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ON INTEGRATED SEMIGROUPS AND AGE STRUCTURED MODELS IN L^p SPACES

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Abstract. In this paper, we first develop some techniques and results for integrated semigroups when the generator is not a Hille-Yosida operator and is non-densely defined. Then we establish a theorem of Da Prato and Sinestrari's type for the nonhomogeneous Cauchy problem and prove a perturbation theorem. In particular, we obtain necessary and sufficient conditions for the existence of mild solutions for non-densely defined non-homogeneous Cauchy problems. Next we extend the results to L^p spaces and consider the semilinear and non-autonomous problems. Finally we apply the results to studying age-structured models with dynamic boundary conditions in L^p spaces. We also demonstrate that neutral delay differential equations in L^p can be treated as special cases of the age-structured models in an L^p space.

1. Introduction

The goal of this paper is to study certain class of non-autonomous and non-densely defined semi-linear equations arising in population dynamics. In order to investigate such problems, we first need to consider a non-densely defined non-homogeneous Cauchy problem:

$$\frac{du}{dt} = Au(t) + f(t),\ t \in [0, \tau_0],\quad u(0) = x \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator in a Banach space X and $f \in L^1((0, \tau_0), X)$. When A is a Hille-Yosida operator and is densely defined (i.e., $\overline{D(A)} = X$), the problem has been extensively studied (see Pazy [39]

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and Yosida [54]). When A is a Hille-Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [16] investigated the existence of several types of solutions for (1.1). They first reformulated (1.1) as a sum of operator problems (i.e., $Bu = Au + f$ with $Bu(t) = \frac{du}{dt}$), and then obtained the existence and uniqueness of integrated solutions of (1.1) for each $x \in \overline{D(A)}$ and each $f \in L^1((0,\tau_0), X)$. In this paper, we first study the existence of mild solutions for the non-homogeneous Cauchy problem (1.1) when A is not a Hille-Yosida operator and its domain is non-densely defined.

A very important and useful approach to investigate such non-densely defined problems is to use the integrated semigroup theory, which was first introduced by Arendt [6, 7]. In the context of Hille-Yosida operators, we have the following relationship between the integrated semigroup and integrated solutions of (1.1). An integrated semigroup $\{S(t)\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X, which commute with the resolvent of A, such that for each $x \in X$ the map $t \to S(t)x$ is an integrated solution of the Cauchy problem

$$\frac{du}{dt} = Au(t) + x, \quad u(0) = 0. \tag{1.2}$$

Arendt [6, 7] proved that if there is a strongly continuous family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on X, which is assumed to be exponentially bounded (see Section 2 for a precise definition), and if

$$(\mu I - A)^{-1}x = \mu \int_0^\infty e^{-\mu t} S(t)x \, dt$$

holds for all $x \in X$ and all $\mu > \omega$ large enough (where $(\omega, \infty) \subset \rho(A)$), then $\{S(t)\}_{t\geq 0}$ is an integrated semigroup and A is called its generator. Kellermann and Hieber [28] further developed the integrated semigroup theory and provided an easy proof of Da Prato and Sinestrari's result [16]. To be more specific, Kellermann and Hieber [28] proved that when A is a Hille-Yosida operator, the map $t \to (S * f)(t) := \int_0^t S(t-s)f(s)ds$ is continuously differentiable and $u(t) = \frac{d}{dt}(S * f)(t)$ is an integrated solution of (1.1). For recent studies on the integrated semigroup theory, we refer to the monographs of Arendt et al. [8], Xiao and Liang [53] and the references cited therein.

The notion of generator has been extended by Thieme [42] to non-exponentially bounded integrated semigroups. The relationship between an exponentially bounded semigroup (not necessarily locally Lipshitz continuous) and its generator has also been studied by Kellermann and Hieber [28] and

Neubrander [38]. We finally mention that it is also possible to study non-densely defined problems by using the extrapolation space technique. The connection between integrated semigroups and extrapolation spaces has been investigated by Thieme [42] and Arendt et al. [9].

As demonstrated by Da Prato and Sinestrari [16], there are many examples of differential operators with non-dense domain. Examples include operators arising from certain constructions which can be used to handle dynamic boundary conditions and non-autonomous differential equations. Thieme [41, 42, 43, 44] also used the integrated semigroup theory to consider various biological models, such as age-structured population models.

We now introduce age-structured models to explain our motivation. Let $H : D(H) \subset Z \to Z$ be a Hille-Yosida operator on a Banach space $(Z, \|\cdot\|_Z)$. Assume that $p \in [1, +\infty), a_0 \in (0, +\infty]$, and $(Y, \|\cdot\|_Y)$ is a Banach space. Consider the following age-structured model

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = A(a)v(t,a) + \mathcal{F}\left(t, x(t), v(t)\right)(a), \quad a \in (0, a_0), \\ v(t,0) = \mathcal{K}(t, x(t), v(t)), \\ \dfrac{dx(t)}{dt} = Hx(t) + \mathcal{G}\left(t, x(t), v(t)\right), \\ x(0) = x_0 \in \overline{D(H)}, \quad v(0,\cdot) = \psi \in L^p\left((0,a_0), Y\right), \end{cases} \tag{1.3}$$

where a represents the age, $\mathcal{F}, \mathcal{K}$, and $\mathcal{G}$ are continuous nonlinear maps from $[0, \tau_0] \times \overline{D(H)} \times L^p\left((0,a_0), Y\right)$ into $L^p\left((0,a_0), Y\right), Y$, and Z, respectively. $\{A(a)\}_{a \in [0,a_0)}$ is a family of linear operators that generates an evolution family $\{U(a,s)\}_{0 \le s < a < a_0}$ on Y. An important example of the operator $A(a)$ is the following

$$A(a)(\varphi)(x) = \sum_{i,j=1}^{n} \partial_{x_i}(d_{ij}(a,x)\partial_{x_j}\varphi(x)) - \mu(a,x)\varphi(x), \ x \in \Omega,$$

where $\Omega \subset L^n$ with $Y = \mathbb{R}^n$.

Age-structured models with diffusion have been studied by a large number of researchers. We refer to Anita [5], Busenberg and Cook [12], Busenberg and Iannelli [13], Gurtin [22], Di Blasio [18], Gurtin and MacCamy [23], Kunisch et al. [29], Langlais [30, 31], Marcati [34], Marcati and Serafini [35], Webb [49], etc. To investigate age-structured models, one can use the classical method, that is, use solutions integrated along the characteristics and work with nonlinear Volterra equations. We refer to the monographs of Webb [49], Metz and Diekmann [37] and Iannelli [26] on this method. A second approach is the variational method; we refer to Anita [5], Aineseba [4] and the references cited therein. One can also regard the problem as a semilinear problem with non-dense domain and use the integrated semigroups method. We refer to Thieme [41, 43, 44], Magal [32], Thieme and Vrabie [45], Magal and Thieme [33], and Thieme and Vosseler [46] for more details on this approach.

The problem (1.3) when $p = 1$ is a classical case and has been extensively studied in the literature using either one of the above-mentioned approaches. Concerning the case when $p > 1$, one can find examples in which either the classical approach or the variational method can be applied. The goal of this paper is to investigate the case when $p \in (1, +\infty)$ by using the integrated semigroups theory. This approach has not been developed for such cases. The main difficulty is that, when $p > 1$, the linear part of (1.3) generates an integrated semigroup, but its generator is not a Hille-Yosida operator and the integrated semigroup is not locally Lipschitz continuous.

In Section 2, we first recall some classical definitions and results about integrated semigroups, then we prove a Kellermann and Hieber's [28] type result for a class of non-locally Lipschitz integrated semigroups. We prove an integrated semigroup formulation of Desch and Schappacher's [17] linear perturbation theorem in Section 3. In Section 4, we obtain some sufficient conditions on the resolvent of A in order to apply Kellermann and Hieber's type result in Section 2, when the space $L^1((0, \tau_0), X)$ is replaced by $L^p((0, \tau_0), X)$ (with $p > 1$). In Section 5, we investigate the existence and uniqueness of a non-autonomous semiflow generated by non-autonomous semi-linear problems. The obtained results are applied to studying age-structured models in L^p spaces in Section 6. Finally, in Section 7 we demonstrate that neutral delay differential equations in L^p can be treated as special cases of the age structured models in the L^p space.

2. Integrated Semigroups

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two Banach spaces. Denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X into Y. When $X = Y$, denote $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$. Let $A : D(A) \subset X \to X$ be a linear operator. Denote by $\rho(A)$ the resolvent set of A, $N(A)$ the null space of A, and $R(A)$ the range of A, respectively. Also denote by X_0 the closure of $D(A)$, and A_0 the part of A in X_0. Note that $A_0 : D(A_0) \subset X_0 \to X_0$ is a linear operator defined by

$$A_0x = Ax, \ \forall x \in D(A_0) = \{y \in D(A) : Ay \in X_0\}.$$

Assume that $(\omega,+\infty)\subset\rho(A)$. Then it is easy to check that for each $\lambda>\omega$,

$$D(A_0)=(\lambda I-A)^{-1}X_0 \text{ and } (\lambda I-A_0)^{-1}=(\lambda I-A)^{-1}|_{X_0}.$$

Moreover, we have the following result.

Lemma 2.1. *Let $(X,\|.\|)$ be a Banach space and let $A:D(A)\subset X\to X$ be a linear operator. Assume that there exists $\omega\in\mathbb{R}$, such that $(\omega,+\infty)\subset\rho(A)$ and*

$$\limsup_{\lambda\to+\infty}\lambda\left\|(\lambda I-A)^{-1}\right\|_{\mathcal{L}(X_0)}<+\infty. \tag{2.1}$$

Then the following assertions are equivalent:

(i) $\lim_{\lambda\to+\infty}\lambda(\lambda I-A)^{-1}x=x,\forall x\in X_0$.
(ii) $\lim_{\lambda\to+\infty}(\lambda I-A)^{-1}x=0,\forall x\in X$.
(iii) $\overline{D(A_0)}=X_0$.

