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On integral transformations associated with a certain Lagrangian
— as a prototype of quantization

Dedicated to the memory of the late Professor H. Kumano-go

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Introduction.

Let M be a finite dimensional manifold and let $L(\gamma, \dot{\gamma})$ be a function on the tangent bundle TM. Our aim is to construct a C^0-*semi* group of bounded linear operators $\boldsymbol{H}_t^\lambda(L)$ associated with $L(\gamma, \dot{\gamma})$ and its infinitesimal generator $A^\lambda(L)$ on the intrinsic Hilbert space $\mathcal{H}(M)$ (see §5 for its definition), where $t\in\boldsymbol{R}_+$ and λ is a positive parameter.

As the above problem is too vague to consider, we restrict ourselves to the following case which seems rather typical.

(M) M is a smooth, simply-connected and connected d-dimensional manifold.

(L. I) $L(\gamma, \dot{\gamma})$ is represented by

$$L(\gamma, \dot{\gamma})=L^0(\gamma, \dot{\gamma})-V(\gamma), \qquad L^0(\gamma, \dot{\gamma})=(1/2)g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j \tag{1}$$

for $(\gamma, \dot{\gamma})\in TM$. (Hereafter, we use Einstein's convention to contract indices.)

Moreover,

(L. II) $ds^2=g_{ij}(x)dx^idx^j$ defines a complete Riemannian metric on M.

(In the following, for such $g_{ij}(x)$, we associate quantities in Riemannian geometry as are used usually.)

(L. III) There exists a constant $k\geqq 0$ such that for any 2-plane π, the sectional curvature K_π satisfies $-k^2\leqq K_\pi\leqq 0$.

(L. IV) Denote by $R_{ijk}{}^h(x)$ the component of curvature tensor $R(\cdot\,,\cdot)$. Then, there exists a constant C_0 such that

$$|\nabla^\alpha R_{ijk}{}^h(x)|\leqq C_0 \qquad \text{for} \quad 0\leqq|\alpha|\leqq 3\,,$$

where $\alpha=(\alpha_1, \cdots, \alpha_d)$ is a multi-index, $\nabla^\alpha=\nabla_1^{\alpha_1}\cdots\nabla_d^{\alpha_d}$ and ∇_j represents the covariant derivation in the direction of x^j for any local chart at $x=(x^1, \cdots, x^d)$.

(L. V) $V\in C_0^\infty(M)$ is real valued.

For any natural measure μ on M which is defined without mentioning Riemannian metric (see § 5 for its definition), we consider the following transformation in $L^2(M, d\mu)$ with parameters $t>0$ and $\lambda>0$. For any $f\in C_0^\infty(M)$ and sufficiently small $t>0$, we put

$$(2)\quad (H_t^\lambda(L;\mu)f)(x)=(2\pi\lambda)^{-d/2}\int_M \rho(L;\mu)(t, x, y)\exp\{-\lambda^{-1}S(L)(t, x, y)\}\cdot f(y)d\mu(y)$$

Here we denote

$$(3)\qquad S(L)(t, x, y)=\inf\left\{\int_0^t L(\gamma(\tau), \dot\gamma(\tau))d\tau \ :\ \gamma(\tau)\in\Omega_{t,x,y}\right\},\quad \dot\gamma(\tau)=d\gamma(\tau)/d\tau,$$

$$(4)\qquad \Omega_{t,x,y}=\left\{\gamma(\cdot)\in C([0, t]\to M)\ :\ \text{absolutely continuous in } \tau \right.$$
$$\left.\text{with } \gamma(0)=y,\ \gamma(t)=x, \text{ and } \int_0^t\langle\dot\gamma(\tau), \dot\gamma(\tau)\rangle_{\gamma(\tau)}d\tau<+\infty\right\}$$

and

$$(5)\qquad \rho(L;\mu)(t, x, y)=[\det(-\partial_{x^i}\partial_{y^\alpha}S(L)(t, x, y))/\mu(x)\mu(y)]^{1/2}$$

where $\mu(x)$ is the density of μ at x, i.e. $d\mu(x)=\mu(x)dx^1\wedge\cdots\wedge dx^d$, ∂_{x^i} denotes the partial derivation in the direction of x^i at $x=(x^1, \cdots, x^d)$, and $\langle X, Y\rangle_x$ is the Riemannian scalar product at x for $X, Y\in T_xM$.

Our theorem is:

THEOREM. *Let M and L be given satisfying assumptions* (M) *and* (L. I)-(L. V). *Then, there exists a positive number $T>0$ such that the followings hold:*

(a) *For any natural measure μ, the operator $H_t^\lambda(L;\mu)$ defines a bounded linear operator in $L^2(M, d\mu)$ for $0<t<T$.*

(b) $$\lim_{t\to 0}\|H_t^\lambda(L;\mu)f-f\|=0 \quad \textit{for all } f\in L^2(M, d\mu).$$

(c) *There exist positive constants C and C' depending on T independent of μ such that*

$$(6)\quad \|H_{t+s}^\lambda(L;\mu)f-H_t^\lambda(L;\mu)H_s^\lambda(L;\mu)f\|\leqq[C\{(t+s)^{3/2}-t^{3/2}+s^{3/2}\}+C'(t+s)s]\|f\|$$

for $0<t+s<T$. Moreover, we take $C'=0$ for $V=0$.

Furthermore, we have:

(d) *There exists a limit $\boldsymbol{H}_t^\lambda(L;\mu)=\lim_{n\to\infty}[H_{t/n}^\lambda(L;\mu)]^n$ in the operator norm in $L^2(M, d\mu)$ for any $t>0$. Moreover, $\{\boldsymbol{H}_t^\lambda(L;\mu)\}_{t\geq 0}$ with $\boldsymbol{H}_0^\lambda(L;\mu)=$the identity operator, forms a C^0 semi-group in $L^2(M, d\mu)$.*

(e) *For any two natural measures μ and ν on M, we have*

$$(7)\qquad \boldsymbol{H}_t^\lambda(L;\mu)=U_{\nu\mu}^{-1}\boldsymbol{H}_t^\lambda(L;\nu)U_{\nu\mu}$$

where $U_{\nu\mu}$ is an isomorphism from $L^2(M, d\mu)$ onto $L^2(M, d\nu)$, defined by

$$(8)\qquad (U_{\nu\mu}f)(x)=f(x)(\mu(x)/\nu(x))^{1/2} \quad \text{for} \quad f\in L^2(M, d\mu).$$

(f) *The infinitesimal generator $\boldsymbol{A}^\lambda(L;\mu)$ of $\boldsymbol{H}_t^\lambda(L;\mu)$ is given by*

$$(9)\qquad \partial_t(\boldsymbol{H}_t^\lambda(L;\mu)f)_{|t=0}=\boldsymbol{A}^\lambda(L;\mu)f=U_{\mu_g\mu}^{-1}\boldsymbol{A}^\lambda(L;\mu_g)U_{\mu_g\mu}f \quad \text{for } f\in C_0^\infty(M),$$

$$(\boldsymbol{A}^\lambda(L;\mu_g)f)(x)=\lambda^2(\Delta_g/2-R(x)/12)f(x)+V(x)f(x).$$

Here Δ_g is the negative Laplace-Beltrami operator and $R(\cdot)$ stands for the scalar curvature.

In other words, the above procedure defines a C^0 semi-group $\boldsymbol{H}_t^\lambda(L)$ and its infinitesimal generator $\boldsymbol{A}^\lambda(L)$ on the intrinsic Hilbert space $\mathcal{H}(M)$ such that if $\mathcal{H}(M)$ is trivialized by a natural measure μ as $L^2(M, d\mu)$, then $\boldsymbol{H}_t^\lambda(L)$ and $\boldsymbol{A}^\lambda(L)$ are represented by $\boldsymbol{H}_t^\lambda(L;\mu)$ and $\boldsymbol{A}^\lambda(L;\mu)$ on $L^2(M, d\mu)$.

The old and debated question whether the Schrödinger equation in the curved space contains the term with $\hbar^2R(\cdot)$ will be solved completely if we could proceed in a similar way as above for $\lambda=i\hbar$.

In § 1, we enumerate the basic properties for the quantities derived from the classical mechanics defined through L. As our configuration space is curved, we cannot apply directly the iteration scheme used in Fujiwara [8] to our case. Instead of it, we use the Morse theory to obtain the estimates of the classical quantities. In §§ 2-4, for the special choice of $\mu=\mu_g$, we give the proof of (a)-(d) of Theorem. As one of the corollaries, we give the covariant property of the operators $\boldsymbol{H}_t^\lambda(L;\mu_g)$ and $\boldsymbol{A}^\lambda(L;\mu_g)$ under a diffeomorphism of M. In § 5, we give the definition of the half-density and the intrinsic Hilbert space on M, which combined with the result in § 4, gives readily the proof of Theorem. Here, the meaning of the appearance of the term $R(\cdot)/12$ is clarified. In spite of this fact, in § 6, we claim that for any number $\beta\in\boldsymbol{R}$, we may produce the term $\lambda^2(1/6-\beta/12)R(\cdot)$ if we change our procedure a little bit. But in this case, we must use the fact that Δ_g is essentially self-adjoint on $C_0^\infty(M)$ not proving it as in § 4. Moreover, there exist no covariance properties unless $\beta=1$.

Some parts of Theorem were already announced in Inoue-Maeda [**11**].

1. Some properties of the classical action — preliminaries from differential geometry.

For any x, $y\in M$, the set $\Omega_{t,x,y}$ introduced in (4), denoted simply by Ω in this section, forms a Hilbert manifold. (See p. 247 of Abraham-Marsden [**1**].) The tangent space $T_\gamma\Omega$ at $\gamma\in\Omega$ may be identified with the space of vector fields Z on M along γ with $Z(0)=Z(t)=0$. Moreover, a scalar product on $T_\gamma\Omega$ is defined by

$$\langle Z_1, Z_2\rangle_\gamma=\int_0^t\langle Z_1(\tau), Z_2(\tau)\rangle_{\gamma(\tau)}d\tau\,. \tag{1.1}$$

(For the notational simplicity, we drop the indices γ and $\gamma(\tau)$ above if there occurs no confusion.)

Now, we introduce two functionals $S(\gamma)$ and $S^0(\gamma)$ on Ω by

$$S(\gamma)=\int_0^t L(\gamma(\tau), \dot\gamma(\tau))d\tau \quad \text{for} \quad \gamma\in\Omega, \tag{1.2}$$

$$S^0(\gamma)=\int_0^t L^0(\gamma(\tau), \dot\gamma(\tau))d\tau \quad \text{for} \quad \gamma\in\Omega\,. \tag{1.3}$$

LEMMA 1.1. *Under assumptions* (M), (L. I), (L. II) *and* (L. V), *we have:*

(i) *$S(\gamma)$ is bounded from below.*

(ii) *$S(\cdot)$ is a smooth functional on Ω, i.e. for any curve $\gamma(t\,;\varepsilon)$ on M satisfying $\gamma(\cdot\,;0)=\gamma(\cdot)\in\Omega$, $(d/d\varepsilon)\gamma(\cdot\,;\varepsilon)_{|\varepsilon=0}=Z\in T_\Omega$, there exists $(d/d\varepsilon)S(\gamma(\cdot\,;\varepsilon))_{|\varepsilon=0}$ $=dS(\gamma)\cdot Z$.*

(iii) *$\gamma\in\Omega$ is a critical point of S, i.e. $dS(\gamma)=0$, iff* (a) *γ belongs to $C^2([0, t]\to M)$ and* (b) *γ satisfies the following equation*

$$\ddot\gamma^i(\tau)+\Gamma_{jk}{}^i(\gamma(\tau))\dot\gamma^j(\tau)\dot\gamma^k(\tau)=-(\nabla V(\gamma(\tau)))^i, \tag{1.4}$$

where $\Gamma_{jk}{}^i$ stands for the Christoffel symbol and $(\nabla V(x))^i=g^{ij}(x)\nabla_j V(x)$.

