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On integer parts of the reciprocal remainders of some sums

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Abstract

We present generalizations of some results on the integer parts of the reciprocal remainders of the zeta function $\zeta(s)$ with $s=2$ and $s=3$, and a very short and elegant proof of a recent result on the integer parts of the reciprocal remainders of the series $\zeta(3)$. We also give some historical and theoretical remarks to problems of this type, conduct some analyses, and make some connections with the theory of linear difference equations with constant coefficients.

MSC: 40A99; 39A06; 11M06

Keywords: Integer part; Remainder of a sum; Asymptotics; Closed-form formula; Linear difference equations

1 Introduction

1.1 Notation

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be the sets of natural numbers, integers, and real numbers, respectively, and $\mathbb{N}_k := \{n \in \mathbb{Z} : n \geq k\}$ where $k \in \mathbb{Z}$ is fixed. For $s, t \in \mathbb{Z}$ such that $s \leq t$, we use the notation $j = \overline{s,t}$, instead of writing $s \leq j \leq t$, for $j \in \mathbb{Z}$. By $[x]$, we denote the integer part of a number $x \in \mathbb{R}$ (the integer part function or the floor function [2, 22, 28]). If $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$, then the quantity denoted by C_j^α, we define as follows

$$C_j^\alpha := \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}.$$

Note that for $\alpha \in \mathbb{N}$ the quantity reduces to the Binomial coefficient C_j^n. If $j = 0$, then we regard that $C_0^\alpha = 1$. Let us mention that when $\alpha \in \mathbb{N}$, many authors use the notation C_n^j (see, e.g., [2, 22, 49]) instead of C_j^n, which we prefer (see, e.g., [40]), since index j is usually the main running variable in a concrete situation, and the notation is more similar to the robust one $\binom{n}{j}$, which is perhaps the most frequently used (see, e.g., [21, 27–29]).

1.2 Motivation

Recently, there has been some interest in calculating the integer parts of the reciprocal remainders of some sums. Motivated by the formulas in [31, 46, 48] for the integer parts of the reciprocal remainders of some sums containing the Fibonacci and Pell numbers ([22, 28, 45]), author of [47] proposed to obtain related formulas for the integer parts of the reciprocal remainders of the Riemann zeta-function, that is, to calculate

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^s}\right)^{-1}\right],\quad n\in\mathbb{N},$$

for some $s\in\mathbb{N}_2$.

It was proved therein that the following formulas hold

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1}\right]=n-1,\quad n\in\mathbb{N},\tag{1}$$

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)^{-1}\right]=2n(n-1),\quad n\in\mathbb{N}.\tag{2}$$

Here, we present some historical facts connected to formulas (1) and (2), conduct some analyses regarding the proofs given in [47], provide some comments on them, present a very short and elegant proof of the main result therein, give another way for obtaining some estimates of the remainders of the Riemann zeta-function in the case $s = 3$, present a useful auxiliary result about the asymptotics of the remainders of the sums, and based on it, we also present some extensions of the formulas (1) and (2).

2 Analyses, main results and some comments

2.1 Few words on the history of the problem

The author of [47] claims that, as far as he knows, it seems that none has studied the problem yet and that he has not seen any related result before. However, problems of this type are matter or folklore and have been relatively popular and circulated among the fans of problems in elementary algebra and analysis. The literature devoted to the elementary problem could be challenging to trace back. The problems of this type have been known to the author of this paper for a long time. Formula (1) is Problem 3.1.46 in book [28]. The book was quite popular and has had several editions. However, unlike many other books by Mitrinović and his collaborators, it was not translated into English, so it predominately circulated among some Slavic countries. Unlike some other problems in [28], Mitrinović and Adamović did not give any information about the source of the problem. At the moment, we do not have an earlier reference. Nevertheless, [28] shows that the problems of the type are known. It should also be mentioned that Problem 3.1.46 in [28] is proposed in an equivalent but camouflaged form. Namely, it is suggested to show that the formula holds

$$\left[\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)^{-1}\right]=n,\quad n\in\mathbb{N}.$$

2.2 On the proof of formula (1) in [47]

It is known that not only that formula (1), but, besides this, the proof of the formula given in [47] is exactly the same as the one in [28]. Namely, for each $n\in\mathbb{N}_2$, we have

$$\begin{aligned}\frac{1}{n}&=\sum_{k=n}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\sum_{k=n}^{\infty}\frac{1}{k(k+1)}<\sum_{k=n}^{\infty}\frac{1}{k^2}<\sum_{k=n}^{\infty}\frac{1}{(k-1)k}\\&=\sum_{k=n}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\frac{1}{n-1}\end{aligned}$$

and consequently

$$n-1<\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{-1}<n$$

for each $n \in \mathbb{N}_2$, from which together with the definition of the integer part function, formula (1) easily follows for any such n.