Proof. Since $D(A_0)=(\lambda I-A)^{-1}X_0$, $\forall\lambda>\omega$, it is clear that (i) $\Rightarrow$ (iii). The proof of (iii) $\Rightarrow$ (i) follows from the argument in the proof of Lemma 3.2 in Pazy [39]. It remains to show that (i) $\Leftrightarrow$ (ii). Assume first that (i) is satisfied, then by using the resolvent formula, we know that for fixed $\mu\in(\omega,+\infty)$ and all $\lambda>\mu$,

$$\begin{aligned}(\lambda I-A)^{-1}&=(\lambda I-A)^{-1}-(\mu I-A)^{-1}+(\mu I-A)^{-1}\\&=-(\lambda-\mu)(\lambda I-A)^{-1}(\mu I-A)^{-1}+(\mu I-A)^{-1}\\&=\left[I-\lambda(\lambda I-A)^{-1}\right](\mu I-A)^{-1}+\mu(\lambda I-A)^{-1}(\mu I-A)^{-1}\end{aligned}$$

and (ii) follows. Assume now that (ii) is satisfied; then, we have

$$\begin{aligned}&\left[\lambda(\lambda I-A)^{-1}-I\right](\mu I-A)^{-1}\\&=\frac{\lambda}{(\mu-\lambda)}\left[(\lambda I-A)^{-1}-(\mu I-A)^{-1}\right]-(\mu I-A)^{-1}.\end{aligned}$$

We obtain that $\lim_{\lambda\to+\infty}\lambda(\lambda I-A)^{-1}x=x,\forall x\in D(A)$, and by using (2.1), we know that (i) is satisfied. □

Recall that A is a *Hille-Yosida operator* if there exist two constants, $\omega\in\mathbb{R}$ and $M\geq1$, such that $(\omega,+\infty)\subset\rho(A)$ and

$$\left\|(\lambda I-A)^{-k}\right\|_{\mathcal{L}(X)}\leq\frac{M}{(\lambda-\omega)^k},\quad\forall\lambda>\omega,\quad\forall k\geq1.$$

In the following, we assume that A satisfies some weaker conditions.

Assumption 2.1. Let $(X, \|\cdot\|)$ be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. Assume that

(a) There exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$, such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathcal{L}(X_0)} \leq \frac{M}{(\lambda - \omega)^k}, \quad \forall \lambda > \omega, \quad \forall k \geq 1;$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0$, $\forall x \in X$.

By using Lemma 2.1 and the Hille-Yosida theorem (see Pazy [39], Theorem 5.3 on p. 20), and the fact that if $\rho(A) \neq \emptyset$ then $\rho(A) = \rho(A_0)$, one obtains the following lemma.

Lemma 2.2. *Assumption* 2.1 *is satisfied if and only if* $\rho(A) \neq \emptyset$, *and* A_0 *is the infinitesimal generator of a* C_0*-semigroup* $\{T_{A_0}(t)\}_{t \geq 0}$ *on* X_0.

Now we give the definition of the integrated semigroups.

Definition 2.3. Let $(X, \|\cdot\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called *an integrated semigroup* if

(i) $S(0) = 0$.

(ii) The map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.

(iii) $S(t)$ satisfies

$$S(s)S(t) = \int_0^s (S(r+t) - S(r))\, dr, \quad \forall t, s \geq 0. \tag{2.2}$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be *non-degenerate* if, whenever $S(t)x = 0$, $\forall t \geq 0$, then $x = 0$. According to Thieme [42], we say that a linear operator $A : D(A) \subset X \to X$ is the *generator* of a non-degenerate integrated semigroup $\{S(t)\}_{t \geq 0}$ on X if and only if

$$x \in D(A),\ y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)y\, ds,\ \forall t \geq 0. \tag{2.3}$$

From [42, Lemma 2.5], we know that if A generates $\{S_A(t)\}_{t \geq 0}$, then for each $x \in X$ and $t \geq 0$,

$$\int_0^t S_A(s)x\, ds \in D(A) \text{ and } S(t)x = A \int_0^t S_A(s)x\, ds + tx.$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be *exponentially bounded* if there exist two constants $\widehat{M} > 0$, and $\widehat{\omega} > 0$, such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq \widehat{M} e^{\widehat{\omega} t}, \forall t \geq 0.$$

When we restrict ourselves to the class of non-degenerate exponentially bounded integrated semigroups, Thieme's notion of generator is equivalent to the one introduced by Arendt [7]. More precisely, combining Theorem 3.1 in Arendt [7] and Proposition 3.10 in Thieme [42], one has the following result.

Theorem 2.4. *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous exponentially bounded family of bounded linear operators on a Banach space $(X, \|\cdot\|)$ and $A : D(A) \subset X \to X$ be a linear operator. Then $\{S(t)\}_{t\geq 0}$ is a non-degenerate integrated semigroup, and A its generator if and only if there exists some $\widehat{\omega} > 0$ such that $(\widehat{\omega}, +\infty) \subset \rho(A)$ and*

$$(\lambda I - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda s} S(s)x\, ds, \ \ \forall \lambda > \hat{\omega}.$$

The following result is well known in the context of integrated semigroups.

Proposition 2.5. *Let Assumption 2.1 be satisfied. Then A generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t\geq 0}$. Moreover, for each $x \in X$, each $t \geq 0$, and each $\mu > \omega$, $S_A(t)x$ is given by*

$$S_A(t)x = \mu \int_0^t T_{A_0}(s) \left(\mu I - A\right)^{-1} x\, ds + \left[I - T_{A_0}(t)\right] \left(\mu I - A\right)^{-1} x. \quad (2.4)$$

Furthermore, the map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \ \ \forall t \geq 0, \ \forall x \in X_0.$$

Proof. The right-hand side of (2.4) defines an exponentially bounded function of t. A short calculation shows that its Laplace transform is $\lambda^{-1}(\lambda I - A)^{-1}$. The result follows by using Theorem 2.4. □

From (iii) in Definition 2.3 one easily deduces that

$$T_{A_0}(r) S_A(t) = S_A(t + r) - S_A(r), \ \forall t, r \geq 0. \quad (2.5)$$

From Proposition 2.5, we deduce that $S_A(t)$ commutes with $(\lambda I - A)^{-1}$ and

$$S_A(t)x = \int_0^t T_{A_0}(l)x\, dl, \ \forall t \geq 0, \ \forall x \in X_0.$$

Hence, $\forall x \in X, \ \forall t \geq 0, \ \forall \mu \in (\omega, +\infty)$,

$$\left(\mu I - A\right)^{-1} S_A(t)x = S_A(t) \left(\mu I - A\right)^{-1} x = \int_0^t T_{A_0}(s) \left(\mu I - A\right)^{-1} x\, ds.$$

We have the following result.

Lemma 2.6. *Let Assumption* 2.1 *be satisfied and let* $\tau_0 > 0$ *be fixed. Denote for each* $f \in C^1([0,\tau_0], X)$,

$$(S_A * f)(t) = \int_0^t S_A(t-s)f(s)ds, \ \forall t \in [0,\tau_0].$$

Then we have the following:

(i) *The map* $t \to (S_A * f)(t)$ *is continuously differentiable on* $[0,\tau_0]$.
(ii) $(S_A * f)(t) \in D(A), \ \forall t \in [0,\tau_0]$.
(iii) *If we set* $u(t) = \frac{d}{dt}(S_A * f)(t)$, *then*

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \forall t \in [0,\tau_0]. \tag{2.6}$$

(iv) *For each* $\lambda \in (\omega, +\infty)$ *and each* $t \in [0,\tau_0]$, *we have*

$$(\lambda I - A)^{-1}\frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds. \tag{2.7}$$

Proof. Let $f \in C^1([0,\tau_0], X)$. Then

$$\frac{d}{dt}(S_A * f)(t) = S_A(t)f(0) + \int_0^t S_A(s)f'(t-s)ds.$$

The result follows from Lemmas 3.2.9 and 3.2.10 in Arendt et al. [8]. □

We now consider the inhomogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \ t \in [0,\tau_0], \ \ u(0) = x \in \overline{D(A)} \tag{2.8}$$

and assume that f belongs to some appropriate subspace of $L^1((0,\tau_0), X)$.

Definition 2.7. A continuous map $u \in C([0,\tau_0], X)$ is called *an integrated solution (or mild solution)* of (2.8) if and only if

$$\int_0^t u(s)ds \in D(A), \ \ \forall t \in [0,\tau_0] \tag{2.9}$$

and

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau_0].$$

Remark 2.8. From (2.9) we know that if u is an integrated solution of (2.8), then $u(t) \in \overline{D(A)}, \ \forall t \in [0,\tau_0]$.

Since A generates a non-degenerate integrated semigroup on X, we can apply Theorem 3.7 in Thieme [42] and obtain the following result.

Lemma 2.9. *Let Assumption* 2.1 *be satisfied. Then for each* $x \in \overline{D(A)}$ *and each* $f \in L^1\left((0,\tau_0), X\right)$, (2.8) *has at most one integrated solution.*

The following lemma can be used to obtain explicit solutions.

Lemma 2.10. *Let Assumption* 2.1 *be satisfied. Let* $v \in C\left([0,\tau_0], X_0\right)$, $f \in L^1\left([0,\tau_0], X\right)$, *and* $\lambda \in (\omega, +\infty)$. *Assume the following:*

(i) $(\lambda I - A)^{-1} v \in W^{1,1}\left([0,\tau_0], X\right)$ *and for almost every* $t \in [0,\tau_0]$,

$$\frac{d}{dt}(\lambda I - A)^{-1} v(t) = \lambda (\lambda I - A)^{-1} v(t) - v(t) + (\lambda I - A)^{-1} f(t).$$

(ii) $t \to (S_A * f)(t)$ *is continuously differentiable on* $[0,\tau_0]$.

Then v *is an integrated solution of* (2.8) *and*

$$v(t) = \left[T_{A_0}(t)v(0) + \frac{d}{dt}(S_A * f)(t)\right], \ \forall t \in [0,\tau_0].$$

Proof. We have for almost every $t \in [0,\tau_0]$ that

$$\begin{aligned}
&\frac{d}{dt}(\lambda I - A)^{-1} v(t) \\
&= \lambda (\lambda I - A)^{-1} v(t) - (\lambda I - A)(\lambda I - A)^{-1} v(t) + (\lambda I - A)^{-1} f(t) \\
&= A_0 (\lambda I - A)^{-1} v(t) + (\lambda I - A)^{-1} f(t).
\end{aligned}$$

So

$$(\lambda I - A)^{-1} v(t) = T_{A_0}(t)(\lambda I - A)^{-1} v(0) + \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds.$$

By (ii),

$$\begin{aligned}
(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) &= \frac{d}{dt}(\lambda I - A)^{-1}(S_A * f)(t) \\
&= \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds, \ \forall t \in [0,\tau_0],
\end{aligned}$$

so we have for all $t \in [0,\tau_0]$ that

$$(\lambda I - A)^{-1} v(t) = (\lambda I - A)^{-1}\left[T_{A_0}(t)v(0) + \frac{d}{dt}(S_A * f)(t)\right].$$

Since $(\lambda I - A)^{-1}$ is injective, the result follows. □

In order to obtain an estimate for the integral solution, we introduce the following assumption.