(iv) *$\gamma\in\Omega$ is a non-degenerate minimum of $S(\cdot)$ iff there is no trivial $J\in T_\gamma\Omega$ satisfying the following V-Jacobi equation,*

$$J''(\tau)+R(J(\tau), \dot\gamma(\tau))\dot\gamma(\tau)=-\nabla^2 V(\gamma(\tau))\cdot J(\tau)\,, \tag{1.5)$_{V.J}$ $$

where $J'(\tau)=(\delta/\delta\tau)J(\tau)=$the covariant derivative of J along $\gamma(\tau)$ and $\nabla^2V(x)$ stands for the matrix whose (i, j) element is given by $\nabla^j\nabla_i V(x)$.

PROOF. See, Milnor [**13**] and modify it slightly, if necessary.

It is well-known that under assumption (L. II)

$$S^0(t, x, y)=d^2(x, y)/2t \quad \text{for} \quad x, y\in M, \tag{1.6}$$

where $d(x, y)$ is the Riemannian distance between x and y. We consider the initial value problem for (1.4) with initial conditions

$$\gamma(0)=y \quad \text{and} \quad \dot\gamma(0)=Y\in T_y M\,. \tag{1.7}$$

As is well-known, for any $Y\in T_y M$, there is an interval I_Y where the solution $\gamma(t)$ of (1.4) with (1.7) exists. Moreover we have:

LEMMA 1.2. *Assume* (M), (L. I), (L. II) *and* (L. V). *There exists a constant $C_1>0$ such that*

$$(1.8) \qquad |Y|-C_1t \leqq |\dot{\gamma}(t)| \leqq |Y|+C_1t \quad \text{for} \quad t\in I_Y,$$

$$(1.9) \qquad \begin{aligned} &t^2[|Y|^2-C_1|Y|t+(C_1^2/3)t^2-4C_1] \\ &\leqq d(y, \gamma(t))^2 \\ &\leqq t^2[|Y|^2+C_1|Y|t+(C_1^2/3)t^2+4C_1] \quad \text{for} \quad t\in I_Y. \end{aligned}$$

Here, $|Y|^2=\langle Y, Y\rangle_y$ *for* $Y\in T_yM$.

PROOF.

$$\begin{aligned}(1/2)(d/dt)|\dot{\gamma}(t)|^2 &= \langle(\delta/\delta t)\dot{\gamma}(t), \dot{\gamma}(t)\rangle \\ &= -\langle\nabla V(\gamma(t)), \dot{\gamma}(t)\rangle \\ &\leqq C_1|\dot{\gamma}(t)|.\end{aligned}$$

This leads us to the second inequality of (1.8). And we have, by definition of $S(t, x, y)$ and $S^0(t, x, y)$,

$$(1.10) \qquad S^0(t, x, y)-C_1t \leqq S(t, x, y) \leqq S^0(t, x, y)+C_1t.$$

Combining this with (1.6), we have

$$\begin{aligned} d^2(y, \gamma(t)) &= 2tS^0(t, y, \gamma(t)) \\ &\leqq 2t\left\{\int_0^t (1/2)|\dot{\gamma}(\tau)|^2 d\tau - \int_0^t V(\gamma(\tau))d\tau + C_1t\right\}. \end{aligned}$$

So, we have the second inequality of (1.9) readily. The other parts are proved analogously. q. e. d.

COROLLARY 1.3. *Same assumptions as above. The solution* $\gamma(t)=\gamma(t; y, Y)$ *of* (1.4) *with* (1.7) *exists for any* t, *that is,* $I_Y=[0, \infty)$. *Moreover,* $\gamma(t; y, Y)$ *depends smoothly on* $(t, y, Y)\in[0, \infty)\times TM$.

We denote $\gamma(t; y, Y)$ by $\Phi_{t,y}(Y)$. By calculating primitively, we have:

LEMMA 1.4. *Assume* (M), (L. I), (L. II) *and* (L. V). *Let* $(d\Phi_{t,y})_Y : T_Y(T_yM)\to T_{\Phi_{t,y}(Y)}M$ *be the differential of* $\Phi_{t,y}$ *at* $Y\in T_yM$. *Then, for any* $W\in T_Y(T_yM)$, $J(t)=(d\Phi_{t,y})_YW$ *satisfies the V-Jacobi equation* (1.5) *with initial conditions*

$$(1.11) \qquad J(0)=0 \quad \text{and} \quad J'(0)=W.$$

LEMMA 1.5. *Under assumptions* (M), (L. I)-(L. III) *and* (L. V), *there exists a constant* $T_1>0$ *such that for any* $W\neq 0$, *the solution* $J(t)$ *of* (1.5) *with* (1.11) *satisfies* $J(t)\neq 0$ *for* $0<t<T_1$.

PROOF.

$$\begin{aligned}(1/2)(d^2/dt^2)|J(t)|^2 &= (d/dt)\langle J(t), J'(t)\rangle \\ &= \langle J'(t), J'(t)\rangle + \langle J(t), J''(t)\rangle \\ &\geqq |J'(t)|^2 - C_1|J(t)|^2\end{aligned}$$

by (L. III) and (L. V). Using $(d/dt)|J(t)|^2=\int_0^t(d^2/d\tau^2)|J(\tau)|^2d\tau$ and $J(t)=\int_0^t J'(\tau)d\tau$, we have readily

$$(1/2)(d/dt)|J(t)|^2\geqq(1-(C_1t^2)/2)\int_0^t|J'(\tau)|^2d\tau\,.$$

Taking T_1 smaller than $\sqrt{2/C_1}$, we have the desired result. q. e. d.

COROLLARY 1.6. *Under the same assumptions in Lemma* 1.5, *we have* $(d\Phi_{t,y})_Y\neq 0$ *for* $0<t<T_1$ *and* $Y\in T_yM$ *where* T_1 *is taken as above.*

PROPOSITION 1.7. *Assume* (M), (L. I)-(L. III) *and* (L. V). *Then, there exists a positive number* T_1 *such that* $\Phi_{t,y}$ *gives a diffeomorphism from* T_yM *onto* M *for* $0<t<T_1$ *and* $y\in M$.

PROOF. As the functional $S(\cdot):\Omega\to\boldsymbol{R}$ has the non-degenerate Hessian at critical points for $0<t<T$ by Corollary 1.6 and (iv) of Lemma 1.1, Ω is homotopically equivalent to a CW-complex with 0-dimensional vertex. On the other hand, as M is assumed to be simply connected, Ω consists of one point. That is, there exists one and only one critical point for $S(\cdot)$ in Ω. This means that $\Phi_{t,y}$ is one to one and onto from T_yM to M and $\Phi_{t,y}$ is differentiable by Corollary 1.3. So we have the desired result. q. e. d.

Take orthonormal bases $\{W_j\}$ and $\{e_j\}$ on $T_Y(T_yM)$ and T_xM respectively. Then, the differential mapping $(d\Phi_{t,y})_Y$ may be expressed by

$$(d\Phi_{t,y})_YW_j=F^i_je_i\,. \tag{1.12}$$

We denote $\det(F^i_j)$ by $\det_g(d\Phi_{t,y})_Y$, which is independent of the choice of orthonormal bases. Defining

$$\Theta(t,x,y)=t^{-d}|\det_g(d\Phi_{t,y})_Y|, \tag{1.13}$$

we want to give the upper and lower estimates of it. Before that, we prepare the following lemma which is a modification of Rauch's comparison lemma (§ 7 of Chap. I of Aubin [2], § 10 of Chap. I of Cheeger-Ebin [4]).

LEMMA 1.8. *Let* $A(\tau)$ *be a smooth real vector field on* $\boldsymbol{R}^d$ *satisfying*

$$\ddot{A}(\tau)+K(\tau)A(\tau)=0 \qquad \textit{with} \quad A(0)=0\,. \tag{1.14}$$

Here $K(\tau)$ *is a real* $d\times d$ *matrix of* $0\leqq\tau\leqq t$.

(i) *If* $-K(\tau)$ *is bounded from below, i.e.* $-K(\tau)\geqq-\kappa_1^2I$ $(\kappa_1\geqq 0)$, *then*

$$\langle\dot{A}(t),A(t)\rangle_0\geqq\kappa_1\cot\kappa_1t\cdot\langle A(t),A(t)\rangle_0 \qquad \textit{for} \quad t<\pi/\kappa_1\,. \tag{1.15}$$

(ii) *If* $-K(\tau)$ *is symmetric and bounded from above, i.e.* $-K(\tau)\leqq\kappa_2^2I$ $(\kappa_2\geqq 0)$, *then*

$$\langle\dot{A}(t),A(t)\rangle_0\leqq\kappa_2\coth\kappa_2t\cdot\langle A(t),A(t)\rangle_0\,, \tag{1.16}$$

for sufficiently small t, where $\langle\ ,\ \rangle_0$ *is the Euclidean scalar product.*

PROOF. (i) We compare the equation (1.4) with

$$\ddot{B}(\tau)+\kappa_1^2 B(\tau)=0 \quad with \quad B(0)=0. \tag{1.17}$$

Let $\{B_j(\tau)\}$ be a solution of (1.17) with $B_j(0)=0$, $\dot{B}_j(0)=\boldsymbol{e}_j$, $\{\boldsymbol{e}_j\}$ stands for the canonical bases of $\boldsymbol{R}^d$. As $\{B_j(\tau)\}$ is linearly independent vectors in $\boldsymbol{R}^d$ for $t<\pi/\kappa_1$, we expand a solution $A(\tau)$ of (1.14) as $A(\tau)=\sum_{j=1}^d a_j(\tau)B_j(\tau)$. Using $\langle B_j(\tau), \dot{B}_j(\tau)\rangle=\langle \dot{B}_j(\tau), B_j(\tau)\rangle$ and integration by parts, we get readily

$$\begin{aligned}\langle \dot{A}(t), A(t)\rangle_0=&\int_0^t \{\langle \dot{A}(\tau), \dot{A}(\tau)\rangle_0-\langle K(\tau)A(\tau), A(\tau)\rangle\}\, d\tau\\ &+\left\langle \sum_{i=1}^d a_i(t)\dot{B}_i(t), \sum_{j=1}^d a_j(t)B_j(t)\right\rangle\\ &+\int_0^t\left\langle \sum_{i=1}^d a_i(\tau)\dot{B}_i(\tau), \sum_{j=1}^d a_j(\tau)\dot{B}_j(t)\right\rangle d\tau.\end{aligned}$$

On the other hand, as any solution of (1.17) with $B(0)=0$ is represented by $B(\tau)=\sum_{j=1}^d \beta_j B_j(\tau)$ ($\beta_j\in\boldsymbol{R}$), we have readily, for $B(\tau)$ satisfying $B(t)=A(t)$,

$$\langle \dot{A}(t), A(t)\rangle_0 \geqq \langle \dot{B}(t), B(t)\rangle_0.$$

Since, $A(t)=B(t)$ implies $\beta_j=a_j(t)$. Defining $B(\tau)=(\sin\kappa_1\tau/\sin\kappa_1 t)A(t)$ we have (1.15).

