Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 560586, 14 pages
doi:10.1155/2012/560586

Research Article

On Integral Inequalities of Hermite-Hadamard Type for s-Geometrically Convex Functions

Tian-Yu Zhang,[1] Ai-Ping Ji,[1] and Feng Qi[2]

[1] *College of Mathematics, Inner Mongolia University for Nationalities, Inner Mongolia Autonomous Region, Tongliao City 028043, China*
[2] *School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China*

Correspondence should be addressed to Feng Qi, qifeng618@gmail.com

Received 12 May 2012; Accepted 19 May 2012

Academic Editor: Yonghong Yao

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The authors introduce the concept of the s-geometrically convex functions. By the well-known Hölder inequality, they establish some integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions and apply these inequalities to special means.

1. Introduction

We firstly list several definitions and some known results.

Definition 1.1. A function $f : I \subset \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f\left(\lambda x + (1-\lambda)y\right) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 (see [1]). A function $f : I \subset \mathbb{R}_0 = [0, +\infty) \to \mathbb{R}_0$ is said to be s-convex if

$$f\left(\lambda x + (1-\lambda)y\right) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for some $s \in (0,1]$, where $x, y \in I$, and $\lambda \in [0,1]$.

If $s = 1$, the s-convex function becomes a convex function on $\mathbb{R}_0$.

Theorem 1.3 ([2], Theorem 2.2). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$, $a < b$.*

(i) *If $|f'(x)|$ is a convex function on $[a, b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.3}$$

(ii) *If $|f'(x)|^{p/(p-1)}$ is a convex function on $[a, b]$, for $p > 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{(1/p)}} \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}. \tag{1.4}$$

Theorem 1.4 ([3], Theorems 2.3 and 2.4). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$, $a < b$. If $|f'(x)|^p$ is convex on $[a, b]$, for $p > 1$, then*

$$\begin{gathered} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|), \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ \times \left[\left(|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\right)^{(p-1)/p} \right. \\ \left. + \left(3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right)^{(p-1)/p} \right]. \end{gathered} \tag{1.5}$$

Theorem 1.5 ([4], Theorem 1–4). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$, $a < b$. If $|f'(x)|^q$ is s-convex on $[a, b]$, for $q > 1$, then*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\quad \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{(q-1)/q} \left(\frac{2+(1/2)^s}{(s+1)(s+2)}\right)^{1/q} \left(|f'(a)|^s + |f'(b)|^s\right)^{1/q}. \end{aligned} \tag{1.6}$$

Theorem 1.6 ([5], Theorem 4). *Let $f : I \to \mathbb{R}_0$ be differentiable on I°, $a, b \in I$, $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some $s \in (0,1]$, and $p, q \geq 1$, such that $(1/q)+(1/p) = 1$ then*

$$\begin{aligned}
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| &\leq 2^{-1/p}\frac{\left(|f'(a)|^q + (s+1)|f'((a+b)/2)|^q\right)^{1/q}}{\{(s+1)(s+2)\}^{1/q}} \\
&\quad + 2^{-1/p}\frac{\left(|f'(b)|^q + (s+1)|f'((a+b)/2)|^q\right)^{1/q}}{\{(s+1)(s+2)\}^{1/q}} \\
&= 2^{-1/p}\left[\left(\beta(s+1,2)|f'(a)|^q + \beta(s+2,1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\
&\quad \left. + \left(\beta(s+1,2)|f'(b)|^q + \beta(s+2,1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right].
\end{aligned} \tag{1.7}$$

Theorem 1.7 ([6], Theorems 2.2–2.4). *Let $f : I \to \mathbb{R}_0$ be differentiable on I°, $a, b \in I$, $a < b$, and $f' \in L([a,b])$.*

(i) *If $|f'(x)|$ is s-convex on $[a,b]$ for some $s \in (0,1]$, then*

$$\begin{aligned}
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| &\leq \frac{b-a}{4(s+1)(s+2)}\left[|f'(a)| + 2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right] \\
&\leq \frac{(2^{2-s}+1)(b-a)}{4(s+1)(s+2)}\left[|f'(a)| + |f'(b)|\right].
\end{aligned} \tag{1.8}$$

(ii) *If $|f'(x)|^{p/(p-1)}$ $(p > 1)$ is a s-convex function on $[a,b]$ for some $s \in (0,1]$, then*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q}\right. \\
&\quad \left. + \left(2^{1-s}|f'(a)|^q|f'(a)|^{p/(p-1)} + \left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right],
\end{aligned} \tag{1.9}$$

where $1/p + 1/q = 1$.