Assumption 2.2. Let $\tau_0 > 0$ be fixed. Let $Z \subset L^1((0,\tau_0),X)$ be a Banach space endowed with some norm $\|\cdot\|_Z$. Assume that $C^1([0,\tau_0],X) \cap Z$ is dense in $(Z, \|.\|_Z)$ and the embedding from $(Z,\|\cdot\|_Z)$ into $\left(L^1((0,\tau_0),X), \|\cdot\|_{L^1}\right)$ is continuous. Also assume that there exists a continuous map $\Gamma : [0,\tau_0] \times Z \to [0,+\infty)$ such that

(a) $\Gamma(t,0) = 0$, $\forall t \in [0,\tau_0]$, and the map $f \to \Gamma(t,f)$ is continuous at 0 uniformly in $t \in [0,\tau_0]$.

(b) We have $\forall t \in [0,\tau_0]$, $\forall f \in C^1([0,\tau_0],X) \cap Z$ that

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq \Gamma(t,f).$$

We now state and prove the main result in this section.

Theorem 2.11. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then for each* $f \in Z$ *the map* $t \to (S_A * f)(t)$ *is continuously differentiable,* $(S_A * f)(t) \in D(A), \forall t \in [0,\tau_0]$, *and if we denote* $u(t) = \frac{d}{dt}(S_A * f)(t)$, *then*

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau_0]$$

and

$$\|u(t)\| \leq \Gamma(t,f), \ \forall t \in [0,\tau_0].$$

Moreover, for each $\lambda \in (\omega,+\infty)$, *we have*

$$(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds.$$

Proof. Consider the linear operator

$$L_{\tau_0} : \left(C^1([0,\tau_0],X) \cap Z, \|\cdot\|_Z\right) \to \left(C([0,\tau_0],X), \|\cdot\|_{\infty,[0,\tau_0]}\right)$$

defined by

$$L_{\tau_0}(f)(t) = \frac{d}{dt}(S_A * f)(t), \ \forall t \in [0,\tau_0], \ \forall f \in C^1([0,\tau_0],X) \cap Z.$$

Then

$$\sup_{t\in[0,\tau_0]} \|L_{\tau_0}(f)(t)\| \leq \sup_{t\in[0,\tau_0]} \Gamma(t,f).$$

Since $C^1([0,\tau_0],X) \cap Z$ is dense in Z, using assumptions (a) and (b), we know that L_{τ_0} has a unique extension $\widehat{L}_{\tau_0}$ on Z and

$$\|\widehat{L}_{\tau_0}(f)(t)\| \leq \Gamma(t,f), \ \forall t \in [0,\tau_0], \ \forall f \in Z.$$

By construction $\widehat{L}_{\tau_0}$ is continuous from $(Z, \|\cdot\|_Z)$ into $(C([0,\tau_0], X), \|\cdot\|_{\infty,[0,\tau_0]})$.

Let $f \in Z$ and let $\{f_n\}_{n\geq 0}$ be a sequence in $C^1([0,\tau_0], X) \cap Z$, such that $f_n \to f$ in Z. We have for each $n \geq 0$ and each $t \in [0,\tau_0]$ that

$$\int_0^t \widehat{L}_{\tau_0}(f_n)(s)ds = \int_0^t L_{\tau_0}(f_n)(s)ds = \int_0^t S_A(t-s)f_n(s)ds.$$

Since the embedding from $(Z, \|\cdot\|_Z)$ into $\left(L^1((0,\tau_0), X), \|\cdot\|_{L^1}\right)$ is continuous, we have that $f_n \to f$ in $L^1((0,\tau_0), X)$ and when $n \to +\infty$,

$$\int_0^t \widehat{L}_{\tau_0}(f)(s)ds = \int_0^t S_A(t-s)f(s)ds, \; \forall t \in [0,\tau_0].$$

Thus, the map $t \to (S_A * f)(t)$ is continuously differentiable and

$$\widehat{L}_{\tau_0}(f)(t) = \frac{d}{dt}\int_0^t S_A(t-s)f(s)ds, \; \forall t \in [0,\tau_0].$$

Finally, by Lemma 2.6, we have for each $n \geq 0$ and each $t \in [0,\tau_0]$ that

$$\widehat{L}_{\tau_0}(f_n)(t) = A\int_0^t \widehat{L}_{\tau_0}(f_n)(s)ds + \int_0^t f_n(s)ds;$$

the result follows from the fact that A is closed. □

In the proof of Theorem 2.11, we basically followed the same method Kellermann and Hieber [28] used to prove the result of Da Prato and Sinestrari [16] (see also [8, Theorem 3.5.2, p. 145]) for Hille-Yosida operators and with $Z = L^1((0,\tau_0), X)$.

By Lemma 2.10 and Theorem 2.11, we obtain the following result.

Corollary 2.12. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then for each* $x \in X_0$ *and each* $f \in Z$, *the Cauchy problem* (2.8) *has a unique integrated solution* $u \in C([0,\tau_0], X_0)$ *given by*

$$u(t) = T_{A_0}(t)x + \frac{d}{dt}(S_A * f)(t), \; \forall t \in [0,\tau_0].$$

Moreover, we have

$$\|u(t)\| \leq Me^{\omega t}\|x\| + \Gamma(t,f), \; \forall t \in [0,\tau_0].$$

3. Bounded Perturbation

In this section we investigate the properties of $A+L : D(A) \subset X \to X$, where L is a bounded linear operator from X_0 into X. If A is a Hille-Yosida operator, it is well known that $A+L$ is also a Hille-Yosida operator (see [8, Theorem 3.5.5]).

The following theorem is closely related to Desch and Schappacher's theorem (see [17] or Engel and Nagel [20, Theorem 3.1, p. 183]). This is in fact an integrated semigroup formulation of this result.

Theorem 3.1. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Assume in addition that* $C([0,\tau_0], X) \subset Z$ *and there exists a constant* $\delta > 0$ *such that*

$$\Gamma(t,f) \leq \delta \sup_{s\in[0,t]} \|f(s)\|, \ \forall f \in C([0,\tau_0], X), \ \forall t \in [0,\tau_0].$$

Let $L \in \mathcal{L}(X_0, X)$ *and assume that* $\|L\|_{\mathcal{L}(X_0,X)} \delta < 1$. *Then* $A+L : D(A) \subset X \to X$ *satisfies Assumptions* 2.1 *and* 2.2. *More precisely, if we denote by* $\{S_{A+L}(t)\}_{t\geq 0}$ *the integrated semigroup generated by* $A+L$, *then* $\forall f \in C^1([0,\tau_0], X)$,

$$\left\| \frac{d}{dt}(S_{A+L} * f)(t) \right\| \leq \frac{1}{1 - \|L\|_{\mathcal{L}(X_0,X)} \delta} \sup_{s\in[0,t]} \Gamma(s,f), \forall t \in [0,\tau_0]. \tag{3.1}$$

Proof. Without loss of generality we can assume that $\tau_0 = \tau_1$. We first prove that there exists $\widehat{\omega} \in \mathbb{R}$ such that $(\widehat{\omega}, +\infty) \subset \rho(A+L)$. We have for $x \in D(A)$ and $y \in X$ that

$$\begin{aligned}(\lambda I - (A+L))\, x = y \quad &\Leftrightarrow \quad (\lambda I - A)\, x = y + Lx \\ &\Leftrightarrow \quad x = (\lambda I - A)^{-1} y + (\lambda I - A)^{-1} Lx.\end{aligned}$$

So $(\lambda I - (A+L))$ is invertible if $\|(\lambda I - A)^{-1} L\|_{\mathcal{L}(X_0,X)} < 1$. Since $\{S_A(t)\}_{t\geq 0}$ is exponentially bounded, by Theorem 2.4, we have for all $\lambda > \widetilde{\omega}$ that

$$(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t) x\, dt, \ \forall x \in X.$$

We obtain that

$$(\lambda I - A)^{-1} Lx = \lambda \int_{\tau_0}^{+\infty} e^{-\lambda t} S_A(t) Lx\, dt + \lambda \int_0^{\tau_0} e^{-\lambda t} S_A(t) Lx\, dt.$$

Since $S_A(t)y = \frac{d}{dt}\int_0^t S_A(t-s)y\, ds, \forall y \in X$, from the assumption we have

$$\|S_A(t)y\| \leq \delta \|y\|, \ \forall t \in [0,\tau_0], \ \forall y \in X.$$

Thus,

$$\left\| \lambda \int_0^{\tau_0} e^{-\lambda t} S_A(t) L x \, dt \right\| \leq \lambda \int_0^{\tau_0} e^{-\lambda t} dt \, \|L\|_{\mathcal{L}(X_0,X)} \, \delta \, \|x\|$$

and

$$\lambda \int_0^{\tau_0} e^{-\lambda t} dt = 1 - e^{-\lambda \tau_0} \to 1 \text{ as } \lambda \to +\infty.$$

Moreover

$$\left\| \lambda \int_{\tau_0}^{+\infty} e^{-\lambda t} S_A(t) L x \, dt \right\| \to 0 \text{ as } \lambda \to +\infty.$$

So we obtain

$$\limsup_{\lambda \to +\infty} \left\| (\lambda I - A)^{-1} L \right\|_{\mathcal{L}(X_0,X)} \leq \|L\|_{\mathcal{L}(X_0,X)} \, \delta < 1.$$

We know that there exists $\widehat{\omega} \in \mathbb{R}$ such that

$$\left\| (\lambda I - A)^{-1} L \right\|_{\mathcal{L}(X_0,X)} < \frac{\|L\|_{\mathcal{L}(X_0,X)} \, \delta + 1}{2}, \ \ \forall \lambda \in (\widehat{\omega}, +\infty).$$