(ii) Instead of (1.17), we use

$$\ddot{C}(\tau)-\kappa_2^2 C(\tau)=0 \quad with \quad C(0)=0. \tag{1.18}$$

Let $\{A_j(\tau)\}$ be solution of (1.14) with $A_j(0)=0$, $\dot{A}_j(0)=\boldsymbol{e}_j$. For sufficiently small t, $\{A_j(\tau)\}$ is linearly independent. Expanding a solution of (1.18) as $C(\tau)=\sum_{j=1}^d a_j(\tau)A_j(\tau)$, we have, by analogous calculation as in proving (i),

$$\begin{aligned}\langle \dot{C}(t), C(t)\rangle_0=&\left\langle \sum_{i=1}^d c_i(t)A_i(t), \sum_{j=1}^d c_j(t)A_j(t)\right\rangle_0\\ &+\int_0^t\left\langle \sum_{i=1}^d c_i(\tau)\dot{A}_i(\tau), \sum_{j=1}^d c_j(\tau)\dot{A}_j(\tau)\right\rangle_0 d\tau.\end{aligned}$$

Here we use the symmetry of $K(\tau)$ to prove $\langle \dot{A}_i(\tau), A_j(\tau)\rangle_0=\langle A_i(\tau), \dot{A}_j(\tau)\rangle_0$. As $A(\tau)=\sum_{i=1}^d c_j(t)A_j(\tau)$ is a solution of (1.14) with $A(t)=C(t)$, we get

$$\langle \dot{C}(t), C(t)\rangle_0 \geqq \langle \dot{A}(t), A(t)\rangle_0.$$

Putting $C(\tau)=(\sinh\kappa_2\tau/\sinh\kappa_2 t)A(t)$, we have (1.16). q.e.d.

Now, using this, we give first of all, the lower bound for $\Theta(t, x, y)$.

PROPOSITION 1.9. *Assume* (M), (L. I)-(L. III) *and* (L. V). *Then, there exists a constant* $T_2>0$ *such that*

(1.19) $\Theta(t, x, y) \geqq (\sin\sqrt{C_1}t/\sqrt{C_1}t)^d$, *for* $0<t<T_2$ *and* $x, y \in M$.

PROOF. Introducing vector fields $\{P_j(\tau)\}$ parallel along $\gamma_c(\tau)$ and orthogonal w.r.t. $\langle , \rangle$, we expand any V-Jabobi field $J(\tau)$ as $J(\tau)=\sum_{j=1}^d a^j(\tau)P_j(\tau)$ where $a^j(\tau)\in \boldsymbol{R}$. Then, $A(\tau)={}^t(a^1(\tau), \cdots, a^d(\tau))$ defines a vector field in $\boldsymbol{R}^d$ and satisfies

$$\ddot{A}(\tau)+K(\tau)A(\tau)=0 .$$

Here (j, k)-element of $K(\tau)$ is given by

(1.20) $$\langle R(P_j(\tau), \dot{\gamma}_c(\tau))\dot{\gamma}_c(\tau), P_k(\tau)\rangle + \langle \nabla^2 V(\gamma_c(\tau))\cdot P_j(\tau), P_k(\tau)\rangle .$$

By assumptions (L.III) and (L.V), there exists $C_1>0$ such that

$$-K(\tau) \geqq -C_1 I .$$

Using (i) of the above lemma, for $t<T_2=\min(T_1, \pi/\sqrt{C_1})$, we have

$$(1/2)(d/dt)\log|J(t)|^2 \geqq \sqrt{C_1}\cot\sqrt{C_1}t .$$

Here, we use the fact $|J(t)|^2=\langle J(t), J(t)\rangle=\langle A(t), A(t)\rangle_0$. Remarking $\lim_{t\to 0}|J(t)|/t=1$, we have $|J(t)|/t \geqq \sin\sqrt{C_1}t/\sqrt{C_1}t$. We get (1.19) by the definition of det. q.e.d.

Analogously, we have the upper bound of $\Theta(t, x, y)$ as

PROPOSITION 1.10 *Under assumptions* (M), (L.I)-(L.III) *and* (L.V), *we have*

(1.21) $\Theta(t, x, y) \leqq [\sinh(k|Y|+C_2)t/(k|Y|+C_2)t]^d$ *for* $0<t<T_2$

where $x=\Phi_{t,y}(Y)$, $C_2=(k^2C_1^2T_1^2+C_1)^{1/2}$ *and* T_2 *is defined in Proposition* 1.9.

PROOF. As before, we consider (1.14) with (1.20). Then, assumptions (L.III) and (L.V), combined with the estimate (1.8) give us an upper bound for $K(\tau)$ as

$$\langle -K(\tau)A(\tau), A(\tau)\rangle_0 \leqq (k|Y|+C_2)^2|A(\tau)|^2 \quad \text{for } 0<\tau<T_2 .$$

Using (ii) of Lemma 1.8, and proceeding as before, we have the desired estimate. q.e.d.

Now, we give elementary properties for $S(t, x, y)$, called the classical action. By Proposition 1.7, there exists the unique critical point γ_c of $S(\cdot)$, called the classical path, which satisfies

(1.22) $$S(t, x, y)=\int_0^t \{(1/2)\langle \dot{\gamma}_c(\tau), \dot{\gamma}_c(\tau)\rangle - V(\gamma_c(\tau))\}\, d\tau .$$

Following fact is rather well-known. (See p. 390 of Berger et al. [3].)

LEMMA 1.11. *Under assumptions* (M), (L.I)-(L.III) *and* (L.V), $S(t, x, y)$ *is a* C^∞*-function on* $(0, T_2)\times M\times M$, *symmetric in* x *and* y, *satisfying the Hamilton-Jacobi equation:*

$(1.23)_{\mathrm{H.J}}$ $\qquad \partial_t S(t, x, y)+(1/2)\langle\nabla_x S(t, x, y), \nabla_x S(t, x, y)\rangle+V(x)=0,$

and for any $Y\in T_y M$

$$(1.24) \qquad (\nabla_y S(t, x, y))(Y)=-\langle\Phi_{t,y}^{-1}(x), Y\rangle .$$

By (1.24), we have, for $X\in T_x M$, $Y\in T_y M$

$$\nabla_x\nabla_y S(t, x, y)(X, Y)=-\langle(d\Phi_{t,y}^{-1})_x(X), Y\rangle ,$$

which gives us the following formula:

$$(1.25) \qquad \det_g(d\Phi_{t,y}^{-1})_x=\det(\nabla_y\nabla_x S(t, x, y))/\sqrt{g(x)}\sqrt{g(y)} \qquad (\sqrt{g(x)}=\mu_g(x)).$$

If we put $\rho(t, x, y)=|\det_g(d\Phi_{t,y}^{-1})_x|^{1/2}$, we have easily $\rho(t, x, y)=t^{-d/2}\Theta^{-1/2}(t, x, y)$. Then, we have

PROPOSITION 1.12. *Assume* (M), (L. I)-(L. III) *and* (L. V). *The function* $\rho(t, x, y)$ *is smooth on* $(0, T_2)\times M\times M$ *and satisfies the following continuity equation:*

$$(1.26)_{\mathrm{C}} \quad \partial_t\rho(t, x, y)+(1/2)\rho(t, x, y)\Delta^{(x)}S(t, x, y)+\langle\nabla_x\rho(t, x, y), \nabla_x S(t, x, y)\rangle=0 .$$

Here, $\Delta^{(x)}$ *stands for the Laplace-Beltrami operator acting on a function of* x.

PROOF. Define $\zeta(t, x, y)=\det_g(d\Phi_{t,y}^{-1})_x$. Then, using the identity $\Phi_{t+s,y}^{-1}=\Phi_{t,y}^{-1}\circ\Phi_{t,y}\circ\Phi_{t+s,y}^{-1}$, we get

$$(d\Phi_{t+s}^{-1})_x=(d\Phi_{t,y}^{-1})_z\cdot(d(\Phi_{t,y}\circ\Phi_{t+s,y}^{-1}))_x , \qquad z=\Phi_{t,y}\circ\Phi_{t+s,y}^{-1}(x) .$$

This means

$$(\partial/\partial s)\zeta(t+s, x, y)=\langle\nabla_z\zeta(t, x, z), \partial_s z\rangle\cdot\det_g(d(\Phi_{t,y}\circ\Phi_{t+s,y}^{-1}))_x$$
$$+\zeta(t, x, y)(\partial/\partial s)\exp\operatorname{trace}\log(d(\Phi_{t,y}\circ\Phi_{t+s}^{-1}))_x .$$

As $\langle\nabla_z\zeta(t, x, y), \partial_s z\rangle_{|s=0}=-\langle\nabla_x\zeta, \nabla_x S(t, x, y)\rangle$ and

$$\partial_s\exp\operatorname{trace}\log(d(\Phi_{t,y}\circ\Phi_{t+s,y}^{-1}))_x=-\operatorname{trace}[\nabla_x^2 S(t, x, y)]$$
$$=-\Delta^{(x)}S(t, x, y),$$

we have

$$\partial_t\zeta(t, x, y)+\langle\nabla_x\zeta(t, x, y), \nabla_x S(t, x, y)\rangle_x+\zeta(t, x, y)\Delta^{(x)}S(t, x, y)=0 .$$

As $\rho(t, x, y)=\zeta(t, x, y)^{1/2}$, we have $(1.26)_{\mathrm{C}}$. q. e. d.

REMARK. As is mentioned before, $S(t, x, y)$ and $\rho(t, x, y)$ are symmetric in x and y. In the following, we use $(1.23)_{\mathrm{H.J}}$, and $(1.26)_{\mathrm{C}}$ with changing the role of x and y if necessary.

Following estimates are easily obtained by differentiating $(1.5)_{\mathrm{V.J}}$ and using the variation of constant for ordinary differential equations (see, Appendix in Maeda [**12**]).

LEMMA 1.13. *Let take $T>0$ smaller than T_2 and satisfying $\Theta(t, x, y)\geqq 1/2$ for any $x, y\in M$ and $0<t<T$. Then, under assumptions* (M) *and* (L. I)-(L. V), *there exists a positive constant C_3 independent of $x, y\in M$ and $0<t<T$ such that for $0\leqq|\alpha|\leqq 3$ and $x=\Phi_{t,y}(Y)$,*

$$|\nabla_y^\alpha\Theta(t, x, y)|\leqq C_3\exp(k|Y|+C_2)t\,, \tag{1.27}$$

$$|\nabla_x^\alpha\rho(t, x, y)|\leqq C_3t^{-d/2}\exp(k|Y|+C_2)t\,. \tag{1.28}$$

2. Some basic properties of $H_t^\lambda(L\,;\mu_g)$.

In this and in §§3 and 4, we denote $H_t^\lambda(L, \mu_g)$ simply by H_t^λ and L^2-norm of $L^2(M, d\mu_g)$ by $\|\cdot\|$. Hereafter, we fix $T>0$ as defined in Lemma 1.13.

PROPOSITION 2.1. *Assume* (M), (L. I)-(L. III) *and* (L. IV). *Then, the operator H_t^λ are stable in $L^2(M, d\mu_g)$. That is, there exists a positive constant $C_4=C_4(\lambda, T)$ such that*

$$\|H_t^\lambda f\|\leqq\exp C_4t\cdot\|f\| \qquad \text{for} \quad 0<t<T \quad \text{and} \quad f\in C_0^\infty(M). \tag{2.1}$$

PROOF. H_t^λ is an integral operator with kernel

$$h^\lambda(t, x, y)=(2\pi\lambda)^{-d/2}\rho(t, x, y)\exp\{-\lambda^{-1}S(t, x, y)\}. \tag{2.2}$$

We claim that there exist constants $C_1'=C_1'(\lambda\,;T)$ and $C_4'=C_4'(\lambda\,;T)$ such that

$$\int_M h^\lambda(t, x, y)d\mu_g(y)\leqq(1+C_4't)\cdot\exp C_1't/\lambda\,, \tag{2.3}$$

$$\int_M h^\lambda(t, x, y)d\mu_g(x)\leqq(1+C_4't)\cdot\exp C_1't/\lambda\,, \tag{2.3$'$}$$

for all $0<t<T$.