Remark 1 Note that the above argument does not hold for $n = 1$, since the quantity $\frac{1}{n-1}$ is not defined for the value of n. However, in this case, formula (1) follows from the Euler formula

$$\sum_{n=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$$

[15] and the observation $\pi^2 > 6$.

2.3 On the proof of formula (2) in [47]

The proof of formula (2) given in [47] is a modification of the above proof of formula (1). First, for $n \in \mathbb{N}_2$, one can get the following well-known estimate

$$\sum_{k=n}^{\infty}\frac{1}{k^3}<\sum_{k=n}^{\infty}\frac{1}{(k-1)k(k+1)}=\sum_{k=n}^{\infty}\frac{1}{2}\left(\frac{1}{(k-1)k}-\frac{1}{k(k+1)}\right)=\frac{1}{2(n-1)n}. \tag{3}$$

At the same time, it was not noticed therein that the last sum is telescoping, so that there are some unnecessary extra calculations in [47].

To get the inequality

$$\frac{1}{2(n-1)n+1}<\sum_{k=n}^{\infty}\frac{1}{k^3} \tag{4}$$

from which, along with inequality (3), formula (2) easily follows, it was used the inequality

$$\frac{1}{2(k-1)k+1}-\frac{1}{2(k+1)(k+2)+1}<\frac{1}{k^3}+\frac{1}{(k+1)^3},\quad k\in\mathbb{N}_2, \tag{5}$$

which was applied to the partial sums of odd and even indices separately. However, it was also not noticed that the inequalities in Lemma 2 and Lemma 3 in [47] can be unified by inequality (5), so there are also some unnecessary extra calculations therein. For this proof of formula (2), we do not have a specific reference at the moment, so it may be new.

2.4 A short and elegant proof of inequality (2)

In order to prove (2), it seems less natural to use the difference

$$\frac{1}{2(k-1)k+1}-\frac{1}{2(k+1)(k+2)+1}$$

for dealing with a telescoping sum in this case.

Note that for $k \in \mathbb{N}_2$, we have

$$\frac{1}{2(k-1)k+1} - \frac{1}{2k(k+1)+1} = \frac{4k}{(2k^2+2k+1)(2k^2-2k+1)} = \frac{4k}{4k^4+1} < \frac{1}{k^3} \tag{6}$$

from which it follows that

$$\frac{1}{2(n-1)n+1} = \sum_{k=n}^{\infty}\left(\frac{1}{2(k-1)k+1} - \frac{1}{2k(k+1)+1}\right) = \sum_{k=n}^{\infty}\frac{4k}{4k^4+1} < \sum_{k=n}^{\infty}\frac{1}{k^3}, \tag{7}$$

which is a very short and elegant proof of inequality (4).

Hence, the estimations in (3) and (7) give a short and elegant proof of formula (2), which is in the spirit of the proof of formula (1) given in [28].

2.5 A refinement of inequality (3)

Inequality (3) is a sort of an elementary set up estimate. A better one can be obtained using the Hermite-Hadamard inequalities

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{8}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function [17, 18].

Namely, from (8), we have

$$f(k) = f\left(\frac{k-\frac{1}{2}+k+\frac{1}{2}}{2}\right) \le \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(t)\,dt$$

from which, it follows that

$$\sum_{k=n}^{\infty} f(k) \le \int_{n-\frac{1}{2}}^{+\infty} f(t)\,dt.$$

Since, in our case,

$$f(t) = \frac{1}{t^3}, \tag{9}$$

we have

$$\sum_{k=n}^{\infty}\frac{1}{k^3} < \int_{n-\frac{1}{2}}^{+\infty}\frac{dt}{t^3} = \frac{1}{2(n-\frac{1}{2})^2} = \frac{1}{2n(n-1)+\frac{1}{2}}. \tag{10}$$

Remark 2 Note that

$$\left[\left(\frac{1}{2n(n-1)+\frac{1}{2}}\right)^{-1}\right] = 2n(n-1),$$

for $n \in \mathbb{N}_2$.

2.6 An estimate from below

If f is a real convex function, then using the second inequality in (8), we have

$$\frac{f(k)+f(k+1)}{2} \geq \int_k^{k+1} f(t)\,dt \tag{11}$$

for each $k \in \mathbb{N}$.