Hence, for all $\lambda \in (\widehat{\omega}, +\infty)$, $(\lambda I - (A+L))$ is invertible,

$$(\lambda I - (A+L))^{-1} y = \sum_{k=0}^{+\infty} \left[(\lambda I - A)^{-1} L \right]^k (\lambda I - A)^{-1} y,$$

and for each $y \in X$,

$$\left\| (\lambda I - (A+L))^{-1} y \right\| \leq \frac{1}{1 - \frac{\|L\|_{\mathcal{L}(X_0,X)} \delta + 1}{2}} \left\| (\lambda I - A)^{-1} y \right\| \to 0 \text{ as } \lambda \to +\infty.$$

To prove Assumption 2.1 it remains to show that $(A+L)_0$, the part of $(A+L)$ in X_0, is a Hille-Yosida operator. Let $x \in X_0$. Define $\Pi, \Psi_x : C([0,\tau_0], X_0) \to C([0,\tau_0], X_0)$ for each $v \in C([0,\tau_0], X_0)$ by

$$\Pi(v)(t) = \frac{d}{dt} (S_A * Lv)(t) \ \text{ and } \ \Psi_x(v)(t) = T_{A_0}(t)x + \Pi(v)(t), \ \forall t \in [0,\tau_0].$$

Then from the assumptions it is clear that Ψ_x is an $\|L\|_{\mathcal{L}(X_0,X)}\delta$-contraction. So Ψ_x has a unique fixed point given by

$$U(t)x = \sum_{k=0}^{\infty} \Pi^k \left(T_{A_0}(\cdot) x \right)(t), \ \forall t \in [0,\tau_0].$$

In particular,

$$\|U(t)x\| \leq \frac{1}{1 - \|L\|_{\mathcal{L}(X_0,X)} \, \delta} M e^{\omega t} \|x\|, \ \ \forall t \in [0,\tau_0].$$

Thus, we obtain $\{U(t)\}_{0\leq t\leq\tau_0}$, a family of bounded linear operators on X_0, such that for each $x \in X_0$, $t \to U(t)x$ is the unique solution of

$$U(t)x = x + A\int_0^t U(s)x\,ds + \int_0^t LU(s)x\,ds,\ \forall t \in [0,\tau_0].$$

Therefore, $U(t+s) = U(t)U(s)$, $\forall t,s \in [0,\tau_0]$ with $t+s \leq \tau_0$. We can define for each integer $k \geq 0$ and each $t \in [k\tau_0,(k+1)\tau_0]$ that $U(t) = U(t-k\tau_0)U(\tau_0)^k$, which yields a C_0-semigroup of X_0 and

$$U(t)x = x + A\int_0^t U(s)x\,ds + \int_0^t LU(s)x\,ds,\ \forall t \geq 0.$$

It remains to show that $(A+L)_0$ is the generator of $\{U(t)\}_{t\geq 0}$. Let $B : D(B) \subset X_0 \to X_0$ be the generator of $\{U(t)\}_{t\geq 0}$. Since $U(t)x$ is the unique solution of

$$U(t)x = x + (A+L)\int_0^t U(s)x\,ds,\ \forall t \geq 0, \forall x \in X_0,$$

we know that $(\lambda I - (A+L))^{-1}$ and $U(t)$ commute, in particular $(\lambda I - (A+L))^{-1}$ and $(\lambda I - B)^{-1}$ commute. On the other hand, we also have

$$B\int_0^t U(s)x = (A+L)\int_0^t U(s)x\,ds,\ \forall t \geq 0, \forall x \in X_0.$$

Thus,

$$(\lambda I - (A+L))^{-1}\int_0^t U(s)x = (\lambda I - B)^{-1}\int_0^t U(s)x\,ds,\ \forall t \geq 0, \forall x \in X_0.$$

Taking the derivative of the last expression at $t = 0$, we obtain for sufficiently large $\lambda \in \mathbb{R}$ that

$$(\lambda I - (A+L))^{-1}x = (\lambda I - B)^{-1}x,\ \forall x \in X_0.$$

Hence, $B = (A+L)_0$ and $(A+L)$ satisfies Assumption 2.1.

Now using Proposition 2.5 we know that $(A+L)$ generates an integrated semigroup $\{S_{A+L}(t)\}_{t\geq 0}$ and

$$S_{A+L}(t)x = (A+L)\int_0^t S_{A+L}(t)x + \int_0^t x\,ds,\ \forall t \geq 0,\ \forall x \in X.$$

So

$$S_{A+L}(t)x = S_A(t)x + \frac{d}{dt}\left(S_A * LS_{A+L}(\cdot)x\right)(t),\ \forall t \in [0,\tau_0],\ \forall x \in X$$

and for each $f \in L^1([0,\tau_0], X)$,

$$\int_0^t S_{A+L}(t-s)f(s)ds = \int_0^t S_A(t-s)f(s)ds + \int_0^t W(t-s)f(s)ds,$$

$\forall t \in [0,\tau_0]$, $\forall x \in X$, where $W(t)x = \frac{d}{dt}\left(S_A * LS_{A+L}(\cdot)x\right)(t)$.

Also notice that

$$\begin{aligned}
&\int_0^t \int_0^l W(l-s)f(s)ds\,dl = \int_0^t \int_s^t W(l-s)f(s)dl\,ds \\
&= \int_0^t \int_0^{t-s} W(l)f(s)dl\,ds = \int_0^t \left(S_A * LS_{A+L}(\cdot)f(s)\right)(t-s)ds \\
&= \int_0^t \int_0^{t-s} S_A(t-s-l)LS_{A+L}(l)f(s)dl\,ds \\
&= \int_0^t \int_s^t S_A(t-l)LS_{A+L}(l-s)f(s)dl\,ds \\
&= \int_0^t \int_0^l S_A(t-l)LS_{A+L}(l-s)f(s)ds\,dl \\
&= \int_0^t S_A(t-l) \int_0^l LS_{A+L}(l-s)f(s)ds\,dl;
\end{aligned}$$

we then have

$$\int_0^t W(t-s)f(s)ds = \frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(\cdot)\right)(t).$$

Thus,

$$\left(S_{A+L} * f\right)(t) = (S_A * f)(t) + \frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(\cdot)\right)(t),\ \forall t \in [0,\tau_0].$$

Let $f \in C^1([0,\tau_0], X)$. The map $t \to L\left(S_{A+L} * f\right)(\cdot)$ is continuously differentiable and

$$\begin{aligned}
&\frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(\cdot)\right)(t) \\
&= S_A(t)L\left(S_{A+L} * f\right)(0) + \Big(S_A * \frac{d}{dt}L\left(S_{A+L} * f\right)(\cdot)\Big)(t),
\end{aligned}$$

so

$$\frac{d}{dt}\left(S_{A+L} * f\right)(t) = \frac{d}{dt}(S_A * f)(t) + \frac{d}{dt}\Big(S_A * L\frac{d}{dt}\left(S_{A+L} * f\right)(\cdot)\Big)(t).$$

Therefore, for each $t \in [0, \tau_0]$, we have

$$\left\| \frac{d}{dt} (S_{A+L} * f)(t) \right\| \leq \Gamma(t,f) + \|L\|_{\mathcal{L}(X_0,X)} \delta \sup_{s\in[0,t]} \left\| \frac{d}{dt} (S_{A+L} * f)(s) \right\|$$

and

$$\sup_{s\in[0,t]} \left\| \frac{d}{dt} (S_{A+L} * f)(s) \right\| \leq \frac{1}{1 - \|L\|_{\mathcal{L}(X_0,X)} \delta} \sup_{s\in[0,t]} \Gamma(s,f).$$

This completes the proof. □

4. The L^p Case

In this section we investigate the case when

$$Z = L^p((0,\tau_0), X) \text{ and } \Gamma(t,f) = \widehat{M} \left\| e^{\widehat{\omega}(t-\cdot)} f(\cdot) \right\|_{L^p((0,t),X)},$$

where $p \in [1, +\infty)$, $\widehat{M} > 0$, $\widehat{\omega} \in \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. From now on, for any Banach space $(Y, \|\cdot\|_Y)$ we denote by Y^* the space of continuous linear functionals on Y. We recall a result which will be used in the sequel (see Diestel and Uhl [19, pp. 97–98]).

Proposition 4.1. *Let Z be a Banach space and $I \subset \mathbb{R}$ be a non-empty open interval. Assume $p, q \in [1, +\infty]$ with $1/p + 1/q = 1$.*

(i) *For each $q \in [1, +\infty]$ and each $\psi \in L^q(I, Z^*) \cap C(I, Z^*)$,*

$$\|\psi\|_{L^q(I,Z^*)} = \sup_{\substack{\varphi\in C_c^\infty(I,Z) \\ \|\varphi\|_{L^p(I,Z)}\leq 1}} \int_{\hat{I}} \psi(s)(\varphi(s))\, ds$$

(ii) *For each $p \in [1, +\infty)$ and for each $\varphi \in L^p\left(\hat{I}, Z\right)$,*

$$\|\varphi\|_{L^p(I,Z)} = \sup_{\substack{\psi\in C_c^\infty(I,Z^*) \\ \|\psi\|_{L^p(I,Z)^*}\leq 1}} \int_{\hat{I}} \psi(s)(\varphi(s))\, ds.$$

From now on, denote

$$abs(f) := \inf\left\{ \delta > 0 : e^{-\delta \cdot} f(\cdot) \in L^1(0, +\infty, X) \right\} < +\infty$$

and define the *Laplace transform* of f by

$$\mathcal{L}(f)(\lambda) = \int_0^{+\infty} e^{-\lambda s} f(s) ds$$

when $\lambda > abs(f)$. We first give a necessary condition for the L^p case when $p \in [1, +\infty]$.