Putting $y=\Phi_{t,x}(X)$, $X\in T_xM$, and remarking $\rho(t, x, y)=t^{-d/2}\Theta(t, x, y)^{-1/2}$, we have

$$\begin{aligned}\int_M & h^\lambda(t, x, y)d\mu_g(y) \\ &=(2\pi\lambda)^{-d/2}t^{-d/2}\int_{T_xM}\exp\{-\lambda^{-1}S(t, x, \Phi_{t,x}(X))\}\cdot\Theta(t, x, \Phi_{t,x}(X))^{1/2}dX\,.\end{aligned} \tag{2.4}$$

Inserting the relation above and (1.21) and using polar coordinate, we have

$$\begin{aligned}&\int_M h^\lambda(t, x, y)d\mu_g(y) \\ &\leqq(2\pi\lambda)^{-d/2}t^{d/2}\mathrm{vol}(S^{d-l}) \\ &\quad\times\int_0^\infty r^{d-1}[\sinh(kr+C_2)t/(kr+C_2)t]^{d/2}\exp[(-r^2t+6C_1t+C_1rt^2)/2\lambda]dr\,.\end{aligned}$$

Using

$$(-r^2+C_1rt^2+6C_1t)/2\lambda=-r^2t/4\lambda+((-r^2t/2)+C_1rt^2+6C_1t)/2\lambda$$
$$\leqq -r^2t/4\lambda+C_1't/\lambda, \qquad \text{for } 0<t<T \text{ and for } r\geqq 0,$$

where $C_1'=3C_1+(C_1^2T^2/4)$, we have

$$\int_M h^\lambda(t, x, y)d\mu_g(y)\leqq \exp(C_1't/\lambda)\cdot F_\lambda(t).$$

Here, we set

$$F_\lambda(t)=(2\pi)^{-d/2}\mathrm{vol}(S^{d-1})\int_0^\infty u^{d-1}[\cosh(ku(\lambda t)^{1/2}+C_2t)]^{d/2}\exp(-u^2/4)du.$$

Using $\sinh r/r\leqq\cosh r$ for $r>0$, we get

$$\begin{aligned} F_\lambda'(t)&=\frac{d}{dt}F_\lambda(t)\\ &\leqq(d/4)(2\pi)^{-d/2}\mathrm{vol}(S^{d-1})\\ &\quad\times\int_0^\infty u^{d-1}\cosh(ku(\lambda t)^{1/2}+C_2t)^{d/2}\\ &\qquad\times[k^2u^2\lambda+3C_2ku(\lambda t)^{1/2}+2C_2^2t]\exp(-u^2/4)du, \end{aligned}$$

there exists a positive constant $C_4'=C_4'(\lambda;T)$ which bounds from above the right-hand side of the inequality above. Combining this with $F_\lambda(0)=1$, we get (2.3). Analogously, we get (2.3)$'$.

Remarking

$$|(H_t^\lambda f)(x)|\leqq\int_M h^\lambda(t, x, y)^{1/2}h^\lambda(t, x, y)^{1/2}|f(y)|d\mu_g(y),$$

and using Schwarz' inequality, we have

$$\|H_t^\lambda f\|^2\leqq\exp(2C_1't/\lambda)\cdot(1+C_4't)^2\|f\|^2.$$

This implies (2.1) by putting $C_4=C_4'+(C_1'T/\lambda)$. q.e.d.

By Proposition 2.1, we may extend H_t^λ acting on $L^2(M, d\mu_g)$. Now, we study the dependence of H_t^λ on t.

PROPOSITION 2.2. *Assume that* (M), (L.I)-(L.III) *and* (L.V) *hold. For* $f\in L^2(M, d\mu_g)$, *we have*

$$\lim_{t\to+0}\|H_t^\lambda f-f\|=0. \tag{2.5}$$

Therefore, putting H_0^λ=*the identity operator, we have the mapping from* $t\in[0, T]$ *to* $H_t^\lambda f\in L^2(M, d\mu_g)$, *strongly continuous in* t *for each* $f\in L^2(M, d\mu_g)$.

PROOF. By Proposition 2.1, it is sufficient to prove (2.6) for each $f\in C_0^\infty(M)$. We take a smooth function $\chi(x)$, $0\leqq\chi(x)\leqq 1$, as

$$\chi(x)=\begin{cases}1 & \text{if} \quad d(x, \operatorname{supp} f)\leqq 2\\ 0 & \text{if} \quad d(x, \operatorname{supp} f)>3.\end{cases}$$

We shall show the following.

$$\lim_{t\to+0}\|H_1(t, \cdot)-f(\cdot)\|=0, \tag{2.6}$$

$$\lim_{t\to+0}\|H_2(t, \cdot)\|=0, \tag{2.7}$$

where $H_1(t, x)=\chi(x)(H_t^\lambda f)(x)$ and $H_2(t, x)=(1-\chi(x))(H_t^\lambda f)(x)$.

Proof of (2.6). Putting $y=\Phi_{t,x}(X)=\gamma(t, x, X)$, $X\in T_x(M)$, we get

$$f(y)=f(x)+f_1(x; t, X) \quad \text{and} \quad \Theta(t, x, y)^{1/2}=1+\Theta_1(x, t, X), \tag{2.8}$$

where

$$f_1(x, t, X)=\int_0^t (d/ds)f(\Phi_{s,x}(X))ds=\int_0^t \langle\nabla_y f(\Phi_{s,x}(X)), \dot\gamma(s, x, X)\rangle ds,$$

$$\begin{aligned}\Theta_1(x; t, X)&=\int_0^s (d/ds)\Theta(t, x, \Phi_{s,x}(X))^{1/2}ds\\ &=(1/2)\int_0^t \Theta(t, x, \Phi_{s,x}(X))^{-1/2}\langle\nabla_y\Theta(t, x, \Phi_{s,x}(X)), \dot\gamma(s, x, X)\rangle ds.\end{aligned}$$

So, using polar coordinate $X=r\omega$, $r\in(0, \infty)$, $\omega\in S^{d-1}$, we have

$$\begin{aligned}H_1(t, x)=\chi(x)(2\pi\lambda)^{-d/2}t^{d/2}\int_0^\infty\int_{S^{d-1}}(f(x)+H_1(x, t, r\omega))\\ \times\exp\{-\lambda^{-1}S(t, x, \Phi_{t,x}(r\omega))\}r^{d-1}drd\omega,\end{aligned} \tag{2.9}$$

where $H_1(x, t, r\omega)=f_1(x, t, r\omega)+f(\Phi_{t,x}(r\omega))\Theta_1(x, t, r\omega)$. Using the estimates (1.8) and (1.27), we have readily, for some constant $C_5=C_5(T)$,

$$|H_1(x; t, r\omega)|\leqq[\sup_{x\in M}|\nabla f|+C_5\sup_{x\in M}|f|\cdot\exp kr][rt+(C_1t^2/2)]. \tag{2.10}$$

Using these estimates and remarking that $\chi(x)$ has compact support, there exists a constant $C_6=C_6(T, \lambda, \chi)$ such that

$$\|H_1(t, \cdot)-f(\cdot)\|\leqq C_6t^{1/2}[\sup_{x\in M}|\nabla f|+\sup_{x\in M}|f|]. \tag{2.11}$$

Proof of (2.7). Choose another function $\varphi(y)\in C_0^\infty(M)$, $0\leqq\varphi\leqq 1$ satisfying

$$\varphi(y)=\begin{cases}1 & \text{if} \quad d(y, \operatorname{supp} f)\leqq 1\\ 0 & \text{if} \quad d(y, \operatorname{supp} f)\geqq 3/2.\end{cases}$$

Remarking $\varphi(y)f(y)=f(y)$, we have

$$H_2(t, x)=\int_M F_\lambda(t, x, y)f(y)d\mu_g(y), \tag{2.12}$$

where

$$F_\lambda(t,\ x,\ y)=(2\pi\lambda)^{-d/2}(1-\chi(x))\varphi(y)\rho(t,\ x,\ y)\exp\{-\lambda^{-1}S(t,\ x,\ y)\}.$$

By the choice of $\chi(x)$ and $\varphi(y)$, we have $F_\lambda(t, x, y)=0$ for $d(x, y)\leqq 1/2$. On the other hand, by the choice of T,

$$\sup_{\substack{0<t<T \\ x,y\in M}} t^{d/2}\rho(t,\ x,\ y)\leqq\sqrt{2} \quad \text{and} \quad \sup_{\substack{0<t<T \\ X\geqq 1/2}}[t^{-d/2}\exp(-X^2/2t)]\leqq C(d)<\infty .$$

So, using (1.6) and (1.10), we get

$$F_\lambda(t,\ x,\ y)\leqq C_7(1-\chi(x))\varphi(y)\exp(-d^2(x,\ y)) \quad \text{for} \quad 0<t<T ,$$

where C_7 is a positive constant, independent of $x, y\in M$ and $0<t<T$. Moreover, as $\lim_{t\to 0}|F_\lambda(t, x, y)|=0$ for each $x, y\in M$, using Lebesgue's dominated convergence theorem and the argument at the end of the proof of Proposition 2.1, we get (2.7). q. e. d.

3. The convergence of the product H_t^λ in the operator norm.

Take $t>0$ arbitrarily. Dividing the interval $[0, t]$ into n-equal subintervals such that $(t/n)<T$, we define an operator $\tilde{H}_n^\lambda(t)$ as $\tilde{H}_n^\lambda(t)=H_{t/n}^\lambda\cdots H_{t/n}^\lambda$ (product of n-times).

PROPOSITION 3.1. *Assume that M and L satisfy* (M), (L. I)-(L. V). *Then, there exists a C^0-semi group $\boldsymbol{H}_t^\lambda(L;\mu_g)$ $(t\geqq 0)$ on $L^2(M, d\mu_g)$ such that, for any $t>0$,*

$$\|\boldsymbol{H}_t^\lambda(L;d\mu_g)-\tilde{H}_n^\lambda(t)\|\leqq(Ct^{3/2}n^{-1/2}+C't^2n^{-1})\cdot\exp C_4t , \tag{3.1}$$

where C and C' are positive constants depending on T, independent of n.

Proof of this proposition is composed of several lemmas.