If the convex function f also satisfies the relation

$$\lim_{t\to+\infty} f(t) = 0, \tag{12}$$

then from (11), it follows that

$$\sum_{k=n}^{\infty} f(k) \geq \frac{f(n)}{2} + \int_n^{+\infty} f(t)\,dt, \quad n \in \mathbb{N}. \tag{13}$$

Since the function in (9) satisfies condition (12), and for the function, inequality (11) is strict, from (13), it follows that

$$\sum_{k=n}^{\infty} \frac{1}{k^3} > \frac{1}{2n^3} + \int_n^{+\infty} \frac{dt}{t^3} = \frac{n+1}{2n^3}.$$

From this and since

$$\left[\frac{2n^3}{n+1}\right] = \left[\frac{2n^3 - 2n + n + 1 + n - 1}{n+1}\right] = 2n(n-1) + 1,$$

we have

$$\widetilde{a}_n := \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^3}\right)^{-1}\right] \leq 2n(n-1) + 1$$

from which, along with (10), we have that the quantity $\widetilde{a}_n$ takes one of the values $2n(n-1)$ and $2n(n-1)+1$. This interesting fact shows the strength of elementary inequality (6).

Remark 3 It is highly expected that a better estimate from below for the sequence $\sum_{k=n}^{\infty} \frac{1}{k^3}$, $n \in \mathbb{N}$ can be obtained using some more refined integral inequalities or using some integral formulas, such as the Euler-Maclaurin formula [14, 19, 20, 26] together with some elementary inequalities. However, since we already have a few elementary proofs for the estimate, we will not conduct further investigation in this direction.

2.7 Connection with difference equations

Bearing in mind that some authors have given formulas for the integer parts of some reciprocal sums of solutions to some linear difference equations with constant coefficients (see, e.g., [31, 46, 48]), it is natural to make some connections between the topic and the problems mentioned above. Recall that solvability of the linear equations has been known to de Moivre [11, 12] and D. Bernoulli [9] and that the theory was later developed by several authors [10, 15, 23–25]. Their solvability implies the solvability of many nonlinear difference equations, including some recent ones (see, e.g., [1, 10, 20, 22, 23, 26, 29, 30, 33, 38–44] and the related references therein).

Note that the sequence $a_n = n^2$ is a solution to the difference equation

$$\Delta^3 a_n = 0, \quad n \in \mathbb{N}, \tag{14}$$

where Δ is the forward difference operator defined by

$$\Delta a_n = a_{n+1} - a_n$$

([20, 26]). General solution to equation (14) is given by

$$a_n = an^2 + bn + c, \quad n \in \mathbb{N}, \tag{15}$$

where $a, b, c \in \mathbb{R}$ are some constants (a method for solving equation (14) was known to D. Bernoulli [9]; a closed-form formula for solutions to equation (14) depending on initial values can be found also in [15]).

Based on these facts and formula (1), it is natural to pose the following interesting problem.

Problem 1 *Let $(a_n)_{n\in\mathbb{N}}$ be the sequence in (15). Find closed-form formulas for the sequence*

$$\left[\left(\sum_{k=n}^{+\infty}\frac{a}{a_k}\right)^{-1}\right],$$

for all $n \in \mathbb{N}$ or for sufficiently large n.

First, note that such formulas need not always exist for all $n \in \mathbb{N}$, since the sequence a_n defined in (15) can be equal to zero for some $n \in \mathbb{N}$ and some values of coefficients a, b, and c. Besides this, the problem makes sense if the series

$$\sum_{k=1}^{+\infty}\frac{1}{ak^2 + bk + c}$$

is convergent. Hence, it must be $a \neq 0$ (see, e.g., [13, 28, 49]).

Of an interest is the case when the sequence a_n is positive, since then it naturally generalizes the case $a_n = n^2$ for which formula (1) holds. A necessary condition for this is $a > 0$. In this case, it must also be $a + b + c > 0$. Since $\Delta^2 a_n = 2a$ for every $n \in \mathbb{N}$, we have $2a = \Delta^2 a_1$ so that the initial values a_j, $j = \overline{1,3}$, have to satisfy the condition $a_3 - 2a_2 + a_1 > 0$. If $a > 0$, then a sufficient condition for this is $\min\{b, c\} \geq 0$.

However, since

$$an^2 + bn + c \sim an^2,$$

as $n \to +\infty$, we will not assume the last condition, but only $a \neq 0$, and will consider the sequence

$$\left(\sum_{k=n}^{+\infty}\frac{a}{ak^2 + bk + c}\right)^{-1}, \quad n \in \mathbb{N}. \tag{16}$$

Since the coefficients a, b, c are arbitrary numbers, the values of (16) can vary drastically for the small values of indices. Thus, the solution to the problem heavily depends on the values of the coefficients, and the requested formulas could be quite complicated if we want them to hold for every $n \in \mathbb{N}$. This is why some authors try to find some closed-form formulas which hold for sufficiently large n.