Lemma 4.2. *Let Assumption* 2.1 *be satisfied and let* $p, q \in [1, +\infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that there exist* $\widehat{M} > 0$ *and* $\widehat{\omega} \in \mathbb{R}$, *so that* $\forall t \geq 0$, $\forall f \in C^1([0,t], X)$,

$$\|(S_A \diamond f)(t)\| \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(\cdot) \right\|_{L^p((0,t),X)}. \tag{4.1}$$

Then there exists a subspace $E \subset X_0^*$ *such that for each* $x^* \in E$ *there exists* $V_{x^*} \in L^q([0,+\infty), X^*) \cap C([0,+\infty), X^*)$ *such that*

$$x^* \left((\lambda - (A - \widehat{\omega} I))^{-1} x \right) = \int_0^{+\infty} e^{-\lambda s} V_{x^*}(s) x \, ds, \tag{4.2}$$

when $\lambda > 0$ *is sufficiently large,*

$$x^* \left(S_{(A - \widehat{\omega} I)}(t) x \right) = \int_0^t V_{x^*}(s) x \, ds, \forall t \geq 0, \tag{4.3}$$

$$\sup_{x^* \in E : \|x^*\|_{X_0^*} \leq 1} \|V_{x^*}\|_{L^q([0,+\infty),X^*)} \leq \widehat{M}, \forall t \geq 0,$$

and

$$\|x\| \leq \sup_{x^* \in E : \|x^*\|_{X_0^*} \leq M} x^*(x), \forall x \in X_0, \tag{4.4}$$

where $M > 0$ *is the constant introduced in Assumption* 2.1.

Proof. We set

$$B = \left\{ (\lambda - \omega)^2 y^* \circ (\lambda I - A_0)^{-2} : y^* \in X_0^*, \ \|y^*\|_{X_0^*} \leq 1, \text{ and } \lambda > \omega \right\}.$$

From Assumption 2.1, we obtain $\sup\{\|x^*\|_{X_0^*} : x^* \in B\} \leq M$ and

$$\lim_{\lambda \to +\infty} (\lambda - \omega)^2 (\lambda I - A_0)^{-2} x = x, \forall x \in X_0.$$

Using the Theorem of Hahn-Banach, we have $\|x\| \leq \sup_{x^* \in B} x^*(x)$. Let E be the subspace of X_0^* generated by B. Then

$$\|x\| \leq \sup_{x^* \in B} x^*(x) \leq \sup_{x^* \in E : \|x^*\|_{X_0^*} \leq M} x^*(x)$$

and (4.4) is satisfied.

Let $y^* \in X_0^*$ be such that $\|y^*\|_{X_0^*} \leq 1$ and let $\mu > \omega$. Set

$$x^* := (\mu - \omega)^2 y^* \circ (\mu I - A_0)^{-2}.$$

Then for $\lambda > \widehat{\omega} + \max(0, \omega)$, we have for each $x \in X$ that

$$x^*((\lambda - (A - \widehat{\omega} I))^{-1} x)$$

$$= (\mu - \omega)^2 y^* \left((\mu I - A_0)^{-1} (\lambda - (A_0 - \widehat{\omega} I))^{-1} (\mu I - A)^{-1} x \right)$$
$$= (\mu - \omega)^2 y^* \left((\mu I - A_0)^{-1} \int_0^{+\infty} e^{-(\lambda + \widehat{\omega})t} T_{A_0}(t) (\mu I - A)^{-1} x \, dt \right).$$

So

$$x^*((\lambda - (A - \widehat{\omega} I))^{-1} x) = \int_0^{\infty} e^{-\lambda t} V_{x^*}(t) x \, dt$$

with

$$V_{x^*}(t) = e^{-\widehat{\omega} t} (\mu - \omega)^2 y^* \circ (\mu I - A_0)^{-1} \circ T_{A_0}(t) \circ (\mu I - A)^{-1}, \forall t \geq 0.$$

Since

$$T_{A_0}(t) x = x + A_0 \int_0^t T_{A_0}(l) x \, dl$$

and $A_0 (\mu I - A_0)^{-1}$ is bounded, it follows that $t \to (\mu I - A_0)^{-1} T_{A_0}(t)$ is continuous from $[0, +\infty)$ into $\mathcal{L}(X_0)$, and is exponentially bounded. Thus, $t \to V_{x^*}(t)$ is Bochner measurable from $[0, +\infty)$ into X^* and belongs to $L^1_{Loc}([0, +\infty), X^*)$. Moreover, for each $f \in C^1([0, t], X)$, we have

$$x^*((S_A \diamond f)(t))$$
$$= (\mu - \omega)^2 \int_0^t x^* \circ (\mu I - A_0)^{-1} \circ T_{A_0}(t - s) \circ (\mu I - A)^{-1} (f(s)) \, ds$$
$$= \int_0^t V_{x^*}(t - s) e^{\widehat{\omega}(t-s)} f(s) ds.$$

Now by using (4.1) it follows that

$$x^*((S_A \diamond f)(t)) = \int_0^t V_{x^*}(t - s) e^{\widehat{\omega}(t-s)} f(s) ds. \tag{4.5}$$

Since E is the set of all the finite linear combinations of elements of B, it follows that (4.2), (4.3) and (4.5) are satisfied for each $x^* \in E$. Let $x^* \in E$ with $\|x^*\|_{X_0^*} \leq 1$. We have from (4.1) that

$$\int_0^t V_{x^*}(t - s) e^{\widehat{\omega}(t-s)} f(s) ds = x^*((S_A \diamond f)(t)) \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(\cdot) \right\|_{L^p((0,t),X)}.$$

Using Proposition 4.1(i), we have

$$\|V_{x^*}\|_{L^q((0,t),X^*)} \leq \widehat{M}, \ \forall t \geq 0.$$

This completes the proof. □

Theorem 4.3. *Let Assumption 2.1 be satisfied. Let* $B : \overline{D(A)} \to Y$ *be a bounded linear operator from* $\overline{D(A)}$ *into a Banach space* $(Y, \|\cdot\|_Y)$ *and* $\chi : (0, +\infty) \to \mathbb{R}$ *a non-negative measurable function with* $abs(\chi) < +\infty$. *Then the following assertions are equivalent:*

(i) $\|B(S_A \diamond f)(t)\| \leq \int_0^t \chi(t-s)\|f(s)\|ds,\ \forall t \geq 0,\ \forall f \in C^1([0,+\infty), X)$.

(ii) $\left\|B(\lambda - A)^{-n}\right\|_{\mathcal{L}(X,Y)} \leq \frac{1}{(n-1)!}\int_0^{+\infty} s^{n-1}e^{-\lambda s}\chi(s)\,ds,\ \forall \lambda > \delta,\ \forall n \geq 1.$

(iii) $\|B[S_A(t+h) - S_A(t)]\|_{\mathcal{L}(X,Y)} \leq \int_t^{t+h} \chi(s)\,ds,\ \forall t, h \geq 0.$

Moreover, if one of the above three conditions is satisfied, then $\chi \in L^q_{Loc}([0,+\infty), \mathbb{R})$ *for some* $q \in [1, +\infty]$, *and* $p \in [1, +\infty)$ *satisfies* $\frac{1}{p} + \frac{1}{q} = 1$, *then for each* $\tau > 0$ *and each* $f \in L^p((0,\tau), X)$, *the map* $t \to B(S_A * f)(t)$ *is continuously differentiable and*

$$\left\|\frac{d}{dt}B(S_A * f)(t)\right\| \leq \int_0^t \chi(t-s)\|f(s)\|\,ds,\ \forall t \in [0,\tau].$$

Proof. Proof of (i)⇒(ii). Let $x \in X$ be fixed. From the formula

$$(\lambda - A)^{-1}x = \lambda \int_0^{+\infty} e^{-\lambda l}S_A(l)x\,dl,\ \forall \lambda > \delta$$

one deduces that

$$n!(\lambda - A)^{-(n+1)}x = (-1)^n \frac{d^n(\lambda - A)^{-1}}{d\lambda^n} = \int_0^{+\infty}\left[\lambda l^n - nl^{n-1}\right]e^{-\lambda l}S_A(l)x\,dl.$$

We also remark that

$$-\int_0^t l^n e^{-\lambda l}S_A(l)x\,dl = \int_0^t S_A(l)f(t, t-l)\,dl = (S_A * f(t,\cdot))(t),$$

where

$$f(t,s) = h(t-s)x \text{ with } h(l) = -l^n e^{-\lambda l}.$$

It follows that

$$-t^n e^{-\lambda t}S_A(t) = \frac{d}{dt}[(S_A * f(t,\cdot))(t)] = (S_A \diamond f(t,\cdot))(t) + \Big(S_A * \frac{\partial f(t,\cdot)}{\partial t}\Big)(t),$$

so for all $\lambda > 0$ large enough

$$\lim_{t\to+\infty}(S_A \diamond f(t,\cdot))(t) = -\lim_{t\to+\infty}\Big(S_A * \frac{\partial f(t,\cdot)}{\partial t}\Big)(t).$$

But

$$\Big(S_A * \frac{\partial f(t,\cdot)}{\partial t}\Big)(t) = \int_0^t S(l)h'(t-(t-l))dl = \int_0^t [\lambda l^n - nl^{n-1}]e^{-\lambda l}S(l)x\,dl,$$

so

$$n!\left(\lambda-A\right)^{-(n+1)}=\lim_{t\to+\infty}\left(S_A*\frac{\partial f\left(t,\cdot\right)}{\partial t}\right)(t)=-\lim_{t\to+\infty}\left(S_A\diamond f\left(t,\cdot\right)\right)(t).$$

Now by using (i), it follows that

$$\begin{aligned}\left\|n!B\left(\lambda-A\right)^{-(n+1)}x\right\| &= \lim_{t\to+\infty}\left\|B\left(S_A\diamond f\left(t,\cdot\right)\right)(t)\right\| \\ &\leq \lim_{t\to+\infty}\int_0^t \chi(l)\left\|f(t,t-l)\right\|dl=\int_0^{+\infty}l^{n-1}e^{-\lambda l}\chi\left(l\right)dl\left\|x\right\|\end{aligned}$$

and (ii) follows.

