W.W. Cooper, "Programming with Linear Fractional Functionals," *Naval Res. Log. Quart.*, **9** (1962), 181-186.
[6] Charnes, A. and W.W. Cooper, "Programming with Linear Fractional Functionals, A Communication," *Naval Res. Log. Quart.*, **10** (1963), 273-274.
[7] Cord, J., "A Method for Allocating Funds to Investment Projects when Returns are subject to Uncertainty," *Management Sci.*, **10** (1964) 335-341.
[8] Dantzig, G.B., *Linear Programming and Extensions*, Prinston Univ. Press, N.J., 1963.
[9] Everett III, H., "Generalized Lagrange Multiplier Method," *Opns. Res.*, **11** (1963), 399-417.
[10] Fogler, H.R., "Ranking Techniques and Capital Budgeting," *The Accounting Review*, **XLVII** (1972), 134-143.
[11] Geoffrion, A., "Solving Bicriterion Mathematical Programs," *Opns. Res.*, **15** (1967), 39-54.
[12] Gilmore, P.C. and R.E. Gomory, "A Linear Programming Approach to the Cutting Stock Problem—Part II," *Opns. Res.*, **11** (1963), 863-888.
[13] Ivanescu, P.L. and S. Rudeaunu, "Pseudo-Boolean Methods for Bivalent Programming," *Lecture Notes in Mathematics*, Vol. 23, Springer-Verlag, Berlin-Heidelberg-New York, 1966.
[14] Jagannathan, R., "On Some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming," *Management Sci.*, **12** (1966), 609-615.
[15] Joksch, H., "Programming with Fractional Linear Objective Functions," *Naval Res. Log. Quart.*, **11** (1964), 197-204.
[16] Martos, S., "Hyperbolic Programming," *Naval Res. Log. Quart.*, **11** (1964), 135-155.
[17] Nemhauser, G. and Z. Ullman, "A Note on the Generalized Lagrange Multiplier Solution to an Integer Programming Problem," *Opns. Res.*, **16** (1968), 450-452.
[18] Shapiro, J.F., "Generalized Lagrange Multipliers in Integer Programming," *Opns. Res.*, **19** (1971), 68-76.
[19] Swarup, K., "Linear Fractional Functionals Programming," *Opns. Res.*, **13** (1965), 1029-1036.
[20] Swarup, K., "Fractional Programming with Non-linear Constraints," *Z.A.M.M.*, **46** (1966), 468-469. As for nonlinear fractional programming,

see also Gupta, R.K. and K. Swarup, *Z.A.M.M.*, **49** (1969), 753-756, and Anand, P. and K. Swarup, *Z.A.M.M.*, **50** (1970), 320-321.

[21] Swarup, K., "Some Aspects of Duality for Linear Fractional Functionals Programming," *Z.A.M.M.*, **47** (1967), 204-205.

[22] Weingartner, H.M., "Capital Budgeting of Interrelated Projects: Survey and Synthesis," *Management Sci.*, **12** (1966), 485-516.