In what follows, we consider Problem 1, as well as the corresponding problem for the sequence

$$a_n = an^3 + bn^2 + cn + d, \quad n \in \mathbb{N}. \tag{17}$$

To do this, we will use some asymptotic methods, which are also used in studying sequences and difference equations from time to time (see, e.g., [3–8, 16, 32, 34–37] and the related references therein).

We need a lemma, which could be of folklore type. We give proof of it for the completeness and benefit of the reader.

Lemma 1 *Let $k > 1$. Then, the following asymptotic formula holds*

$$\sum_{j=n}^{+\infty} \frac{1}{j^k} = \frac{1}{(k-1)n^{k-1}} + \frac{1}{2n^k} + \frac{k}{12n^{k+1}} - \frac{k(k+1)(k+2)}{720n^{k+3}} + o\left(\frac{1}{n^{k+3}}\right). \tag{18}$$

Proof First, recall that the following known asymptotic relation holds

$$\sum_{j=n}^{+\infty} \frac{1}{j^k} \sim \int_n^{+\infty} \frac{dt}{t^k} = \frac{1}{(k-1)n^{k-1}}, \tag{19}$$

when $k > 1$ (see, e.g., [13, 28, 49]).

Let

$$x_n := \sum_{j=n}^{+\infty} \frac{1}{j^k} - \frac{1}{(k-1)n^{k-1}}. \tag{20}$$

Since $k > 1$, the remainder of the series in (20) converges to zero, implying the convergence to zero of the sequence x_n.

Further, by some calculations and using the well-known asymptotic relation

$$(1+x)^\alpha = 1 + C_1^\alpha x + C_2^\alpha x^2 + \cdots + C_l^\alpha x^l + o\left(x^l\right), \quad \text{as } x \to 0, \tag{21}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $l \in \mathbb{N}$ (see, e.g., [13, 21, 49]), we have

$$\begin{aligned} x_j - x_{j+1} &= \frac{1}{j^k} - \frac{1}{(k-1)j^{k-1}} + \frac{1}{(k-1)(j+1)^{k-1}} = \frac{1}{j^{k-1}}\left(\frac{1}{j} - \frac{1}{k-1} + \frac{(1+\frac{1}{j})^{1-k}}{k-1}\right) \\ &= \frac{1}{j^{k-1}}\left(\frac{1}{j} - \frac{1}{k-1} + \frac{1}{k-1}\left(1 + \frac{1-k}{j} + \frac{(1-k)(-k)}{2j^2} + O\left(\frac{1}{j^3}\right)\right)\right) \\ &= \frac{k}{2j^{k+1}} + O\left(\frac{1}{j^{k+2}}\right) \end{aligned}$$

from which, together with (19), it follows that

$$x_n=\sum_{j=n}^{+\infty}(x_j-x_{j+1})=\sum_{j=n}^{+\infty}\left(\frac{k}{2j^{k+1}}+O\left(\frac{1}{j^{k+2}}\right)\right)=\frac{1}{2n^k}+o\left(\frac{1}{n^k}\right).$$

Hence,

$$\sum_{j=n}^{+\infty}\frac{1}{j^k}=\frac{1}{(k-1)n^{k-1}}+\frac{1}{2n^k}+o\left(\frac{1}{n^k}\right).$$

Let

$$y_n:=\sum_{j=n}^{+\infty}\frac{1}{j^k}-\frac{1}{(k-1)n^{k-1}}-\frac{1}{2n^k}.$$

Then, using some calculations and (21), we have

$$\begin{aligned}
y_j-y_{j+1}&=\frac{1}{j^k}-\frac{1}{(k-1)j^{k-1}}-\frac{1}{2j^k}+\frac{1}{(k-1)(j+1)^{k-1}}+\frac{1}{2(j+1)^k}\\
&=\frac{1}{j^{k-1}}\left(\frac{1}{2j}-\frac{1}{k-1}+\frac{(1+\frac{1}{j})^{1-k}}{k-1}+\frac{(1+\frac{1}{j})^{-k}}{2j}\right)\\
&=\frac{1}{j^{k-1}}\left(\frac{1}{2j}-\frac{1}{k-1}+\frac{1}{k-1}\left(1+\frac{1-k}{j}+\frac{C_2^{1-k}}{j^2}+\frac{C_3^{1-k}}{j^3}+O\left(\frac{1}{j^4}\right)\right)\right.\\
&\quad\left.+\frac{1}{2j}\left(1-\frac{k}{j}+\frac{C_2^{-k}}{j^2}+O\left(\frac{1}{j^3}\right)\right)\right)=\frac{k(k+1)}{12j^{k+2}}+O\left(\frac{1}{j^{k+3}}\right)
\end{aligned}$$

from which, together with (19), it follows that

$$y_n=\sum_{j=n}^{+\infty}(y_j-y_{j+1})=\sum_{j=n}^{+\infty}\left(\frac{k(k+1)}{12j^{k+2}}+O\left(\frac{1}{j^{k+3}}\right)\right)=\frac{k}{12n^{k+1}}+o\left(\frac{1}{n^{k+1}}\right).$$