Proof of (ii)⇒(i). Let $f\in C^1\left(\left[0,+\infty\right),X\right)$ be fixed. Without loss of generality we can assume that f is exponentially bounded. We remark that

$$\begin{aligned}\left(\lambda-A\right)^{-1}\mathcal{L}\left(f\right)\left(\lambda\right) &= \lambda\int_0^{+\infty}e^{-\lambda l}S_A(l)dl\int_0^{+\infty}e^{-\lambda l}f(l)dl \\ &= \lambda\int_0^{+\infty}e^{-\lambda l}\left(S_A*f\right)(l)dl.\end{aligned}$$

Integrating by parts we obtain that

$$\int_0^{+\infty}e^{-\lambda l}\left(S_A\diamond f\right)(l)dl=\left(\lambda-A\right)^{-1}\mathcal{L}\left(f\right)\left(\lambda\right).$$

Then

$$\frac{d^n}{d\lambda^n}\int_0^{+\infty}e^{-\lambda l}\left(S_A\diamond f\right)(l)dl=\sum_{k=0}^{n}C_n^k\frac{d^{n-k}\left(\lambda-A\right)^{-1}}{d\lambda^{n-k}}\frac{d^k}{d\lambda^k}\mathcal{L}\left(f\right)\left(\lambda\right)$$

and

$$\begin{aligned}&\left\|\frac{d^n}{d\lambda^n}\int_0^{+\infty}e^{-\lambda l}B\left(S_A\diamond f\right)(l)dl\right\| \\ &\quad\leq\sum_{k=0}^{n}C_n^k\left\|\frac{d^{n-k}B\left(\lambda-A\right)^{-1}}{d\lambda^{n-k}}\frac{d^k\mathcal{L}\left(f\right)\left(\lambda\right)}{d\lambda^k}\right\| \\ &\quad=\sum_{k=0}^{n}C_n^k\left(n-k\right)!\left\|B\left(\lambda-A\right)^{-(n-k+1)}\right\|\left(-1\right)^k\frac{d^k\mathcal{L}\left(\left\|f\right\|\right)\left(\lambda\right)}{d\lambda^k}.\end{aligned}$$

Now using (ii) it follows that

$$\left\|\frac{d^n}{d\lambda^n}\int_0^{+\infty}e^{-\lambda l}B\left(S_A\diamond f\right)(l)dl\right\|$$

$$\leq (-1)^n \sum_{k=0}^{n} C_n^k \frac{d^{n-k}\mathcal{L}(\chi)(\lambda)}{d\lambda^{n-k}} \frac{d^k \mathcal{L}(\|f\|)(\lambda)}{d\lambda^k}$$
$$= (-1)^n \frac{d^n}{d\lambda^n} \int_0^{+\infty} e^{-\lambda l} (\chi * \|f\|)(l) dl$$

and by the Post-Widder theorem (see Arendt et al. [8]) we obtain

$$\|B(S_A \diamond f)(t)\| \leq (\chi * \|f\|)(t), \ \forall t \geq 0.$$

So we obtain (i) for all the maps f in $C^1([0,+\infty), X)$.

We now prove (iii)⇒(ii). First assume that $n = 1$. We have

$$B(\lambda - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda s} BS(s)x \, ds.$$

Using (iii), we obtain

$$\left\| B(\lambda - A)^{-1} \right\| \leq \lambda \int_0^{+\infty} e^{-\lambda s} \int_0^s \chi(l) \, dl \, ds$$

and by integrating by parts (ii) follows. Next assume that $n \geq 2$. We have

$$\begin{aligned} B(\lambda - A)^{-n} &= B(\lambda - A_0)^{-(n-1)} (\lambda - A)^{-1} \\ &= \frac{(-1)^{n-2}}{(n-2)!} \lambda B \Big(\frac{d^{n-2}(\lambda - A_0)^{-1}}{d\lambda^{n-2}} \Big) \int_0^{+\infty} e^{-\lambda s} S_A(s) ds \\ &= \frac{\lambda}{(n-2)!} B \int_0^{+\infty} s^{n-2} e^{-\lambda s} T_{A_0}(s) ds \int_0^{+\infty} e^{-\lambda s} S_A(s) ds \\ &= \frac{\lambda}{(n-2)!} B \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} T_{A_0}(s-l) S_A(l) dl \, ds. \end{aligned}$$

But $T_{A_0}(s-l)S_A(l) = S_A(s) - S_A(s-l)$, so

$$B(\lambda - A)^{-n} = \frac{\lambda}{(n-2)!} \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} [BS_A(s) - BS_A(s-l)] \, dl \, ds.$$

From (iii), we obtain

$$\left\| B(\lambda - A)^{-n} \right\|_{\mathcal{L}(X)} \leq \frac{\lambda}{(n-2)!} \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} \int_{s-l}^s \chi(r) \, dr \, dl \, ds.$$

Notice that

$$\int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} \int_{s-l}^s \chi(r) dr \, dl \, ds = \int_0^{+\infty} e^{-\lambda s} \int_0^s l^{n-2} \int_l^s \chi(r) dr \, dl \, ds$$

$$= \int_0^{+\infty} e^{-\lambda s} \int_0^s \int_0^r l^{n-2} dl \chi(r) dr\, ds = \frac{1}{n-1} \int_0^{+\infty} e^{-\lambda s} \int_0^s r^{n-1} \chi(r) dr\, ds;$$

integrating by parts, we have

$$\int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-1} \int_{s-l}^s \chi(r) dr\, dl\, ds = \frac{1}{(n-1)\lambda} \int_0^{+\infty} s^{n-1} \chi(s) e^{-\lambda s} ds.$$

It follows that

$$\left\| B(\lambda - A)^{-n} \right\|_{\mathcal{L}(X)} \leq \frac{1}{(n-1)!} \int_0^{+\infty} s^{n-1} \chi(s) e^{-\lambda s} ds.$$

It remains to prove (i)⇒(iii). Let $h > 0$ and $t > h$ be fixed. We have

$$\begin{aligned} \frac{d}{dt} \left(S_A * 1_{[0,h]}(\cdot) x \right)(t) &= \frac{d}{dt} \int_0^t S_A(t-s) 1_{[0,h]}(s) x\, ds = \frac{d}{dt} \int_0^h S_A(t-s) x\, ds \\ &= \frac{d}{dt} \int_{t-h}^t S_A(s) x\, ds = S_A(t)x - S_A(t-h)x. \end{aligned}$$

Let $\{\phi_n\}_{n \geq 0} \subset C^1(\mathbb{R}_+, \mathbb{R})$ be a sequence of non-increasing functions such that

$$\phi_n(t) = \begin{cases} 1, & \text{if } t \in [0,h], \\ \in [0,h], & \text{if } t \in [h, h + \frac{1}{n+1}], \\ 0, & \text{if } t \geq h + \frac{1}{n+1}. \end{cases}$$

We can always assume that $\phi_{n+1} \leq \phi_n$, $\forall n \geq 0$. Then we have

$$\begin{aligned} &\frac{d}{dt} (S_A * \phi_n(\cdot) x)(t) = \frac{d}{dt} \int_0^t S_A(s) \phi_n(t-s) x\, ds \\ &= S_A(t) \phi_n(0) x + \int_0^t S_A(s) \phi_n'(t-s) x\, ds = S_A(t)x + \int_0^t S_A(t-s) \phi_n'(s) x\, ds \\ &= S_A(t)x + \int_0^{h + \frac{1}{n+1}} S_A(t-s) \phi_n'(s) x\, ds. \end{aligned}$$

By the continuity of $t \to S_A(t)x$, it follows that

$$\lim_{n \to +\infty} \frac{d}{dt} (S_A * \phi_n(\cdot) x)(t) = S_A(t)x - S_A(t-h)x.$$

On the other hand, we have $\chi\mid_{[0,t]} \in L^1((0,t), \mathbb{R})$, and $s \to \chi(t-s)\phi_n(s)$ is a non-increasing sequence in $L^1((0,t), \mathbb{R})$. So by the Beppo-Levi (monotone convergence) theorem, we obtain

$$\lim_{n\to+\infty}\int_0^t \chi(t-s)\phi_n(s)ds = \int_0^t \chi(t-s)1_{[0,h]}(s)ds = \int_0^h \chi(t-s)ds = \int_{t-h}^t \chi(l)dl,$$

and (iii) follows from (i). The proof of the last part of the theorem is similar to the proof of Theorem 2.11. □

Remark 4.4. When $B = I$, the previous theorem provides an extension of the Hille-Yosida case. Unfortunately, this kind property is not satisfied in the context of age-structured models. Because if property (iii) were satisfied for some function $\chi \in L^q_{Loc}([0,+\infty),\mathbb{R})$, this would imply that $t \to S_A(t)$ is locally of bounded L^q-variation from $[0,+\infty)$ into $\mathcal{L}(X)$, but this is not true in such a context (see Remark 4.8(d)).

Inspired by the paper of Bochner and Taylor [11] we now consider functions of bounded L^p-variation. Let I be an interval in $\mathbb{R}$. Let $H : \overset{\circ}{I} \to X$ be a map. If $p \in [1,+\infty)$, set

$$VL^p(I,H) = \sup_{\substack{t_0<t_1<\cdots<t_n \\ t_i\in\overset{\circ}{I},\forall i=1,\ldots,n}} \left\{ \Big(\sum_{i=1}^n \frac{\|H(t_i)-H(t_{i-1})\|^p}{|t_i-t_{i-1}|^{p-1}} \Big)^{1/p} \right\},$$

where the supremum is taken over all finite strictly increasing sequences in $\overset{\circ}{I}$. If $p=+\infty$, set

$$VL^\infty(I,H) = \sup_{t,s\in\overset{\circ}{I}} \left\{ \frac{\|H(t)-H(s)\|}{|t-s|} \right\}.$$

We will say that H is of *bounded L^p-variation on I* if $VL^p(I,H) < +\infty$.

Let $(Y, \|\cdot\|_Y)$ be a Banach space. Let $H : I \to \mathcal{L}(X,Y)$ and $f : I \to X$. If π is a finite sequence $t_0 < t_1 < \cdots < t_n$ in $\overset{\circ}{I}$ and $s_i \in [t_{i-1}, t_i]$ $(i = 1,\ldots,n)$, we denote by

$$S(dH,f,\pi) = \sum_{i=1}^n (H(t_i)-H(t_{i-1}))\, f(s_i) \ \text{ and } \ |\pi| = \max_{i=0,\ldots,n} |t_i - t_{i-1}|.$$

We will say that f is *Riemann-Stieltjes integrable with respect to H* if

$$\int_a^b dH(t)f(t) := \lim_{|\pi|\to 0 \text{ with } t_0\to\inf I \text{ and } t_n\to\sup I} S(dH,f,\pi) \text{ exists.}$$

Then we have the following result (see Section 1.9 in Arendt et al. [8] and Section III.3.3 in Hille and Phillips [25] for more details).