LEMMA 3.2. *The function $h^\lambda(t, x, y)$ satisfies the following:*

$$(\lambda\partial_t-(\lambda^2/2)\Delta^{(x)})h^\lambda(t,\ x,\ y) \tag{3.2}$$
$$=-(2\pi\lambda)^{-d/2}[(\lambda^2/2)\Delta^{(x)}-V(x)]\rho(t,\ x,\ y)\cdot\exp\{-\lambda^{-1}S(t,\ x,\ y)\},$$

$$(\lambda\partial_\sigma+(\lambda^2/2)\Delta^{(y)})h^\lambda(t-\sigma,\ x,\ y) \tag{3.3}$$
$$=(2\pi\lambda)^{-d/2}[(\lambda^2/2)\Delta^{(y)}-V(y)]\rho(t-\sigma,\ x,\ y)\cdot\exp\{-\lambda^{-1}S(t-\sigma,\ x,\ y)\}.$$

PROOF. By the formula of $\Delta^{(y)}$ acting on the product of functions, we have

$$\Delta^{(y)}h^\lambda(t,\ x,\ y)=(2\pi\lambda)^{-d/2}[(\Delta^{(y)}\rho(t,\ x,\ y))\exp\{-\lambda^{-1}S(t,\ x,\ y)\} \tag{3.4}$$
$$-2\lambda^{-1}\langle\nabla_y\rho(t,\ x,\ y),\ \nabla_yS(t,\ x,\ y)\rangle\exp\{-\lambda^{-1}S(t,\ x,\ y)\}$$
$$+\rho(t,\ x,\ y)\Delta^{(y)}\exp\{-\lambda^{-1}S(t,\ x,\ y)\}].$$

Combining $(1.23)_{H.J}$ and $(1.26)_C$, we have easily (3.2). Analogously, we have (3.3). q. e. d.

For $f\in C_0^\infty(M)$ and t, $s>0$, $\lambda>0$, we may write

$$(3.5)\qquad (H_{t+s}^\lambda f)(x)-(H_t^\lambda H_s^\lambda f)(x)=\int_M \tilde{h}_\lambda(t, s; x, y)f(y)d\mu_g(y),$$

where

$$(3.6)\qquad \tilde{h}_\lambda(t, s; x, y)=h^\lambda(t+s; x, y)-\int_M h^\lambda(t, x, z)h^\lambda(s, z, y)d\mu_g(z).$$

Since $\tilde{h}^\lambda(t, s; x, y)$ has a singularity at $t=0$, we define, for any positive ε,

$$(3.7)\qquad \tilde{h}_\lambda^\varepsilon(t, s; x, y)=\int_\varepsilon^s\Big[(d/d\sigma)\int_M h^\lambda(t+s-\sigma, x, z)h^\lambda(\sigma, z, y)d\mu_g(z)\Big]d\sigma$$

which satisfies $\lim_{\varepsilon\to+0}\tilde{h}_\lambda^\varepsilon(t, s; x, y)=\tilde{h}_\lambda(t, s; x, y)$ for any (t, s, x, y), $t>s$. Exchanging $d/d\sigma$ and the integration, we have using Lemma 3.2 and integration by parts,

$$(3.8)\qquad \lambda\tilde{h}_\lambda^\varepsilon(t, s; x, y)=-(\lambda^2/2)(2\pi\lambda)^{-d}\int_\varepsilon^s\{(t+s-\sigma)^{-d/2}\sigma^{-d/2}$$
$$\times\int_M[\tilde{\rho}(\sigma, z, y)\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)-\tilde{\rho}(t+s-\sigma, x, z)\Delta^{(z)}\tilde{\rho}(\sigma, z, y)]$$
$$\times\exp\{-\lambda^{-1}(S(t+s-\sigma, x, z)+S(\sigma, z, y))\}d\mu_g(z)\}d\sigma,$$

where we put $\tilde{\rho}(t, x, y)=t^{d/2}\rho(t, x, y)$.

LEMMA 3.3. *Assume that* (M), (L. I)-(L. V) *hold. For* $T>0$ *defined in Lemma* 1.13, *there exist positive constants* C_8 *and* C_9 *depending on* λ *and* T *such that*

$$(3.9)\qquad \lim_{\varepsilon\to 0}\int_M\lambda|\tilde{h}_\lambda^\varepsilon(t, s; x, y)|d\mu_g(y)\leqq C_8\{(t+s)^{3/2}-t^{3/2}+s^{3/2}\}+C_9(t+s)s,$$
$$\lim_{\varepsilon\to 0}\int_M\lambda|\tilde{h}_\lambda^\varepsilon(t, s; x, y)|d\mu_g(x)\leqq C_8\{(t+s)^{3/2}-t^{3/2}+s^{3/2}\}+C_9(t+s)s.$$

PROOF. Remark at first that

$$(3.10)\qquad \tilde{\rho}(\sigma, z, y)\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)-\tilde{\rho}(t+s-\sigma, x, z)\Delta^{(z)}\tilde{\rho}(\sigma, z, y)$$
$$=\tilde{\rho}(\sigma, z, y)[\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)-\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)_{|z=x}]$$
$$+[\tilde{\rho}(\sigma, z, y)\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)_{|z=x}-\tilde{\rho}(t+s-\sigma, x, z)\Delta^{(z)}\tilde{\rho}(\sigma, z, y)_{|z=y}]$$
$$+\tilde{\rho}(t+s-\sigma, x, z)[\Delta^{(z)}\tilde{\rho}(\sigma, z, y)_{|z=y}-\Delta^{(z)}\tilde{\rho}(\sigma, z, y)].$$

Take the geodesic $\gamma_g(\tau)$ with $\gamma_g(0)=x$, $\gamma_g(r)=z$ where $r=d(x, z)$. Then, we have, using (1.28),

$$(3.11)\qquad \begin{aligned} &|\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)-\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)_{|z=x}| \\ &\leqq \int_0^{\tau}|\nabla_z\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, \gamma_g(\tau))|\,d\tau \\ &\leqq C_8\, d(x, z)\exp \tilde{k}d(x, z)\,, \end{aligned}$$

where C_8 and $\tilde{k}$ are positive constants independent of $x, z\in M$. Analogously

$$(3.12)\qquad |\Delta^{(z)}\tilde{\rho}(\sigma, z, y)-\Delta^{(z)}\tilde{\rho}(\sigma, z, y)_{|z=y}|\leqq C_8 d(y, z)\exp \tilde{k}d(y, z)\,.$$

In order to estimate the second term of the right-hand side of (3.10), we use

LEMMA 3.4. *$\Delta^{(z)}\tilde{\rho}(t, x, z)_{|z=x}=R(x)/6+tO(t, x)$, where the function $|O(t, x)|$ is bounded from above $C_9>0$ for any $0<t<T$ and for any $x\in M$.*

Retaining the proof of the above lemma later, we have, by assumption (L. III),

$$(3.13)\qquad \begin{aligned} &|\tilde{\rho}(\sigma, z, y)\Delta^{(z)}\tilde{\rho}(t+s-\sigma, x, z)_{|z=x}-\tilde{\rho}(t+s-\sigma, x, z)\Delta^{(z)}\tilde{\rho}(\sigma, z, y)_{|z=y}| \\ &=|\{(\tilde{\rho}(\sigma, z, y)-1)R(x)/6\}+\{(R(x)-R(y))/6\} \\ &\quad+\{(1-\tilde{\rho}(t+s-\sigma, x, z))R(y)/6\} \\ &\quad+\{(t+s-\sigma)\tilde{\rho}(\sigma, z, y)O(t+s-\sigma, x)-\sigma\tilde{\rho}(t+s-\sigma, x, z)O(\sigma, y)\}| \\ &\leqq C_8[d(y, z)\exp \tilde{k}d(y, z)+d(x, y)+d(x, z)\exp \tilde{k}d(x, z)]+C_9(t+s)\,, \end{aligned}$$

where C_9 is a positive constant independent of $x\in M$.

Inserting these estimate in (3.8), we get

$$(3.14)\qquad \int_M\lambda|\tilde{h}_\lambda^{\varepsilon}(t, s, x, y)|\,d\mu_g(y)\leqq(-\lambda^2/2)\int_{\varepsilon}^{s}[L_1(\sigma)+L_2(\sigma)+L_3(\sigma)]\,d\sigma\,,$$

where

$$(3.15)\qquad \begin{aligned} L_1(\sigma)=&C_8(2\pi\lambda(t+s-\sigma))^{-d/2}(2\pi\lambda\sigma)^{-d/2}\int_M\int_M[d(x, z)+d(x, z)\exp \tilde{k}d(x, z)] \\ &\times\exp\{-\lambda^{-1}(S(t+s-\sigma, x, z)+S(\sigma, z, y))\}\,d\mu_g(z)d\mu_g(y)\,, \end{aligned}$$

$$(3.16)\qquad \begin{aligned} L_2(\sigma)=&C_8(2\pi\lambda(t+s-\sigma))^{-d/2}(2\pi\lambda\sigma)^{-d/2}\int_M\int_M[d(y, z)+d(y, z)\exp \tilde{k}d(y, z)] \\ &\times\exp\{-\lambda^{-1}(S(t+s-\sigma, x, z)+S(\sigma, z, y))\}\,d\mu_g(z)d\mu_g(y) \end{aligned}$$

and

$$(3.17)\qquad \begin{aligned} L_3(\sigma)=&C_9(2\pi\lambda(t+s-\sigma))^{-d/2}(2\pi\lambda\sigma)^{-d/2}(t+s) \\ &\times\int_M\int_M\exp\{-\lambda^{-1}(S(t+s-\sigma, x, z)+S(\sigma, z, y))\}\,d\mu_g(z)d\mu_g(y)\,. \end{aligned}$$

Using (1.8), we have

$$L_1(\sigma)\leqq(2\pi\lambda(t+s-\sigma))^{-d/2}(2\pi\lambda\sigma)^{-d/2}\exp C_1(t+s)/\lambda \times\int_M[d(x, z)+d(x, z)\exp \tilde{k}d(x, z)]\exp(-d^2(x, z)/2\lambda(t+s-\sigma)) \times\left\{\int_M\exp(-d^2(z, y)/2\lambda\sigma)\cdot d\mu_g(y)\right\}d\mu_g(z)\,. \tag{3.18}$$

Then, calculating the integral in $\{\cdots\}$, using the normal coordinate at z and then using the normal coordinate at x, we get readily

$$L_1(\sigma)\leqq C_8(t+s-\sigma)^{1/2} \tag{3.19}$$

for $0<\sigma$, $t+s-\sigma<T$ with some constant C_8 depending on λ, T. Analogously, we have

$$L_2(\sigma)\leqq C_8\sigma^{1/2}. \tag{3.20}$$

Lastly,

$$L_3(\sigma)\leqq C_9(t+s)\,. \tag{3.21}$$

Substituting these into (3.14), we get

$$\int_M\lambda|\tilde{h}_\lambda^\xi(t, s, x, y)|d\mu_g(y)\leqq C_8((t+s)^{3/2}-t^{3/2}+s^{3/2})+C_9(t+s)s\,. \tag{3.22}$$

PROOF OF LEMMA 3.4. In order to calculate this, we take special bases in defining $\Theta(t, x, y)=t^{-d}\det_g(d\Phi_{t,x})_X$ for $\mathrm{Exp}_xX=y$. That is, taking the normal coordinate $(X^1, \cdots, X^d)$ at x, we have

$$\tilde{\rho}(t, x, y)=[\det(A_i^\alpha(t, x))/\sqrt{\det(g_{ij}(X))}]^{1/2}, \tag{3.23}$$

where $A_i^\alpha(t, X)$ is the component of $(d\Phi_{t,x})_X$ with respect to $(X^1, \cdots, X^d)$. As $A_i^\alpha(0, X)=\delta_i^\alpha$, putting the remainder term of the Taylor expansion of $\det(A_i^\alpha(t, X))$ at $t=0$ as $O(t, x, X)$, we get

$$\Delta^{(z)}\tilde{\rho}(t, x, y)=\Delta^{(X)}(\det(g_{ij}(X)))^{-1/2}+tO(t, x, X)\,. \tag{3.24}$$

On the other hand,

$$\Delta^{(X)}(\det(g_{ij}(X)))^{-1/2}|_{X=0}=-R(x)/6\,. \tag{3.25}$$

(See p. 593 of Sakai [13] and p. 97 of Berger et al. [2].) Also, as $\tilde{\rho}(t, x, y)=t^{d/2}\rho(t, x, y)$, the boundedness of $O(t, x)=O(t, x, X)|_{X=0}$ is easily obtained by Lemma 1.14. q. e. d.