Let

$$z_n:=\sum_{j=n}^{+\infty}\frac{1}{j^k}-\frac{1}{(k-1)n^{k-1}}-\frac{1}{2n^k}-\frac{k}{12n^{k+1}}. \tag{22}$$

Then, using some calculations and formula (21), we have

$$\begin{aligned}
z_j-z_{j+1}&=\frac{1}{2j^k}-\frac{1}{(k-1)j^{k-1}}-\frac{k}{12j^{k+1}}+\frac{1}{(k-1)(j+1)^{k-1}}+\frac{1}{2(j+1)^k}+\frac{k}{12(j+1)^{k+1}}\\
&=\frac{1}{j^{k-1}}\left(\frac{1}{2j}-\frac{1}{k-1}-\frac{k}{12j^2}+\frac{(1+\frac{1}{j})^{1-k}}{k-1}+\frac{(1+\frac{1}{j})^{-k}}{2j}+\frac{k(1+\frac{1}{j})^{-k-1}}{12j^2}\right)\\
&=\frac{1}{j^{k-1}}\left(\frac{1}{2j}-\frac{1}{k-1}-\frac{k}{12j^2}\right.\\
&\quad\left.+\frac{1}{k-1}\left(1+\frac{C_1^{1-k}}{j}+\frac{C_2^{1-k}}{j^2}+\frac{C_3^{1-k}}{j^3}+\frac{C_4^{1-k}}{j^4}+\frac{C_5^{1-k}}{j^5}+O\left(\frac{1}{j^6}\right)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2j}\left(1+\frac{C_1^{-k}}{j}+\frac{C_2^{-k}}{j^2}+\frac{C_3^{-k}}{j^3}+\frac{C_4^{-k}}{j^4}+O\left(\frac{1}{j^5}\right)\right)\\
&+\frac{k}{12j^2}\left(1+\frac{C_1^{-k-1}}{j}+\frac{C_2^{-k-1}}{j^2}+\frac{C_3^{-k-1}}{j^3}+O\left(\frac{1}{j^4}\right)\right)\Bigg)\\
=&-\frac{k(k+1)(k+2)(k+3)}{720j^{k+4}}+O\left(\frac{1}{j^{k+5}}\right)
\end{aligned}
$$

from which, together with (19), it follows that

$$
\begin{aligned}
z_n&=\sum_{j=n}^{+\infty}(z_j-z_{j+1})=-\sum_{j=n}^{+\infty}\left(\frac{k(k+1)(k+2)(k+3)}{720j^{k+4}}+O\left(\frac{1}{j^{k+5}}\right)\right)\\
&=-\frac{k(k+1)(k+2)}{720n^{k+3}}+o\left(\frac{1}{n^{k+3}}\right).
\end{aligned}
\tag{23}
$$

From (22) and (23), asymptotic formula (18) follows. □

2.8 Case $a_n = an^2 + bn + c$

Now, we consider Problem 1. First, note that using (21), it follows that

$$
\begin{aligned}
\frac{a}{a_n}&=\frac{a}{an^2}\left(1+\frac{b}{an}+\frac{c}{an^2}\right)^{-1}\\
&=\frac{1}{n^2}\left(1-\frac{b}{an}+\frac{b^2-ac}{a^2n^2}+O\left(\frac{1}{n^3}\right)\right)\\
&=\frac{1}{n^2}-\frac{b}{an^3}+\frac{b^2-ac}{a^2n^4}+O\left(\frac{1}{n^5}\right),
\end{aligned}
$$

as $n\to+\infty$, and consequently

$$
\sum_{j=n}^{+\infty}\frac{a}{a_j}=\sum_{j=n}^{+\infty}\left(\frac{1}{j^2}-\frac{b}{aj^3}+\frac{b^2-ac}{a^2j^4}+O\left(\frac{1}{j^5}\right)\right).
\tag{24}
$$

From Lemma 1, we have

$$
\sum_{j=n}^{+\infty}\frac{1}{j^2}=\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{6n^3}+O\left(\frac{1}{n^5}\right),
\tag{25}
$$

$$
\sum_{j=n}^{+\infty}\frac{1}{j^3}=\frac{1}{2n^2}+\frac{1}{2n^3}+O\left(\frac{1}{n^4}\right),
\tag{26}
$$

$$
\sum_{j=n}^{+\infty}\frac{1}{j^4}=\frac{1}{3n^3}+O\left(\frac{1}{n^4}\right),
\tag{27}
$$

and

$$
\sum_{j=n}^{+\infty}O\left(\frac{1}{j^5}\right)=O\left(\frac{1}{n^4}\right).
\tag{28}
$$