(iii) *If* $|f'(x)|^q$ $(q \geq 1)$ *is s-convex on* $[a,b]$ *for some* $s \in (0,1]$, *then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left(\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q} + \left(2^{1-s}|f'(a)|^q|f'(a)|^q + \left(2^{1-s}+1\right)|f'(b)|^q\right)^{1/q}\right]. \tag{1.10}$$

Now we introduce the definition of the s-geometrically convex function.

Definition 1.8. A function $f : I \subset \mathbb{R}_+ = (0,+\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^\lambda y^{1-\lambda}\right) \leq [f(x)]^\lambda [f(y)]^{1-\lambda} \tag{1.11}$$

for $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.9. A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if

$$f\left(x^\lambda y^{1-\lambda}\right) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s} \tag{1.12}$$

for some $s \in (0,1]$, where $x, y \in I$ and $\lambda \in [0,1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions and then apply these inequalities to special means.

2. A Lemma

In order to prove our results, we need the following lemma.

Lemma 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I°, *and* $a, b \in I$, *with* $a < b$. *If* $f' \in L([a,b])$, *then*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{4}\int_0^1 \left[tf'\left((1-t)a + t\frac{a+b}{2}\right) + (t-1)f'\left((1-t)\frac{a+b}{2} + tb\right)\right]dt,$$

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{b-a}{4}\int_0^1\left[(t-1)f'\left((1-t)a+t\frac{a+b}{2}\right)+tf'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt. \tag{2.1}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned}
&\int_0^1\left[tf'\left((1-t)a+t\frac{a+b}{2}\right)+(t-1)f'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt\\
&\quad=\frac{2}{b-a}\left[tf\left((1-t)a+t\frac{a+b}{2}\right)\Big|_0^1-\int_0^1 f\left((1-t)a+t\frac{a+b}{2}\right)dt\right]\\
&\qquad+\frac{2}{b-a}\left[(t-1)f\left((1-t)\frac{a+b}{2}+tb\right)\Big|_0^1-\int_0^1 f\left((1-t)\frac{a+b}{2}+tb\right)dt\right]\\
&\quad=\frac{4}{b-a}\left(f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right),\\
&\int_0^1\left[(t-1)f'\left((1-t)a+t\frac{a+b}{2}\right)+tf'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt\\
&\quad=\frac{2}{b-a}\left[(t-1)f\left((1-t)a+t\frac{a+b}{2}\right)\Big|_0^1-\int_0^1 f\left((1-t)a+t\frac{a+b}{2}\right)dt\right]\\
&\qquad+\frac{2}{b-a}\left[tf\left((1-t)\frac{a+b}{2}+tb\right)\Big|_0^1-\int_0^1 f\left((1-t)\frac{a+b}{2}+tb\right)dt\right]\\
&\quad=\frac{4}{b-a}\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right).
\end{aligned} \tag{2.2}$$

This completes the proof of Lemma 2.1. □

3. Main Results

Theorem 3.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a,b \in I$, with $a<b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q \ge 1$ and $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1(s,q;g_1(\alpha),g_2(\alpha)) \tag{3.1}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1(s,q;g_2(\alpha),g_1(\alpha)), \tag{3.2}$$

where

$$g_1(\alpha)=\begin{cases}\dfrac{1}{2}, & \alpha=1,\\ \dfrac{\alpha\ln\alpha-\alpha+1}{[\ln\alpha]^2}, & \alpha\neq 1,\end{cases}\qquad g_2(\alpha)=\begin{cases}\dfrac{1}{2}, & \alpha=1,\\ \dfrac{\alpha-\ln\alpha-1}{[\ln\alpha]^2} & \alpha\neq 1,\end{cases}\tag{3.3}$$

$$\alpha(u,v)=\left|f'(a)\right|^{-u}\left|f'(b)\right|^{v},u,v>0,\tag{3.4}$$

$$\begin{aligned}&G_1(s,q;g_1(\alpha),g_2(\alpha))\\&=\begin{cases}\left|f'(a)\right|^{s}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}+\left|f'(a)f'(b)\right|^{s/2}\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}, & \left|f'(a)\right|\leq 1,\\ \left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right)\right]^{1/q}+\left|f'(a)\right|^{1/2s}\left|f'(b)\right|^{s/2}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right)\right]^{1/q}, & \left|f'(b)\right|\leq 1\leq\left|f'(a)\right|,\\ \left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q}+\left|f'(a)f'(b)\right|^{1/2s}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q}, & 1\leq\left|f'(b)\right|.\end{cases}\end{aligned}\tag{3.5}$$