Lemma 4.5. *Assume* $p, q \in [1, +\infty]$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f \in C^1([a, b], X)$. *Let* $H : [a, b] \to \mathcal{L}(X, Y)$ *be a bounded and strongly continuous map. Then* f *is Riemann-Stieltjes integrable with respect to* H *and*

$$\int_a^b dH(t)f(t) = H(b)f(b) - H(a)f(a) - \int_a^b H(t)f'(t)dt,$$

where the last integral is a Riemann integral. If we assume in addition that H *is of bounded* L^q*-variation on* $[a, b]$, *then we have*

$$\left\| \int_a^b dH(t)f(t) \right\| \leq VL^q([a, b], H) \, \|f\|_{L^p((a,b),X)}.$$

Motivated by Lemma 4.2, we introduce the following definition.

Definition 4.6. Let $(Y, \|.\|_Y)$ be a Banach space. Let E be a subspace of Y^*. E is called a *norming space* of Y if the map $|\cdot|_E : Y \to R_+$ defined by

$$|y|_E = \sup_{\substack{y^* \in E \\ \|y^*\|_{Y^*} \leq 1}} y^*(y), \forall y \in Y$$

is a norm equivalent to $\|.\|_Y$.

The main result of this section is the following theorem.

Theorem 4.7. *Let Assumption* 2.1 *be satisfied. Let* $p, q \in [1, +\infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\widehat{\omega} \in \mathbb{R}$. *Then the following properties are equivalent:*

(i) *There exists* $\widehat{M} > 0$, *such that for each* $\tau_0 \geq 0$, $\forall f \in C^1([0, \tau_0], X)$,

$$\|(S_A \diamond f)(t)\| \leq \widehat{M} \left\| e^{\widehat{\omega}(t-\cdot)} f(\cdot) \right\|_{L^p((0,t),X)}, \forall t \in [0, \tau_0].$$

(ii) *There exists a norming space* E *of* X_0, *such that for each* $x^* \in E$ *the map* $t \to x^* \circ S_{A+\omega I}(t)$ *is of bounded* L^q*-variation from* $[0, +\infty)$ *into* X^* *and*

$$\sup_{x^* \in E : \|x^*\|_{X_0^*} \leq 1} \lim_{t \to +\infty} VL^q([0, t], x^* \circ S_{A - \widehat{\omega} I}(\cdot)) < +\infty. \tag{4.6}$$

(iii) *There exists a norming space* E *of* X_0, *such that for each* $x^* \in E$ *there exists* $\chi_{x^*} \in L^q_+((0, +\infty), \mathbb{R})$,

$$\|x^* \circ S_{A-\widehat{\omega} I}(t+h) - x^* \circ S_{A-\widehat{\omega} I}(t)\|_{X^*} \leq \int_t^{t+h} \chi_{x^*}(s)\, ds, \forall t, h \geq 0 \tag{4.7}$$

and

$$\sup_{x^*\in E:\|x^*\|_{X_0^*}\leq 1} \|\chi_{x^*}\|_{L^q((0,+\infty),\mathbb{R})} < +\infty. \tag{4.8}$$

Proof. The proof of (i)⇒(iii) is an immediate consequence of Lemma 4.2. The proof of (iii)⇒(ii) is an immediate consequence of the fact that (4.7) implies

$$VL^q([0,t]\,, x^* \circ S_{A+\widehat{\omega}I}(\cdot)) \leq \|\chi_{x^*}\|_{L^q((0,t),\mathbb{R})}\,,\ \forall t \geq 0.$$

So it remains to prove (ii)⇒(i). Let $x^* \in E$ and $f \in C^1((0,\tau_0)\,, X)$ be fixed. By Lemma 4.5, we have for each $t \in [0,\tau_0]$ that

$$\frac{d}{dt}\left(S_A * f\right)(t) = S_A(t)f(0) + \int_0^t S_A(s)f'(t-s)ds = \int_0^t dS_A(s)f(t-s)ds.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}\left(S_A * f\right)(t) &= \lim_{\lambda\to+\infty} \lambda\left(\lambda I - A_0\right)^{-1} \frac{d}{dt}\left(S_A * f\right)(t) \\
&= \lim_{\lambda\to+\infty} \lambda \int_0^t T_{A_0}(t-s)\left(\lambda I - A\right)^{-1} f(s)ds \\
&= \lim_{\lambda\to+\infty} \lambda \int_0^t T_{A_0-\widehat{\omega}I}(t-s)\left(\lambda I - A\right)^{-1} e^{\widehat{\omega}(t-s)} f(s)ds \\
&= \frac{d}{dt}\left(S_{A+\widehat{\omega}I} * e^{\widehat{\omega}(t-\cdot)} f(\cdot)\right)(t) = \int_0^t dS_{A+\widehat{\omega}I}(s)e^{\widehat{\omega}(t-s)} f(t-s)ds.
\end{aligned}$$

By using the last part of Lemma 4.5, we have

$$\begin{aligned}
x^*\Big(\frac{d}{dt}\left(S_A * f\right)(t)\Big) &= \int_0^t d\left(x^* \circ S_{A-\widehat{\omega}I}\right)(s)e^{\widehat{\omega}(t-s)} f(t-s) \\
&\leq VL^q([0,t]\,, (x^* \circ S_{A-\widehat{\omega}I})(\cdot))\left\|e^{\widehat{\omega}(t-\cdot)} f(\cdot)\right\|_{L^p((0,t),X_1)}.
\end{aligned}$$

Hence, $\forall t \in [0,\tau_0]$ we have

$$x^*\Big(\frac{d}{dt}\left(S_A * f\right)(t)\Big) \leq VL^q([0,+\infty)\,,\ \ (x^* \circ S_{A-\widehat{\omega}I})(\cdot))\left\|e^{\widehat{\omega}(t-\cdot)} f\right\|_{L^p((0,t),X)},$$

and the result follows from the fact that E is a norming space. □

Remark 4.8. (a) We can use Theorem 4.3 to replace (4.6) by the equivalent condition, $\forall\lambda > \delta$, $\forall n \geq 1$,

$$\left\|x^* \circ \left(\lambda - (A + \widehat{\omega}I)\right)^{-n}\right\|_{X^*} \leq \frac{1}{(n-1)!}\int_0^{+\infty} s^{n-1}e^{-\lambda s}\chi_{x^*}(s)\,ds. \tag{4.9}$$

(b) From Thieme [42], we know that

$$(\lambda I - A)^{-1}x = \lambda \int_0^{+\infty} e^{-\lambda s} S_A(s)x\, ds$$

for $\lambda > 0$ sufficiently large. So we can also apply the results of Weis [50] to verify assertion (iii) of Theorem 4.7.

(c) In the Hille-Yosida case, assertions (ii) and (iii) of Theorem 4.7 are satisfied for $q = +\infty$, $E = X_0^*$, and $\chi_{x^*}(s) = M$, $\forall s \geq 0$.

(d) In the context of age-structured problems in L^p spaces the property (iii) holds. But in some cases (see Remark 6.5) we have

$$\|S_{A+\omega I}(t+h) - S_{A+\omega I}(t)\|_{\mathcal{L}(X)} \geq \left(\int_t^{t+h} e^{p\omega l} dl \right)^{1/p}, \ \ \forall t, h \geq 0.$$

So $t \to S_{A+\omega I}(t)$ is not of bounded L^q-variation. Nevertheless, we will see that (4.8) and (4.9) are satisfied. This shows that a dual approach is necessary in general.

5. The Semilinear Problem

In this section we investigate some properties of the non-autonomous semiflow generated by the following equation:

$$U(t,s)x = x + A\int_s^t U(l,s)x\, dl + \int_s^t F(l, U(l,s)x)dl, \ \ t \geq s \geq 0. \quad (5.1)$$

We consider the problem

$$U(t,s)x = T_{A_0}(t-s)x + \frac{d}{dt}(S_A * F(\cdot + s, U(\cdot + s, s)x)(t-s), \ t \geq s \geq 0. \quad (5.2)$$

The results presented here are inspired by the results proved in Cazenave and Haraux [14, Chapter 4] concerning autonomous semilinear equations with dense domain. We also refer to Segal [40] and Weissler [51] for more general results concerning autonomous and non-autonomous densely defined semi-linear equations.

Assumption 5.1. Assume that $A : D(A) \subset X \to X$ is a linear operator satisfying Assumptions 2.1 and 2.2, $C([0, \tau_0], X) \subset Z$, and there exists a map $\delta : [0, \tau_0] \to [0, +\infty)$ such that

$$\Gamma(t, f) \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \ \ \forall t \in [0, \tau_0], \quad \text{and} \quad \lim_{t \to 0^+} \delta(t) = 0.$$

Assume that $F : [0,+\infty) \times \overline{D(A)} \to X$ is a continuous map such that for each $\tau_0 > 0$ and each $\xi > 0$, there exists $K(\tau_0, \xi) > 0$ such that

$$\|F(t,x) - F(t,y)\| \leq K(\tau_0, \xi) \|x - y\|$$

whenever $t \in [0, \tau_0]$, $y, x \in X_0$, and $\|x\| \leq \xi$, $\|y\| \leq \xi$.

First note that without loss of generality we can assume that $\delta(t)$ is non-decreasing. Moreover, using Theorem 3.1 (or direct arguments) and for each $\alpha \in R$ replacing τ_0 by some $\tau_\alpha \in (0, \tau_0)$ such that $\delta(\tau_\alpha) |\alpha| < 1$, we obtain that $A + \alpha I$ satisfies Assumptions 2.1 and 2.2. Replacing A by $A - \omega I$ and $F(t, \cdot)$ by $F(t, \cdot) + \omega I$, we can assume that $\omega = 0$. From now on we assume that $\delta(t)$ is non-decreasing and $\omega = 0$. In the sequel, we will use the norm $|\cdot|$ on X_0 defined by

$$|x| = \sup_{t \geq 0} \|T_{A_0}(t)x\|, \ \forall x \in X_0.$$

Then we have $\|x\| \leq |x| \leq M \|x\|$ and $|T_{A_0}(t)x| \leq |x|$, $\forall x \in X_0$, $\forall t \geq 0$. Remark that by the assumption, for each $f \in C([0, \tau_0], X)$, $\frac{d}{dt}(S_A * f)(t)$ is well defined $\forall t \in [0, \tau_0]$. Let $f \in C^1([0, 2\tau_0], X)$. Then, for $t \in [\tau_0, 2\tau_0]$,

$$\begin{aligned}\frac{d}{dt}(S_A * f)(t) &= \lim_{\mu \to +\infty} \int_0^t T_{A_0}(t-s)\mu(\mu I - A)^{-1} f(s)ds \\ &= \frac{d}{dt}(S_A * f(\cdot + \tau_0))(t - \tau_0) + T_{A_0}(t - \tau_0)\frac{d}{dt}(S_A * f(\cdot))(\tau_0),\end{aligned}$$

so

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq \delta(t - \tau_0) \sup_{l \in [\tau_0, t - \tau_0]} \|f(l)\| + \delta(t - \tau_0) \sup_{l \in [0, \tau_0]} \|f(l)\|.$$

Thus, Assumption 2.2 is satisfied with $Z = C([0, 2\tau_0], X)$; we deduce that $\frac{d}{dt}(S_A * f)(t)$ is well defined for all $t \in [0, 2\tau_0]$ and satisfies the conclusions of Theorem 2.11. By induction, we obtain that for each $\tau_0 > 0$ and each $f \in C([0, \tau_0], X)$, $t \to (S_A * f)(t)$ is continuously differentiable on $[0, \tau_0]$, $(S_A * f)(t) \in D(A), \forall t \in [0, \tau_0]$, and if we denote $u(t) = \frac{d}{dt}(S_A * f)(t)$, then

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0, \tau_0].$$

In the following definition τ is the blow-up time of maximal solutions of (5.1).