Employing (25)–(28) in (24), we have

$$\sum_{j=n}^{+\infty}\frac{a}{a_j}=\frac{1}{n}+\frac{a-b}{2an^2}+\frac{a^2+2b^2-3ab-2ac}{6a^2n^3}+o\left(\frac{1}{n^3}\right). \tag{29}$$

From (29) and using (21), it follows that

$$\begin{aligned}\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]&=\left[n\left(1+\frac{a-b}{2an}+\frac{a^2+2b^2-3ab-2ac}{6a^2n^2}+o\left(\frac{1}{n^2}\right)\right)^{-1}\right]\\&=\left[n+\frac{b-a}{2a}+\frac{a^2-b^2+4ac}{12a^2n}+o\left(\frac{1}{n}\right)\right].\end{aligned} \tag{30}$$

From (30) and the definition of the integer part function, the following theorem easily follows.

Theorem 1 *Let $a\in\mathbb{R}\setminus\{0\}$, $b,c\in\mathbb{R}$, and $a_n=an^2+bn+c$. Then, the following statements hold.*

(a) *If $\frac{b-a}{2a}\notin\mathbb{Z}$, then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]=n+\left[\frac{b-a}{2a}\right], \tag{31}$$

for sufficiently large n.

(b) *If $\frac{b-a}{2a}\in\mathbb{Z}$, and $a^2-b^2+4ac>0$, then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]=n+\frac{b-a}{2a},$$

for sufficiently large n.

(c) *If $\frac{b-a}{2a}\in\mathbb{Z}$, and $a^2-b^2+4ac<0$, then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]=n+\frac{b-3a}{2a},$$

for sufficiently large n.

Corollary 1 *Let $a_n=n^2$. Then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{1}{a_j}\right)^{-1}\right]=n-1,$$

for sufficiently large n.

Proof First, note that the sequence a_n is obtained from (15) for $a=1$, $b=0$ and $c=0$. Since, in this case,

$$\frac{b-a}{2a}=-\frac{1}{2}\notin\mathbb{Z},$$

we can use formula (31) and get

$$\left[\left(\sum_{j=n}^{+\infty}\frac{1}{a_j}\right)^{-1}\right]=n+\left[-\frac{1}{2}\right].$$

From this and since $[-\frac{1}{2}]=-1$, the corollary follows. □

Remark 4 Since Corollary 1 is obtained from Theorem 1, it proves formula (1) for not all n. However, Theorem 1 holds for any $a\in\mathbb{R}\setminus\{0\}$, $b,c\in\mathbb{R}$. It is technically very difficult to find the exact value of $n_0=n_0(a,b,c)$ such that any of the formulas in the theorem hold for $n\geq n_0$.

2.9 Case $a_n = an^3 + bn^2 + cn + d$

To consider this case, we use the method that we have employed in the case of the sequence $a_n = an^2 + bn + c$. Due to more parameters (here, we have four parameters a, b, c, and d), our consideration in the case will have more calculations, so that will be more technical than the previous one.

Using some calculations and relation (21), it follows that

$$\begin{aligned}\frac{a}{a_n}&=\frac{a}{an^3}\left(1+\frac{b}{an}+\frac{c}{an^2}+\frac{d}{an^3}\right)^{-1}\\&=\frac{1}{n^3}\left(1-\frac{b}{an}+\frac{b^2-ac}{a^2n^2}+\frac{2abc-a^2d-b^3}{a^3n^3}+O\left(\frac{1}{n^4}\right)\right)\\&=\frac{1}{n^3}-\frac{b}{an^4}+\frac{b^2-ac}{a^2n^5}+\frac{2abc-a^2d-b^3}{a^3n^6}+O\left(\frac{1}{n^7}\right),\end{aligned}$$

as $n\to+\infty$.