Proof. (1) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq\frac{b-a}{4}\int_0^1\left[t\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|+|t-1|\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|\right]dt\\&\leq\frac{b-a}{4}\left\{\left(\int_0^1 tdt\right)^{1-1/q}\left[\int_0^1 t\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right]^{1/q}\right.\\&\qquad\left.+\left(\int_0^1(1-t)dt\right)^{1-1/q}\left[\int_0^1(1-t)\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q dt\right]^{1/q}\right\}\\&\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 t\left|f'\left(a^{(2-t)/2}b^{t/2}\right)\right|^q dt\right]^{1/q}\right.\\&\qquad\left.+\left[\int_0^1(1-t)\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|^q dt\right]^{1/q}\right\}\end{aligned}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 t\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 (1-t)\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt\right]^{1/q}\right\}. \tag{3.6}$$

If $0<\mu\leq 1\leq\eta, 0<\alpha, s\leq 1$, then

$$\mu^{\alpha^s}\leq\mu^{\alpha s},\quad \eta^{\alpha^s}\leq\eta^{\alpha/s}. \tag{3.7}$$

(i) If $|f'(a)|\leq 1$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt\\
&\quad\leq\int_0^1 t\left(\left|f'(a)\right|^{(sq/2)(2-t)}\left|f'(b)\right|^{(sqt/2)}\right)dt=\left|f'(a)\right|^{sq}g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right),\\
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt\\
&\quad\leq\int_0^1 (1-t)\left(\left|f'(a)\right|^{(sq/2)(1-t)}\left|f'(b)\right|^{(sq/2)(1+t)}\right)dt=\left|f'(a)f'(b)\right|^{sq/2}g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.8}$$

(ii) If $|f'(b)|\leq 1\leq|f'(a)|$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt\\
&\quad\leq\int_0^1 t\left(\left|f'(a)\right|^{(q/2s)(2-t)}\left|f'(b)\right|^{(sq/2)t}\right)dt=\left|f'(a)\right|^{q/s}g_1\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right),\\
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt\\
&\quad\leq\int_0^1 (1-t)\left(\left|f'(a)\right|^{(q/2s)(1-t)}\left|f'(b)\right|^{(sq/2)(1+t)}\right)dt=\left|f'(a)\right|^{q/2s}\left|f'(b)\right|^{sq/2}g_2\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.9}$$

(iii) If $1\leq|f'(b)|$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt\\
&\quad\leq\int_0^1 t\left(\left|f'(a)\right|^{(q/2s)(2-t)}\left|f'(b)\right|^{(q/2s)t}\right)dt=\left|f'(a)\right|^{q/s}g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right),
\end{aligned}$$

$$\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt$$
$$\leq \int_0^1 (1-t)\left(\left|f'(a)\right|^{(q/2s)(1-t)}\left|f'(b)\right|^{(q/2s)(1+t)}\right)dt = \left|f'(a)f'(b)\right|^{q/2s} g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{3.10}$$

From (3.6) to (3.10), (3.1) holds.

(2) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\
&\qquad \left.+\left(\int_0^1 t dt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q dt\right)^{1/q}\right] \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 (1-t)\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt\right]^{1/q}\right. \\
&\qquad \left.+\left[\int_0^1 t\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt\right]^{1/q}\right\}.
\end{aligned} \tag{3.11}$$

(i) If $|f'(a)|\leq 1$, by (3.7), we have

$$\begin{aligned}
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \leq \left|f'(a)\right|^{sq} g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \\
&\int_0^1 t\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \leq \left|f'(a)f'(b)\right|^{sq/2} g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.12}$$