For fixed t, s and $x\in M$, we have

$$(3.26)\qquad |\tilde{h}_\lambda^\varepsilon(t,\ s,\ x,\ y)|\leqq\int_0^s\{(2\pi(t+s-\sigma))^{-d/2}(2\pi\sigma)^{-d/2}$$
$$\times\int_M|\tilde{\rho}(\sigma,\ z,\ y)\Delta^{(z)}\tilde{\rho}(t+s-\sigma,\ x,\ z)-\tilde{\rho}(t+s-\sigma,\ x,\ z)\Delta^{(z)}\tilde{\rho}(\sigma,\ z,\ y)|$$
$$\times\exp\{-\lambda^{-1}(S(t+s-\sigma,\ x,\ z)+S(\sigma,\ z,\ y))\}\cdot d\mu_g(z)d\sigma\,.$$

By Fubini's theorem, we see that the right-hand side of the above inequality is L^1 function with respect to y-variables. Thus, using Lebesgue's dominated convergence theorem and $\lim_{\varepsilon\to 0}\tilde{h}_\lambda^\varepsilon(t,\ s,\ x,\ y)=\tilde{h}_\lambda(t,\ s,\ x,\ y)$ a.e., wet get

$$(3.27)\qquad \int_M\lambda|\tilde{h}_\lambda(t,\ s,\ x,\ y)|d\mu_g(y)\leqq C_8((t+s)^{3/2}-t^{3/2}+s^{3/2})+C_9(t+s)s\,,$$

because of Lemma 3.3. By the same argument, we have

$$(3.28)\qquad \int_M\lambda|\tilde{h}_\lambda(t,\ s,\ x,\ y)|d\mu_g(y)\leqq C_8((t+s)^{3/2}-t^{3/2}+s^{3/2})+C_9(t+s)s\,.$$

So, by the same computation as in the proof of Proposition 2.1, we have, by putting $C=C_8$, $C'=C_9$,

LEMMA 3.5. *Under assumptions* (M) *and* (L. I)-(L. V), *for any* $0<t,\ s,\ t+s<T$ *and any* $f\in L^2(M,\ d\mu_g)$, *we have*

$$(3.29)\qquad \|H_{t+s}^\lambda f-H_t^\lambda H_s^\lambda f\|\leqq[C((t+s)^{3/2}-t^{3/2}+s^{3/2})+C'(t+s)s]\|f\|\,.$$

To prove Proposition 3.1, we prepare

LEMMA 3.6. $\{\tilde{H}_n^\lambda(t)\}_n$ *forms a Cauchy sequence in* $\mathcal{B}(L^2(M,\ d\mu_g))$ *in operator norm, uniformly in t on any finite interval, where* $\mathcal{B}(L^2(M,\ d\mu_g))$ *denotes the space of bounded linear operators in* $L^2(M,\ d\mu_g)$ *with the operator norm. Moreover, its limit* $\boldsymbol{H}_t^\lambda$ *satisfies the estimate* (3.1).

PROOF. As we have, for $s\in[0,\ T]$,

$$H_s^\lambda-(H_{s/n}^\lambda)^n=\sum_{j=0}^{n-2}[H_{(n-j)s/n}^\lambda-H_{(n-j-1)s/n}^\lambda\ H_{s/n}^\lambda]\cdot(H_{s/n}^\lambda)^j,$$

we get

$$(3.30)\qquad \|(H_s^\lambda-(H_{s/n}^\lambda)^n)f\|\leqq\exp C_4s\cdot[C(s^{3/2}+(n-1)(s/n)^{3/2})+C's^2]\|f\|\,,$$

by (3.29) and (2.1). Using (3.30), we have

$$(3.31)\qquad \|(H_{t/n}^\lambda)^n f-(H_{t/nm}^\lambda)^{nm}f\|$$
$$\leqq\sum_{j=0}^{n-1}\|((H_{t/n}^\lambda)^{n-j-1}\cdot H_{t/n}^\lambda\cdot(H_{t/nm}^\lambda)^{jm}-(H_{t/n}^\lambda)^{n-j-1}\cdot(H_{t/nm}^\lambda)^{(j+1)m})f\|$$
$$\leqq\sum_{j=0}^{n-1}\exp\{C_4(n-j-1)(t/n)\}\cdot\|(H_{t/n}^\lambda\cdot(H_{t/nm}^\lambda)^{jm}-(H_{t/nm}^\lambda)^{(j+1)m}f)\|$$

$$\leqq \exp C_4 t\cdot\left[C\sum_{j=0}^{n-1}\{(t/n)^{3/2}+(m-1)(t/nm)^{3/2}\}+C'\sum_{j=0}^{n-1}(t/m)^2\right]$$

$$\leqq \exp C_4 t\cdot[Ct^{3/2}(n^{-1/2}+(nm)^{-1/2})+C't^2n^{-1}]\|f\|.$$

Therefore, we get

$$\begin{aligned}(3.32)\qquad &\|((H^\lambda_{t/n})^n-(H^\lambda_{t/m})^m)f\|\\ &\leqq\|((H^\lambda_{t/nm})^n-(H^\lambda_{t/nm})^{nm})f\|+\|((H^\lambda_{t/nm})^{nm}-(H^\lambda_{t/m})^m)f\|\\ &\leqq \exp C_4 t\cdot[Ct^{3/2}(n^{-1/2}+m^{-1/2}+2(nm)^{-1/2})+C't^2(n^{-1}+m^{-1})].\end{aligned}$$

Thus, $\tilde{H}^\lambda_n(t)=(H^\lambda_{t/n})^n$ is a Cauchy sequence uniformly in t on any finite interval in the operator norm. Therefore, it converges to a limit $\boldsymbol{H}^\lambda_t$. Letting m tend to ∞ in (3.32), we get (3.1). q. e. d.

REMARK. The above proof is a slight modification of the proof of Theorem 5.3, p. 240 of Chorin et al. [5]. This simplifies greatly the proof of Lemma 5.7, p. 79 of Fujiwara [8] which seems rather difficult to follow.

Now, we generalize Proposition 3.1 a little bit:

PROPOSITION 3.7. *Let us assume that M and L satisfy* (M) *and* (L. I)-(L. IV). *Take $T>0$ as in Lemma* 1.13. *Assume also that two subintervals*

$$\Delta_1:\ 0=t_0<t_1<\cdots<t_n=t,\qquad \delta(\Delta_1)=\max_j|t_j-t_{j-1}|,$$

and

$$\Delta_2:\ 0=s_0<s_1<\cdots<s_m=t,\qquad \delta(\Delta_2)=\max_j|s_j-s_{j-1}|,$$

are given as $\delta(\Delta_1)$ and $\delta(\Delta_2)$ are smaller than T.

Define $H(\Delta_1:t)=H^\lambda_{t_n-t_{n-1}}\cdots H^\lambda_{t_1}$, and $H(\Delta_2:t)=H^\lambda_{s_m-s_{m-1}}\cdots H^\lambda_{s_1}$. Then, we have

$$\begin{aligned}(3.33)\qquad &\|H(\Delta_1:t)-H(\Delta_2:t)\|\\ &\leqq \exp C_4 t\cdot[Ct(\delta(\Delta_1)^{1/2}+\delta(\Delta_2)^{1/2})+2t^{1/2}\delta(\Delta_1)^{1/2}\delta(\Delta_2)^{1/2}+C't(\delta(\Delta_1)+\delta(\Delta_2))],\end{aligned}$$

$$(3.34)\qquad \|H(\Delta_1:t)-\boldsymbol{H}^\lambda_t\|\leqq t\cdot\exp C_4 t\cdot(C\delta(\Delta_1)^{1/2}+C'\delta(\Delta_1)).$$

PROOF. Following the proof of Lemma 5.8, p. 81 of [8], we get the result.

By the above argument, we obtain (d) of Theorem in case of $\mu=\mu_g$.

4. Computation of the infinitesimal generator of $\boldsymbol{H}^\lambda_t$.

Our object in this section is to prove:

PROPOSITION 4.1. *Assume* (M) *and* (L. I)-(L. V). *Then, for any $f\in C_0^\infty(M)$,*

$$(4.1)\qquad \lambda\partial_t(\boldsymbol{H}^\lambda_t f)(x)_{|t=0}=\lambda^2[\Delta_g/2-R(x)/12]f(x)+V(x)f(x).$$

This follows from:

LEMMA 4.2. *Under the same assumptions as above, we have*

(i) $\partial_t(\boldsymbol{H}_t^\lambda f)(x)_{|t=0}=\partial_t(H_t^\lambda f)(x)_{|t=0}$ *for* $f\in C_0^\infty(M)$.

(ii) $(H_t^\lambda f)(x)-f(x)=t(\boldsymbol{A}^\lambda f)(x)+tG(t, \lambda, f)(x)$ *for* $f\in C_0^\infty(M)$,

and $\lim_{t\to 0}\|G(t, \lambda, f)\|=0$.

PROOF. For each n, we have

$$(\boldsymbol{H}_t^\lambda f)(x)-f(x)=(\boldsymbol{H}_t^\lambda f)(x)-(\tilde{H}_n^\lambda(t)f)(x)+\sum_{j=1}^n[(H_{j/n}^\lambda)^{n-j}(H_{t/n}^\lambda-I)f](x).$$

Dividing both sides of the above by t, taking n sufficiently large and making t tend to 0, we get (i). Here we used the estimate in Proposition 3.1.

Proof of (ii). Combining $(1.23)_{\mathrm{H.J}}$ and $(1.26)_{\mathrm{C}}$ with the definition of $h^\lambda(t, x, y)$, we get readily

$$\begin{aligned}\lambda\partial_t(H_t^\lambda f)(x)&=\lambda(2\pi\lambda)^{-d/2}\int_M[\partial_t\rho(t, x, y)-\lambda^{-1}\rho(t, x, y)\partial_t S(t, x, y)]\\&\qquad\times\exp\{-\lambda^{-1}S(t, x, y)\}\cdot f(y)d\mu_g(y)\\&=\{H_t^\lambda[((\lambda^2/2)\Delta+V)f]\}(x)\\&\quad-(\lambda^2/2)(2\pi\lambda)^{-d/2}\int_M\Delta^{(y)}\rho(t, x, y)\exp\{-\lambda^{-1}S(t, x, y)\}\cdot f(y)d\mu_g(y).\end{aligned}$$

Therefore, by Lemma 3.4,

$$\begin{aligned}&\lambda\partial_t(H_t^\lambda f)(x)-\lambda^2(\Delta/2-R(x)/12)f(x)-V(x)f(x)\\&=[(H_t^\lambda-I)((\lambda^2/2)\Delta+V)f](x)\\&\quad-(\lambda^2/2)(2\pi\lambda t)^{-d/2}\int_M(\Delta^{(y)}\rho(t, x, y)-R(x)/12)\\&\qquad\times\exp\{-\lambda^{-1}S(t, x, y)\}f(y)d\mu_g(y)\\&=\tilde{G}(t;\lambda, f)(x).\end{aligned}\tag{4.2}$$

Using Proposition 2.2, we have

$$\|(H_t^\lambda-I)((\lambda^2/2)\Delta+V)f\|\to 0 \qquad \text{for} \quad f\in C_0^\infty(M).$$

Also, by Lemma 3.4 and similar computations as in proving Lemma 3.3, we get

$$\begin{aligned}&\lim_{t\to 0}\|(2\pi\lambda t)^{-d/2}\int_M(\Delta^{(y)}\rho(t, \cdot, y)-R(\cdot)/12)\\&\qquad\times\exp\{-\lambda^{-1}S(t, \cdot, y)\}f(y)d\mu_g(y)\|=0, \qquad \text{for} \quad f\in C_0^\infty(M).\end{aligned}$$

Remarking $(H_t^\lambda f)(x)-f(x)=\int_0^t\partial_\sigma(H_\sigma^\lambda f)(x)d\sigma$ and putting $G(t, \lambda, f)(x)=(1/t)\int_0^t\tilde{G}(\sigma, \lambda, f)(x)d\sigma$, we obtain the desired result. q.e.d.