Hence, we have

$$\sum_{j=n}^{+\infty}\frac{a}{a_j}=\sum_{j=n}^{+\infty}\left(\frac{1}{j^3}-\frac{b}{aj^4}+\frac{b^2-ac}{a^2j^5}+\frac{2abc-a^2d-b^3}{a^3j^6}+O\left(\frac{1}{j^7}\right)\right).\tag{32}$$

From Lemma 1, we have

$$\sum_{j=n}^{+\infty}\frac{1}{j^3}=\frac{1}{2n^2}+\frac{1}{2n^3}+\frac{1}{4n^4}+O\left(\frac{1}{n^6}\right),\tag{33}$$

$$\sum_{j=n}^{+\infty}\frac{1}{j^4}=\frac{1}{3n^3}+\frac{1}{2n^4}+\frac{1}{3n^5}+O\left(\frac{1}{n^7}\right),\tag{34}$$

$$\sum_{j=n}^{+\infty}\frac{1}{j^5}=\frac{1}{4n^4}+\frac{1}{2n^5}+O\left(\frac{1}{n^6}\right),\tag{35}$$

$$\sum_{j=n}^{+\infty}\frac{1}{j^6}=\frac{1}{5n^5}+O\left(\frac{1}{n^6}\right),\tag{36}$$

and

$$\sum_{j=n}^{+\infty} O\left(\frac{1}{j^7}\right) = O\left(\frac{1}{n^6}\right). \tag{37}$$

Employing the relations in (33)–(37) in (32), after some standard calculation, it follows that

$$\begin{aligned}\sum_{j=n}^{+\infty}\frac{a}{a_j} &= \frac{1}{2n^2}+\frac{3a-2b}{6an^3}+\frac{a^2+b^2-ac-2ab}{4a^2n^4}\\&\quad+\frac{15ab^2-10a^2b-15a^2c+12abc-6a^2d-6b^3}{30a^3n^5}+O\left(\frac{1}{n^6}\right).\end{aligned} \tag{38}$$

From (38) and using (21), it follows that

$$\begin{aligned}&\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]\\&\quad=\left[2n^2\left(1+\frac{3a-2b}{3an}+\frac{a^2+b^2-ac-2ab}{2a^2n^2}\right.\right.\\&\qquad\left.\left.+\frac{15ab^2-10a^2b-15a^2c+12abc-6a^2d-6b^3}{15a^3n^3}+O\left(\frac{1}{n^4}\right)\right)^{-1}\right]\\&\quad=\left[2n^2\left(1-\frac{3a-2b}{3an}-\frac{a^2+b^2-ac-2ab}{2a^2n^2}\right.\right.\\&\qquad-\frac{15ab^2-10a^2b-15a^2c+12abc-6a^2d-6b^3}{15a^3n^3}\\&\qquad\left.\left.+\frac{(3a-2b)^2}{9a^2n^2}+\frac{2(3a-2b)((a-b)^2-ac)}{6a^3n^3}-\frac{(3a-2b)^3}{27a^3n^3}+O\left(\frac{1}{n^4}\right)\right)\right]\\&\quad=\left[2n^2+\frac{4b-6a}{3a}n+\frac{9a^2-6ab-b^2+9ac}{9a^2}+\frac{2f(a,b,c,d)}{a^3n}+O\left(\frac{1}{n^2}\right)\right],\end{aligned} \tag{39}$$

where

$$\begin{aligned}f(a,b,c,d)&=\frac{10a^2b-15ab^2+15a^2c-12abc+6a^2d+6b^3}{15a^3}\\&\quad+\frac{(3a-2b)((a-b)^2-ac)}{3a^3}-\frac{(3a-2b)^3}{27}.\end{aligned}$$

From (39), the following theorem follows.

Theorem 2 *Let $a\in\mathbb{R}\setminus\{0\}$, $b,c,d\in\mathbb{R}$, and $a_n = an^3+bn^2+cn+d$. Then, the following statements hold.*

(a) *If $af(a,b,c,d)>0$, then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right]=\left[2n^2+\frac{4b-6a}{3a}n+\frac{9a^2-6ab-b^2+9ac}{9a^2}\right],$$

for sufficiently large n.

(b) *If* $af(a,b,c,d) < 0$, *then*

$$\left[\left(\sum_{j=n}^{+\infty}\frac{a}{a_j}\right)^{-1}\right] = \left[2n^2 + \frac{4b-6a}{3a}n + \frac{9a^2-6ab-b^2+9ac}{9a^2}\right] - 1,$$

for sufficiently large n.

Remark 5 It is interesting that formula (39) cannot be used for the following values of the coefficients

$$a = 1, \qquad b = c = d = 0,$$

to get formula (2) for sufficiently large n. Namely, a direct calculation shows that from (39), we have

$$f(a,0,0,0) = 0, \tag{40}$$

for any $a \in \mathbb{R}$.

Using relation (40) together with formula (39), we have that the following relation holds

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)^{-1}\right] = \left[2n^2 - 2n + 1 + O\left(\frac{1}{n^2}\right)\right].$$

However, the sing of the asymptotic quantity $O(\frac{1}{n^2})$ appearing in formula (40) is not determined.