(ii) If $|f'(b)|\leq 1\leq |f'(a)|$, by (3.7), we have

$$\begin{aligned}
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \leq \left|f'(a)\right|^{q/s} g_2\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right), \\
&\int_0^1 t\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \leq \left|f'(a)\right|^{q/2s}\left|f'(b)\right|^{sq/2} g_1\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.13}$$

(iii) If $1 \leq |f'(b)|$, by (3.7), we have

$$\int_0^1 (1-t)\left(|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\right)dt \leq |f'(a)|^{q/s} g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right),$$
$$\int_0^1 t\left(|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\right)dt \leq |f'(a)f'(b)|^{q/2s} g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{3.14}$$

From (3.11) to (3.14), (3.2) holds. This completes the required proof. □

Applying Theorem 3.1 to $q = 1, s = 1$, respectively, results in the following corollary.

Corollary 3.2. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $s \in (0,1]$, then*

(i) *when $q = 1$, one has*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}G_1(s,1;g_1(\alpha),g_2(\alpha)),$$
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}G_1(s,1;g_2(\alpha),g_1(\alpha)). \tag{3.15}$$

(ii) *when $s = 1$, one has*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}G_1(1,q;g_1(\alpha),g_2(\alpha)),$$
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}G_1(1,q;g_2(\alpha),g_1(\alpha)), \tag{3.16}$$

where $g_1(\alpha)$, $g_2(\alpha)$, $\alpha(u,v)$, $G_1(s,q;g_2(\alpha),g_1(\alpha))$ are same with (3.3)–(3.5).

Theorem 3.3. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, a, $b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, for $q > 1$ and $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}G_2(s,q;g_3(\alpha)), \tag{3.17}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}G_2(s,q;g_3(\alpha)), \tag{3.18}$$

where

$$G_2(s,q;g_3(\alpha)) =$$

$$\begin{cases} \left(|f'(a)|^s + |f'(a)f'(b)|^{s/2}\right)\left[g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}, & |f'(a)| \le 1, \\ \left(|f'(a)|^{1/s} + |f'(a)|^{1/2s}|f'(b)|^{s/2}\right)\left[g_3\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right)\right]^{1/q}, & |f'(b)| \le 1 \le |f'(a)|, \\ \left(|f'(a)|^{1/s} + |f'(a)f'(b)|^{1/2s}\right)\left[g_3\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q}, & 1 \le |f'(b)|, \end{cases} \tag{3.19}$$

$$g_3(\alpha) = \begin{cases} 1, & \alpha = 1, \\ \dfrac{\alpha-1}{\ln\alpha}, & \alpha \ne 1, \end{cases}$$

and $\alpha(u,v)$ *is the same as in* (3.4).

Proof. (1) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{b-a}{4}\int_0^1\left[t\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right| + |t-1|\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|\right]dt \\ &\le \frac{b-a}{4}\left[\left(\int_0^1 t^{q/(q-1)}dt\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\ &\quad \left.+\left(\int_0^1 (1-t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|^q dt\right)^{1/q}\right] \\ &\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left(\int_0^1 |f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}dt\right)^{1/q}\right. \\ &\quad \left.+\left(\int_0^1 |f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}dt\right)^{1/q}\right]. \end{aligned} \tag{3.20}$$

(i) If $|f'(a)| \le 1$, we have

$$\begin{aligned} \int_0^1 |f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}dt &\le |f'(a)|^{sq}g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \\ \int_0^1 |f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}dt &\le |f'(a)f'(b)|^{sq/2}g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \end{aligned} \tag{3.21}$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, we have

$$\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right),$$
$$\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \leq |f'(a)|^{q/2s} |f'(b)|^{sq/2} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right). \tag{3.22}$$

(iii) If $1 \leq |f'(b)|$, we have

$$\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right),$$
$$\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \leq |f'(a)f'(b)|^{q/2s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right). \tag{3.23}$$

From (3.20) to (3.23), (3.17) holds.

(2) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left(\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt\right)^{1/q}\right. \\ &\qquad \left. + \left(\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt\right)^{1/q}\right]. \end{aligned} \tag{3.24}$$

From (3.24) and (3.21) to (3.23), (3.18) holds. This completes the proof. □

If taking $s = 1$ in Theorem 3.3, we can derive the following corollary.