Now, we have proved (a)-(d) and (f) of Theorem for $\mu=\mu_g$. As corollaries of the arguments in proving parts of Theorem, we have

COROLLARY 4.3. *There exists a distribution kernel* $\boldsymbol{H}^\lambda(L, \mu_g)(t, x, y)$ *of* $\boldsymbol{H}_t^\lambda(L, \mu_g)$. *That is, for any* $f, h \in C_0^\infty(M)$, *we have*

$$(\boldsymbol{H}_t^\lambda(L, \mu_g)f, h)=\langle \boldsymbol{H}^\lambda(L, \mu_g)(t, x, y), h(x)\otimes f(y)\rangle \tag{4.3}$$

where $\langle , \rangle$ *stands for the duality between* $\mathcal{D}(M\times M)$ *and* $\mathcal{D}'(M\times M)$.

This follows by applying Schwartz' kernel theorem to the left-hand side of (4.3).

PROPOSITION 4.4. *Let* M *and* $\tilde{M}$ *be smooth manifolds diffeomorphic to each other. (We denote the diffeomorphism from* $\tilde{M}$ *onto* M *by* Φ.) *On* M, *there exists a Lagrangian* L *satisfying* (L. I)-(L. V) *also with* (M). *We induce the Lagrangian and the Riemannian metric on* $\tilde{M}$ *by* $\tilde{L}=\Phi^*L$ *and* $\tilde{g}=\Phi^*g$, *where* Φ^* *denotes the pull-back by* Φ. *Then, we have*

$$(\Phi^*)^{-1}\boldsymbol{H}_t^\lambda(\tilde{L}\,;\mu_{\tilde{g}})\Phi^*=\boldsymbol{H}_t^\lambda(L\,;\mu_g)\,. \tag{4.4}$$

In other words,

$$\boldsymbol{H}^\lambda(\tilde{L}, \mu_{\tilde{g}})(t, \Phi^{-1}(x), \Phi^{-1}(y))=\boldsymbol{H}^\lambda(L, \mu_g)(t, x, y) \qquad \textit{for} \quad x, y\in M. \tag{4.4'}$$

PROOF. By the definition of $H_t^\lambda(L, \mu_g)$, we have readily

$$(H_t^\lambda(L\,;\mu_g)f)(x)=[H_t^\lambda(\tilde{L}\,;\mu_{\tilde{g}})(\Phi^*f)](\Phi^{-1}(x)) \qquad \text{for} \quad x\in M,\ f\in C_0^\infty(M)\,.$$

Moreover, as (M, g) is isometric to $(\tilde{M}, \tilde{g})$, we get the assertion. q. e. d.

COROLLARY 4.5. *The differential operator* $\boldsymbol{A}^\lambda(L\,;\mu_g)$ *defines a self-adjoint operator in* $L^2(M, d\mu_g)$ *if* M *and* L *satisfy* (M) *and* (L. I)-(L. V). *Moreover, we may define* $\{U_\tau\}_{\tau\in R}$, *a* C^0-*group of unitary operators on* $L^2(M, d\mu_g)$, *as*

$$U_\tau f=\text{s-}\lim_{\varepsilon\to 0} \boldsymbol{H}_{\varepsilon+i\tau}f \qquad \textit{for} \quad f\in C_0^\infty(M)\,. \tag{4.5}$$

PROOF. By the symmetry in x and y of kernel $h^\lambda(t, x, y)$, $(H_t^\lambda(L\,;\mu_g))^*=H_t^\lambda(L, \mu_g)$. This gives $(\boldsymbol{H}_t^\lambda(L\,;\mu_g))^*=\boldsymbol{H}_t^\lambda(L\,;\mu_g)$ which asserts $(\boldsymbol{A}^\lambda(L\,;\mu_g))^*=\boldsymbol{A}^\lambda(L\,;\mu_g)$. Moreover $\boldsymbol{A}^\lambda(L\,;\mu_g)$ is bounded from below, using Theorem 7.9.1 of Hille-Phillips [**10**], we have the desired assertion. q. e. d.

5. Proof of Theorem and its interpretation.

We first remember the following definition.

DEFINITION (p. 427 of [**1**]). (i) Two measures on M are said to be *equivalent* provided that each is absolutely continuous with respect to the other.

(ii) A measure on M is called *natural* if it is equivalent to the Lebesgue measure in every coordinate chart of M.

(iii) Consider the set of all pairs (f, μ), where μ is a natural measure on M and $f\in L^2(M, d\mu)$. Two pairs (f, μ) and (g, ν) will be called *equivalent* provided that $f(d\mu/d\nu)^{1/2}=g$, where $d\mu/d\nu$ is the Radon-Nikodym derivative of μ w.r.t. ν. We denote the equivalence class of (f, μ) by $f\sqrt{d\mu}$.

(iv) An equivalence class $f\sqrt{d\mu}$ is called a *half-density* on M and the set of all half-densities is regarded as sections of 1-dimensional vector bundle $\Lambda^{1/2}(M)$, called the half density bundle on M, which may be trivialized by choosing a natural measure μ on M.

(v) We denote $\mathcal{H}(M)$ *the intrinsic Hilbert space* on M, by the set of all such equivalence class (f, μ). As $\mathcal{H}(M)$ is trivialized by choosing a natural measure μ on M, $\mathcal{H}(M)$ is isomorphic to $L^2(M, d\mu)$ by the isomorphism $U_\mu : f\in L^2(M, d\mu) \to U_\mu f=f\sqrt{d\mu}\in\mathcal{H}(M)$. So, $U_{\mu\nu}$ defined by (8) is represented by

$$U_{\mu\nu}=U_\mu U_\nu^{-1}. \tag{5.1}$$

Moreover, $U_{\mu\nu}$ is obtained by the transition function of the half-density bundle $\Lambda^{1/2}(M)$.

PROOF OF THEOREM. For $f\in C_0^\infty(M)$, we rewrite $H_t^\lambda(L;\mu)$ by

$$\begin{aligned}(5.2)\quad &(H_t^\lambda(L, \mu)f)(x)\\ &=(2\pi\lambda)^{-d/2}\int_M[\det[-\partial_{x^i}\partial_{y^\alpha}S(L)(t, x, y)]/\sqrt{g(x)}\sqrt{g(y)}]^{1/2}\\ &\quad\times\exp\{-\lambda^{-1}S(L)(t, x, y)\}(\sqrt{g(x)}/\mu(x))^{1/2}(\sqrt{g(y)}/\mu(y))^{1/2}f(y)d\mu(y)\\ &=(\sqrt{g(x)}/\mu(x))^{1/2}(2\pi\lambda)^{-d/2}\int_M\rho(L;\mu_g)(t, x, y)\\ &\quad\times\exp\{-\lambda^{-1}S(L)(t, x, y)\}(\mu(y)/\sqrt{g(y)})^{1/2}f(y)d\mu_g(y).\end{aligned}$$

As $\|f\|_\mu=\|f(\cdot)(\mu(\cdot)/\sqrt{g(\cdot)})^{1/2}\|_{\mu_g}$ for $f\in L^2(M, d\mu)$, we get

$$\|H_t^\lambda(L;\mu)f\|_\mu=\|H_t^\lambda(L;\mu_g)(f(\cdot)(\mu(\cdot)/\sqrt{g(\cdot)})^{1/2})\|_{\mu_g}.$$

So, the isomorphism $U_{\mu_g\mu}: L^2(M, d\mu)\to L^2(M, d\mu_g)$ given by

$$(U_{\mu_g\mu}f)(x)=f(x)(\mu(x)/\sqrt{g(x)})^{1/2}\qquad\text{for}\quad f\in L^2(M, d\mu)$$

leads us to

$$H_t^\lambda(L;\mu)f=U_{\mu_g\mu}^{-1}H_t^\lambda(L;\mu_g)U_{\mu_g\mu}. \tag{5.3}$$

Since $(H_{t/n}^\lambda(L;\mu))^n=U_{\mu_g\mu}^{-1}(H_{t/n}^\lambda(L;\mu_g))^nU_{\mu_g\mu}$, we get $\boldsymbol{H}_t^\lambda(L;\mu)=U_{\mu_g\mu}^{-1}\boldsymbol{H}_t^\lambda(L;\mu_g)U_{\mu_g\mu}$. The end of the proof of Theorem.

REMARK 1. By putting $C_{t,x,y}=\{\gamma(\cdot)\in C([0, t]\to M) : \gamma(0)=y, \gamma(t)=x\}$, we propose to regard Feynman's expression (Feynman [7])

$$\int_{C_{t,x,y}} \exp\left\{-\lambda^{-1}\int_0^t L(\gamma(\tau), \dot\gamma(\tau))d\tau\right\} d_F(\gamma) \tag{5.4}$$

as it stands for the 'distribution kernel' of the operator $\boldsymbol{H}_t^\lambda(L)$ on $\mathcal{H}(M)$ which is represented concretely by using the trivialization of the bundle $\Lambda^{1/2}(M)$. That is, for any natural measure μ on M, we define (5.4) as

$$\begin{aligned}&\left\{\int_{C_{t,x,y}} \exp\left\{-\lambda^{-1}\int_0^t L(\gamma(\tau), \dot\gamma(\tau))d\tau\right\} d_F(\gamma)\right\}_\mu \\ &\quad = \boldsymbol{H}_t^\lambda(L\,;\mu)(t, x, y)|d\mu(x)|^{1/2}\otimes|d\mu(y)|^{1/2},\end{aligned} \tag{5.5}$$

where $\boldsymbol{H}_t^\lambda(L\,;\mu)(t, x, y)$ is the distribution kernel of the operator $H_t^\lambda(L\,;\mu)$ whose existence is assumed by Schwartz' kernel theorem.

In the separate paper [12], Maeda proves that $\boldsymbol{H}_t^\lambda(L\,;\mu)(t, x, y)$ is in fact a smooth function on $M\times M$ for $t>0$.

REMARK 2. As is already noticed, the term $R(x)/12$ in the expression (9) is necessary to consider the operators $\boldsymbol{H}_t^\lambda(L)$ and $\boldsymbol{A}^\lambda(L)$ in the intrinsic Hilbert space. And the choice of a natural measure seems to correspond to fix the measurement of the physical system described in the intrinsic Hilbert space.

But in any way, to answer completely why we consider the problem in L^2 scheme, it is necessary to put $\lambda=i\hbar$. (See, de Witt [6] concerning the term $R(\cdot)/12$.)

6. We may produce any multiple of $R(\cdot)$ in the infinitesimal generator.

In stead of the argument in the previous section, we may produce any multiple of $R(\cdot)$, if we change the order of our procedure and we content ourselves with the convergence of $\tilde H_n^\lambda(t)$ only in the strong sense.

To make our point clear, we consider the case where $V=0$.