To find a closed-form formula for the sequence $[(\sum_{k=n}^{\infty}\frac{1}{k^3})^{-1}]$, more members in the corresponding asymptotic expansions should be taken.

Indeed, by Lemma 1, we have

$$\sum_{j=n}^{+\infty}\frac{1}{j^3} = \frac{1}{2n^2} + \frac{1}{2n^3} + \frac{1}{4n^4} - \frac{1}{12n^6} + O\left(\frac{1}{n^7}\right).$$

Hence,

$$\begin{aligned}
&\left[\left(\sum_{j=n}^{+\infty}\frac{1}{j^3}\right)^{-1}\right] \\
&\quad = \left[2n^2\left(1 + \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^4} + O\left(\frac{1}{n^5}\right)\right)^{-1}\right] \\
&\quad = \left[2n^2\left(1 - \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^4} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{4n^4} - \frac{1}{n^3} - \frac{3}{2n^4} + \frac{1}{n^4} + O\left(\frac{1}{n^5}\right)\right)^{-1}\right] \\
&\quad = \left[2n^2 - 2n + 1 - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right)\right].
\end{aligned}$$

From this and since

$$1 - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right) < 1$$

for sufficiently large n, we obtain

$$\left[\left(\sum_{j=n}^{+\infty}\frac{1}{j^3}\right)^{-1}\right] = 2n(n-1),$$

for sufficiently large n, as desired.

Remark 6 The method employed in proving above theorems can be applied to the sequences of the form

$$a_n = P_k(n), \quad n \in \mathbb{N},$$

where

$$P_k(t) = \sum_{j=0}^{k} c_j t^j,$$

$k \in \mathbb{N}_2$, $c_j \in \mathbb{R}$, $j = \overline{0,k}$, $c_k \neq 0$, that is, to the sequences defined by polynomials of degree greater than or equal to two.

Remark 7 Above type of problems for the case when a sequence is a solution to the homogeneous linear difference equation with constant coefficients

$$a_{n+k} + c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad n \in \mathbb{N}, \tag{41}$$

where $k \in \mathbb{N}$, $c_j \in \mathbb{R}$, $j = \overline{0,k-1}$, $c_0 \neq 0$, such that the characteristic polynomial associated to the equation

$$P_k(\lambda) = \lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_1\lambda + c_0, \tag{42}$$

has a unique dominant zero is dealt with in a relatively simple way and are essentially folklore. Namely, it is well known that the general solution to equation (41) has the form

$$a_n = \sum_{j=1}^{l} Q_j(n)\lambda_j^n, \quad n \in \mathbb{N}, \tag{43}$$

where $1 \leq l \leq k$, λ_j, $j = \overline{1,l}$, are the distinct zeros of polynomial (42), and $Q_j(t)$ is a polynomial of degree $s_j - 1$, where $s_j \in \mathbb{N}$ is the multiplicity of the zero λ_j (see, e.g., [23, 25, 26, 29, 30]).

Without loss of generality, we may assume that

$$|\lambda_1| > \max_{j=\overline{2,l}} |\lambda_j|, \tag{44}$$

which implies that $s_1 = 1$. Then from (43) and (44), we have

$$a_n \sim c\lambda_1^n, \tag{45}$$

as $n \to +\infty$, for some constant $c \in \mathbb{R} \setminus \{0\}$, since, in this case, $Q_1(t)$ is a constant polynomial.

From (45), we see that in many cases, the problem reduces to dealing with a simpler sequence (instead of a_n, we can consider the sequence $c\lambda_1^n$). Such a situation appears, e.g., in the case of the Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$, which is the solution to the difference equation

$$a_{n+2} = a_{n+1} + a_n, \quad n \in \mathbb{N},$$

with the initial values $a_1 = a_2 = 1$ ([22, 28, 45]). The characteristic polynomial associated to the equation is

$$P_2(t) = t^2 - t - 1,$$

and it has two different zeros

$$t_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad t_2 = \frac{1-\sqrt{5}}{2}.$$

Since,

$$f_n = \frac{t_1^n - t_2^n}{t_1 - t_2}, \quad n \in \mathbb{N},$$

and $t_1 > t_2$, it follows that

$$f_n \sim \frac{t_1^n}{t_1 - t_2}, \tag{46}$$

as $n \to +\infty$.

Asymptotic relation (46) is essentially the reason why integer parts of the reciprocal remainders of many sums containing the Fibonacci sequences can be found for sufficiently large n.

Using this simple idea, it can be found many closed-form formulas for integer parts of the reciprocal remainders of sums containing solutions to homogeneous linear difference equations with constant coefficients, which hold for sufficiently large n. When such a formula is found, then it, together with some technical algebraic manipulations, can be used in trying to find all the values of index n for which the formula holds.

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