Corollary 3.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is geometrically convex and monotonically decreasing on $[a,b]$ for $q > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(1,q;g_3(\alpha)),$$
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(1,q;g_3(\alpha)), \tag{3.25}$$

where $\alpha(u,v)$, $G_2(s,q;g_3(\alpha))$, and $g_3(\alpha)$ are the same as in Theorem 3.3.

4. Application to Special Means

Let

$$A(a,b) = \frac{a+b}{2}, \qquad L(a,b) = \frac{b-a}{\ln b - \ln a} \quad (a \neq b),$$
$$L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, \quad a \neq b, \ p \in R, \ p \neq 0, \ -1 \tag{4.1}$$

be the arithmetic, logarithmic, generalized logarithmic means for $a, b > 0$ respectively.

Let $f(x) = x^s/s, \ x \in (0,1], \ 0 < s < 1$, $q \geq 1$, and then the function

$$|f'(x)|^q = x^{(s-1)q} \tag{4.2}$$

is monotonically decreasing on $(0,1]$. For $\lambda \in [0,1]$, we have

$$(s-1)q(\lambda^s - \lambda) \leq 0, \qquad (s-1)q\big((1-\lambda)^s - (1-\lambda)\big) \leq 0. \tag{4.3}$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0 < s < 1$.

Theorem 4.1. *Let* $0 < a < b \leq 1$, $0 < s < 1$, *and* $q \geq 1$. *Then*

$$\begin{aligned}
\left|[A(a,b)]^s - [L_s(a,b)]^s\right| &\leq \frac{(b-a)^{1-1/q}s}{8}\left(\frac{4s}{(1-s)q}L(a,b)\right)^{1/q} \\
&\quad\times \left[a^{(s-1)/(2s)}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q}\right. \\
&\quad\left. + b^{(s-1)/(2s)}\left\{a^{(s-1)q/(2s)} - \left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q}\right] \\
\left|A(a^s,b^s) - [L_s(a,b)]^s\right| & \\
&\leq \frac{(b-a)^{1-1/q}s}{8}\left(\frac{4s}{(1-s)q}L(a,b)\right)^{1/q} \\
&\quad\times \left[b^{(s-1)/(2s)}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q}\right. \\
&\quad\left. + a^{(s-1)/(2s)}\left\{a^{(s-1)q/(2s)} - \left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q}\right].
\end{aligned} \tag{4.4}$$

In particular, if $q = 1$*, one has*

$$\left|[A(a,b)]^s - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{4}\left[L_{(s-1)/(2s)-1}(a,b)\right]^{(s-1)/s-2}[L(a,b)]^2$$

$$\left|A(a^s,b^s) - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{4}L(a,b)\left\{2\left[L_{(s-1)/s-1}(a,b)\right]^{(s-1)/s-1} - \left[L_{(s-1)/(2s)-1}(a,b)\right]^{(s-1)/s-2}L(a,b)\right\}. \quad (4.5)$$

Proof. Let $f(x) = x^s/s,\ x \in (0,1],\ 0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \ge 1$ and

$$\left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}a^{(s-1)/2s}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q},$$

$$\left|f'(a)f'(b)\right|^{1/(2s)}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}b^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{a^{(s-1)q/(2s)} - \left[L_{(s-1)q/((2s)-1)}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q},$$

$$\left|f'(a)f'(b)\right|^{1/(2s)}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}b^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q},$$

$$\left|f'(a)\right|^{1/s}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}a^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left[a^{(s-1)q/(2s)} - \left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right]^{1/q}. \quad (4.6)$$

By Theorem 3.1, Theorem 4.1 is thus proved. □

Theorem 4.2. *Let* $0 < a < b \le 1,\ s \in (0,1)$*, and* $q > 1$*. Then one has*

$$\left|[A(a,b)]^s - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}A\left(a^{(s-1)/(2s)},b^{(s-1)/(2s)}\right) \times\left[\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/2s-1}L(a,b)\right]^{1/q}$$

$$\left|A(a^s,b^s)-[L_s(a,b)]^s\right| \leq \frac{(b-a)s}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q} A\left(a^{(s-1)/(2s)},b^{(s-1)/(2s)}\right) \times \left[\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1} L(a,b)\right]^{1/q}. \tag{4.7}$$

Proof. Let $f(x)=x^s/s$, $x\in(0,1]$, $0<s<1$. Then $|f'(a)|=a^{s-1}>b^{s-1}=|f'(b)|\geq 1$ and

$$g_3\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=a^{-(s-1)q/(2s)}\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1} L(a,b). \tag{4.8}$$

Using Theorem 3.3, Theorem 4.2 is thus proved. □

Acknowledgments

The research was supported by Science Research Funding of Inner Mongolia University for Nationalities (Project no. NMD1103).