For any $\beta\in\boldsymbol{R}$, we define an operator $H_t^\lambda(\beta)$ as

$$(H_t^\lambda(\beta)f)(x)=(2\pi\lambda t)^{-d/2}\int_M \rho^0(x, y)^\beta \exp(-\lambda^{-1}S^0(t, x, y))\cdot f(y)\,d\mu_g(y)\,, \tag{6.1}$$

for $f\in C_0^\infty(M)$, where $t^{d/2}\rho^0(t, x, y)$ is independent of t and simply denoted by $\rho^0(x, y)$. In this case, as we may put $\lambda=1$ without loss of generality, we denote $H_t^1(\beta)$ simply by $H_t(\beta)$. And we drop the super index 0 above for notational simplicity.

THEOREM 6.1. *Under assumptions* (M), (L. I)-(L. IV), *we have the following: Fix $T>0$ arbitrarily. For any $\beta\in\boldsymbol{R}$,*

(a) *$H_t(\beta)$ defines a bounded linear operator in $L^2(M, d\mu_g)$ for $0<t<T$. Moreover, there exists a constant C_{10} such that*

(6.2) $$\|H_t(\beta)f\|\leqq \exp C_{10}t\cdot\|f\|$$

for $0<t<T$ *and* $f\in C_0^\infty(M)$.

(b) $$\lim_{t\to 0}\|H_t(\beta)f-f\|=0 \qquad \textit{for } f\in L^2(M, d\mu_g).$$

(c) $$\partial_t(H_t(\beta)f)(x)_{|t=0}=[\Delta/2-(1-(\beta/2))R(x)/6]f(x) =(\boldsymbol{A}_\beta f)(x) \quad \textit{for } f\in C_0^\infty(M).$$

(d) *There exists a limit* s-$\lim_{n\to\infty}(H_{t/n}(\beta))^n f$, *denoted by* $\boldsymbol{H}_t(\beta)f$ *for each* $f\in C_0^\infty(M)$. $\{\boldsymbol{H}_t(\beta)\}_{t\geqq 0}$ *with* $\boldsymbol{H}_0(\beta)=$*the identity operator, forms a* C^0*-semi group in* $L^2(M, d\mu_g)$ *with the infinitesimal generator given in* (c).

REMARK. Comparing the above theorem with Theorem, we remark that the order of statements is changed. And in proving (d), we use the fact that the Laplace-Beltrami operator Δ is self-adjoint in $L^2(M, d\mu_g)$ under our assumptions. (This fact is proved in the previous sections but we need that fact in order to prove (d).)

PROOF OF (a), (b). In our case, $t^d\Theta^0(t, x, y)$ is independent of t and denoted simply by $\Theta(x, y)$. We may rewrite the operator $H_t(\beta)$ by using normal polar coordinate at x and $\mathrm{Exp}_x X=\Phi_{t,x}(x)$ as

(6.1)′ $$(H_t(\beta)f)(x) =(2\pi t)^{d/2}\int_0^\infty\int_{S^{d-1}}\Theta(x, \mathrm{Exp}_x r\omega)^{1-(\beta/2)}\exp(-d^2(x, \mathrm{Exp}_x r\omega)/2t)r^{d-1}dr\,d\omega.$$

To prove the statements (a) and (b), we proceed analogously as proving Propositions 2.1 and 2.2. But for $\beta\geqq 2$, we use the fact $\Theta(x, y)\geqq 1$ for estimating $\Theta(x, \mathrm{Exp}_x r\omega)^{1-(\beta/2)}$. (As $V=0$, we may take $\Theta(x, y)\geqq 1$ in Proposition 1.10.)

PROOF OF (c). Take a function $\nu(x, y)\in C^\infty(M\times M)$, $0\leqq\nu(x, y)\leqq 1$, satisfying

$$\nu(x, y)=\begin{cases}1 & \text{if } d(x, y)\leqq 1\\ 0 & \text{if } d(x, y)\geqq 3.\end{cases}$$

Define operators $H_1(t, \beta)$ and $H_2(t, \beta)$ as follows:

(6.3) $$(H_1(t, \beta)f)(x)=(2\pi t)^{-d/2}\int_M\nu(x, y)\rho(x, y)^\beta\exp(-d^2(x, y)/2t)f(y)d\mu_g(y),$$

(6.4) $$(H_2(t, \beta)f)(x)=(2\pi t)^{-d/2}\int_M(1-\nu(x, y))\rho(x, y)^\beta\exp(-d^2(x, y)/2t)f(y)d\mu_g(y).$$

Now, we claim the following:

(6.5) $$(H_1(t, \beta)f)(x)=f(x)+t(\boldsymbol{A}_\beta f)(x)+tG_1(t, f)(x) \qquad \text{for } f\in C_0^\infty(M),$$

(6.6) $$\lim_{t\to 0}\|G_1(t, f)(\cdot)\|=0,$$

and

(6.7) $$\lim_{t\to 0}\|t^{-1}(H_2(t,\ \beta))f(\cdot)\|=0\,.$$

By Taylor's expansion, we get

(6.8) $$f(y)=f(x)+(\partial_{X^i}f)(x)X^i+(1/2)(\partial_{X^i}\partial_{X^j}f)(x)X^iX^j+F(x,\ X)\,,$$

where $y=\mathrm{Exp}_x X$. $(\partial_{X^i}f)(x)=\partial_{X^i}f(\mathrm{Exp}_x X)_{|x=0}$, and

$$F(x,\ X)=(1/6)\int_0^1[\partial_{X^i}\partial_{X^j}\partial_{X^k}f(\mathrm{Exp}_x sX)]ds X^iX^jX^k.$$

Then, it is clear that $F(x,\ X)=\nu(x,\ \mathrm{Exp}_x X)F(x,\ X)$ is a smooth function in x and X with compact support.

Analogously, we have

(6.9) $$\begin{aligned}\Theta(x,\ y)^{1-(\beta/2)}&=1-(1/6)(1-(\beta/2))R_{ij}(x)Y^iY^j+\Theta_\beta(x,\ y)\\&=1+\Theta_\beta(x,\ y)\,,\end{aligned}$$

where

$$\Theta_\beta(x,\ y)=(1/6)\int_0^1\partial_{Y^i}\partial_{Y^j}\partial_{Y^k}\Theta(x,\ \mathrm{Exp}_x sY)^{1-(\beta/2)}ds Y^iY^jY^k.$$

By assumption (L.IV), there exist constants C_{11} and κ such that

(6.10) $$|\Theta_\beta(x,\ y)|\leqq C_{11}\exp\kappa|Y|$$

for any $x\in M$ and any $Y\in T_x M$. Inserting (6.8) and (6.9) into (6.3), we get (6.5) by defining $G_1(t,\ f)$ as

(6.11) $tG_1(t,\ f)$

$$\begin{aligned}&=-f(x)(2\pi t)^{-d/2}\int_{T_xM}(1-\nu(x,\ y))[1+(1/6)(1-(\beta/2))R_{ij}(x)Y^iY^j]e^{-|Y|^2/2t}dY\\&+(\partial_{Y^i}f)(x)(2\pi t)^{-d/2}\int_{T_xM}\nu(x,\ y)Y^i\Theta(x,\ y)^{1-(\beta/2)}e^{-|Y|^2/2t}dY\\&-(1/2)(\partial_{Y^i}\partial_{Y^j}f)(x)(2\pi t)^{-d/2}\int_{T_xM}[(1-\nu(x,\ y))+\nu(x,\ y)\Theta_\beta(x,\ y)]Y^iY^je^{-|Y|^2/2t}dY\\&+(2\pi t)^{-d/2}\int_{T_xM}F(x,\ Y)\nu(x,\ Y)^{1-(\beta/2)}e^{-|Y|^2/2t}dY\,,\end{aligned}$$

where $\nu(x,\ y)=\nu(x,\ \mathrm{Exp}_x Y)$ etc.

By (6.10) and the property of $F(x,\ y)$, we have the estimate in (6.6) readily. The estimate (6.7) is an easy consequence of the introduction of $\nu(x,\ y)$.

PROOF OF (d). Under assumptions (M), (L.I)-(L.IV), it is well-known that Δ is self-adjoint in $L^2(M,\ d\mu_g)$. So $\boldsymbol{A}_\beta$ is also self-adjoint. Moreover as $\boldsymbol{A}_\beta$ is bounded from below, $\boldsymbol{A}_\beta$ generates a C^0-semi group. This and the facts (a)-(c) guarantee us to apply the generalized Lax theorem to our case (cf. p. 214, Chorin et al. [**4**]). So we proved our Theorem 6.1. q.e.d.

Note added.

After this paper had been submitted, we were informed of the paper of Darling [Stochastics, Vol. 12, No. 3+4 (1984), pp. 277-301].

In the introduction of this paper, without having asked us for a detailed proof, he claims that our argument in [**11**] contains an error. As this type of response concerning the term $R(x)/12$ is rather general, especially in physics literature, it seems meaningful to refute his claim.

From our point of view, neither his paper nor ours contains any mathematical error. Only for each the understanding of the notorious Feynman measure $d_F\gamma$ is different.

In general, one writes formally the Wiener measure $d_W\gamma$ as

$$d_W\gamma=\exp\left(-\int_0^t L(\gamma(\tau),\ \dot{\gamma}(\tau)d\tau\right)d_F\gamma \qquad \text{for } M=\boldsymbol{R}^n \text{ and } L(\gamma,\ \dot{\gamma})=(1/2)|\dot{\gamma}|^2. \tag{1}$$

This stems from the Feynman-Kac formula representing the fundamental solution of the heat equation. Here, we try to give a meaning directly to the expression

$$\int_{C_{t,x,y}} \exp\left(-\int_0^t L(\gamma(\tau),\ \dot{\gamma}(\tau))d\tau\right)d_F\gamma\,, \tag{2}$$

for M=a suitable manifold and $L(\gamma,\ \dot{\gamma})=(1/2)g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j$ by tracing backward the original argument given by Feynman to derive his famous formula

$$F(t,\ x,\ y)=\int_{C_{t,x,y}} \exp\left((i/\hbar)\int_0^t L(\gamma(\tau),\ \dot{\gamma}(\tau))d\tau\right)d_F\gamma \tag{3}$$

for $M=\boldsymbol{R}^n$ and $L(\gamma,\ \dot{\gamma})=(1/2)|\dot{\gamma}|^2-V(\gamma)$. Here $F(t,\ x,\ y)$ is the fundamental solution of the corresponding Schrödinger equation. So, if we want to regard (2) as a 'definition' of measure $\exp\left(-\int_0^t L(\gamma(\tau),\ \dot{\gamma}(\tau))d\tau\right)d_F\gamma$, then Darling's result states that it does not equal to $d_W\gamma$ when M is curved. Actually, we may recover the formula (1) formally by putting

$$d_W\gamma=\exp\left(-\int_0^t L_{\text{eff}}(\gamma(\tau),\ \dot{\gamma}(\tau))d\tau\right)d_F\gamma \tag{1$'$}$$

for $L_{\text{eff}}(\gamma,\ \dot{\gamma})=L(\gamma,\ \dot{\gamma})-(1/12)R(\gamma)$. Here, $L_{\text{eff}}(\gamma,\ \dot{\gamma})$ is called an effective Lagrangian. But we have, for the time being, no a priori reason to consider $L_{\text{eff}}(\gamma,\ \dot{\gamma})$ instead of $L(\gamma,\ \dot{\gamma})$ before quantization.

We think in this sense that the origin of the debate concerning the term $R(x)/12$ has been clarified.

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