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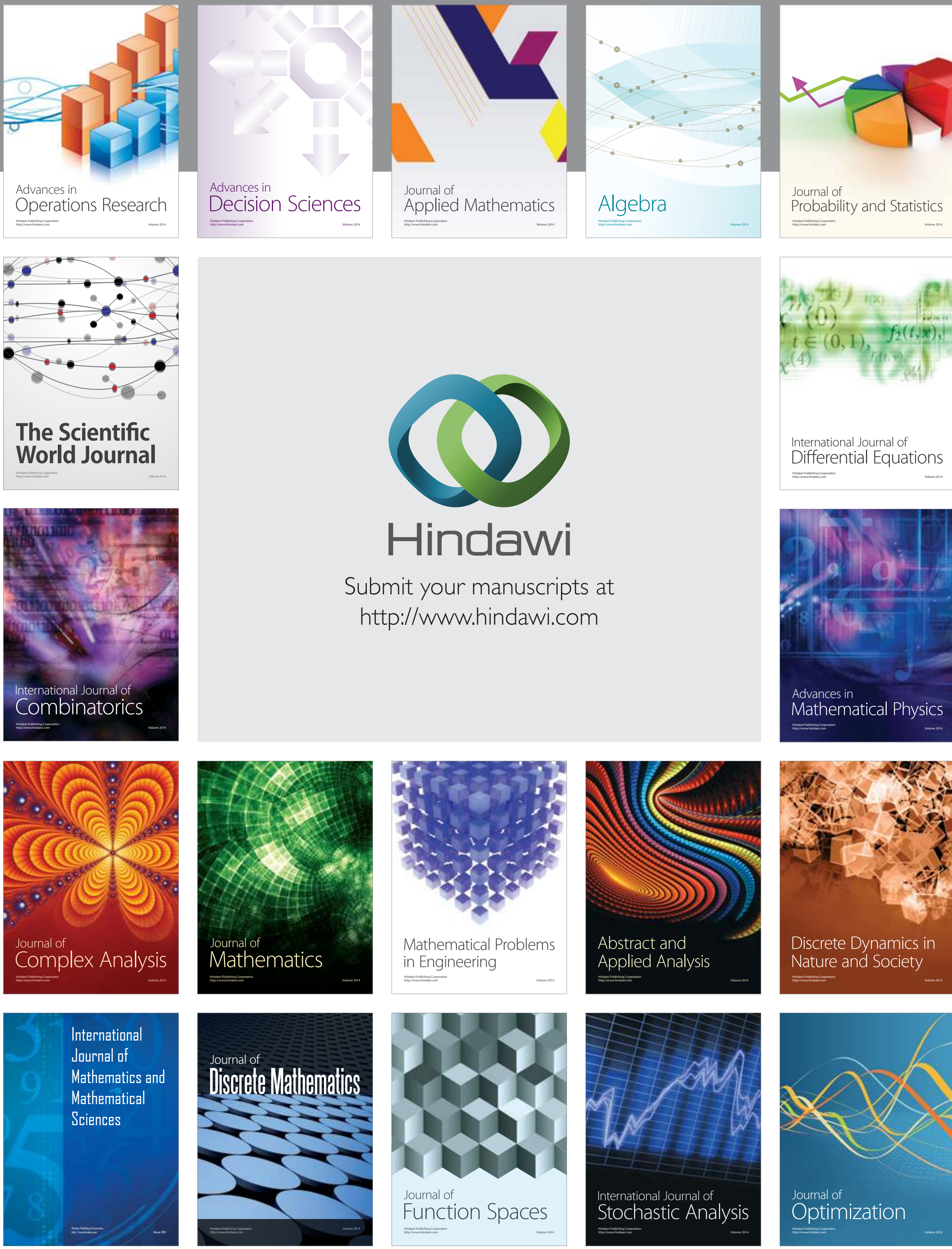
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