Definition 5.1. Consider two maps $\tau : [0, +\infty) \times X_0 \to (0, +\infty]$ and $U : D_\tau \to X_0$, where $D_\tau = \{(t, s, x) \in [0, +\infty)^2 \times X_0 : s \leq t < s + \tau(s, x)\}$. We say that U is *a maximal non-autonomous semiflow on* X_0 if U satisfies the following properties:

(i) $\tau(r, U(r,s)x) + r = \tau(s,x) + s, \forall s \geq 0, \forall x \in X_0, \forall r \in [s, s+\tau(s,x))$.
(ii) $U(s,s)x = x, \forall s \geq 0, \forall x \in X_0$.
(iii) $U(t,r)U(r,s)x = U(t,s)x, \forall s \geq 0, \forall x \in X_0, \forall t, r \in [s, s+\tau(s,x))$ with $t \geq r$.
(iv) If $\tau(s,x) < +\infty$, then $\lim_{t \to (s+\tau(s,x))^-} \|U(t,s)x\| = +\infty$.

Set $D = \{(t,s,x) \in [0,+\infty)^2 \times X_0 : t \geq s\}$. The main result of this section is the following theorem, which is a generalization of Theorem 4.3.4 in [14].

Theorem 5.2. *Let Assumption* 5.1 *be satisfied. Then there exist a map* $\tau : [0,+\infty) \times X_0 \to (0,+\infty]$ *and a maximal non-autonomous semiflow* $U : D_\tau \to X_0$, *such that for each* $x \in X_0$ *and each* $s \geq 0$, $U(\cdot,s)x \in C([s, s+\tau(s,x)), X_0)$ *is a unique maximal solution of* (5.1) (*or equivalently a unique maximal solution of* (5.2)). *Moreover,* D_τ *is open in* D *and the map* $(t,s,x) \to U(t,s)x$ *is continuous from* D_τ *into* X_0.

In order to prove Theorem 5.2 we need some lemmas.

Lemma 5.3. (*Uniqueness*) *Let Assumption* 5.1 *be satisfied. Then for each* $x \in X_0$, *each* $s \geq 0$, *and each* $\tau > 0$, *equation* (5.1) *has at most one solution* $U(\cdot,s)x \in C([s, \tau+s], X_0)$.

Proof. Assume that there exist two solutions of equation (5.1), $u, v \in C([s,\tau+s], X_0)$, with $u(s) = v(s)$. Define $t_0 = \sup\{t \geq s : u(l) = v(l), \forall l \in [s,t]\}$. Then, for each $t \geq t_0$, we have

$$u(t) - v(t) = A\int_{t_0}^{t}[u(l) - v(l)]dl + \int_{t_0}^{t}[F(l,u(l)) - F(l,v(l))]dl.$$

It follows that

$$\begin{aligned}(u-v)(t-t_0+t_0) = A&\int_0^{t-t_0}(u-v)(l+t_0)dl \\ &+ \int_0^{t-t_0}[F(l+t_0, u(l+t_0)) - F(l+t_0, v(l+t_0))]dl;\end{aligned}$$

thus,

$$u(t) - v(t) = \frac{d}{dt}(S_A * (F(\cdot+t_0, u(\cdot+t_0)) - F(\cdot+t_0, v(\cdot+t_0))))(t-t_0).$$

Let $\xi = \max(\|u\|_{\infty,[s,\tau+s]}, \|v\|_{\infty,[s,\tau+s]})$. Thus, we have for each $t \in [t_0, t_0+\tau_0]$ that

$$\|u(t) - v(t)\| \leq \delta(t)K(\tau + s, \xi) \sup_{l\in[t_0,t_0+t]} \|u(l) - v(l)\|.$$

Let $\varepsilon > 0$ be such that $\delta(\varepsilon)K(\tau + s, \xi) < 1$. We obtain that

$$\sup_{l\in[t_0,t_0+\varepsilon]} \|u(l) - v(l)\| \leq \delta(\varepsilon)K(\tau + s, \xi) \sup_{l\in[t_0,t_0+\varepsilon]} \|u(l) - v(l)\|.$$

So $u(t) = v(t)$, $\forall t \in [t_0, t_0 + \varepsilon]$, which gives a contradiction with the definition of t_0. □

Lemma 5.4. (*Local Existence*) *Let Assumption* 5.1 *be satisfied. Then for each* $\tau > 0$, *each* $\beta > 0$, *and each* $\xi > 0$, *there exists* $\gamma(\tau, \beta, \xi) \in (0, \tau_0]$ *such that for each* $s \in [0, \tau]$ *and each* $x \in X_0$ *with* $\|x\| \leq \xi$, *equation* (5.1) *has a unique solution* $U(\cdot, s)x \in C([s, s + \gamma(\tau, \beta, \xi)], X_0)$ *which satisfies*

$$\|U(t, s)x\| \leq (1 + \beta)\xi, \ \forall t \in [s, s + \gamma(\tau, \beta, \xi)].$$

Proof. Let $s \in [0, \tau]$ and $x \in X_0$ with $\|x\| \leq \xi$ fixed. Let $\gamma(\tau, \beta, \xi) \in (0, \tau_0]$ such that

$$\delta(\gamma(\tau, \beta, \xi)) M[1 + \widehat{\xi}_{\tau+\tau_0} + (1 + \beta)\xi K(\tau + \tau_0, (1 + \beta)\xi)] \leq \beta\xi$$

with $\widehat{\xi}_\alpha = \sup_{s\in[0,\alpha]} \|F(s, 0)\|$, $\forall \alpha \geq 0$. Set

$$E = \{u \in C([s, s+\delta(\gamma(\tau, \beta, \xi))], X_0) : \|u(t)\| \leq (1+\beta)\xi, \forall t \in [s, s+\gamma(\tau, \beta, \xi)]\}.$$

Consider the map

$$\Phi_{x,s} : C([s, s + \delta(\gamma(\tau, \beta, \xi))], X_0) \to C([s, s + \delta(\gamma(\tau, \beta, \xi))], X_0)$$

defined for each $t \in [s, s + \delta(\gamma(\tau, p, C))]$ by

$$\Phi_{x,s}(u)(t) = T_{A_0}(t - s)x + \frac{d}{dt}(S_A * F(\cdot + s, u(\cdot + s)))(t - s).$$

We have $\forall u \in E$ that

$$\begin{aligned} |\Phi_{x,s}(u)(t)| &\leq \xi + M\left\|\frac{d}{dt}(S_A * F(\cdot + s, u(\cdot + s)))(t - s)\right\| \\ &\leq \xi + M\delta(\gamma(\tau, \beta, \xi)) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} \|F(t, u(t))\| \\ &\leq \xi + M\delta(\gamma(\tau, \beta, \xi)) \left[\widehat{\xi}_\alpha + K(\tau + \tau_0, (1 + \beta)\xi) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} \|u(t)\|\right] \\ &\leq (1 + \beta)\xi. \end{aligned}$$

Hence, $\Phi_{x,s}(E) \subset E$. Moreover, for all $u, v \in E$, we have

$$\begin{aligned}
&|\Phi_{x,s}(u)(t) - \Phi_{x,s}(v)(t)| \\
&\quad \leq M\delta\left(\gamma\left(\tau,\beta,\xi\right)\right) K(\tau+\tau_0,(1+\beta)\,\xi) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)| \\
&\quad \leq \frac{K(\tau+\tau_0,(1+\beta)\,\xi)\beta\xi}{[1+\widehat{\xi}_\alpha + K(\tau+\tau_0,(1+\beta)\,\xi)\,(1+\beta)\,\xi]} \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)| \\
&\quad \leq \frac{\beta}{1+\beta} \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)|\,.
\end{aligned}$$

Therefore, $\Phi_{x,s}$ is a $(\frac{\beta}{1+\beta})$-contraction on E and the result follows. □

For each $s \geq 0$ and each $x \in X_0$, define

$$\tau\left(s,x\right) = \sup\left\{t \geq 0 : \exists U(\cdot,s)x \in C\left(\left[s,s+t\right],X_0\right) \text{ a solution of (5.1)}\right\}.$$

From Lemma 5.3 we already knew that $\tau\left(s,x\right) > 0,\ \forall s \geq 0,\ \forall x \in X_0$. Moreover, we have the following lemma.

Lemma 5.5. *Let Assumption* 5.1 *be satisfied. Then* $U : D_\tau \to X_0$ *is a maximal non-autonomous semiflow on* X_0.

Proof. Let $s \geq 0$ and $x \in X_0$ be fixed. We first prove assertions (i)-(iii) of Definition 5.1. Let $r \in \left[s,s+\tau\left(s,x\right)\right)$ be fixed. Then, for all $t \in \left[r,s+\tau\left(s,x\right)\right)$,

$$\begin{aligned}
U(t,s)x &= x + A\int_s^t U(l,s)x\,dl + \int_s^t F(l,U(l,s)x)dl \\
&= U(r,s)x + A\int_s^t U(l,s)x\,dl + \int_s^t F(l,U(l,s)x)dl.
\end{aligned}$$

By Lemma 5.3, we obtain that $U(t,s)x = U(t,r)U(r,s)x,\ \forall t \in [r,s+\tau(s,x))$. So $\tau\left(r,U(r,s)x\right) + r \geq \tau\left(s,x\right) + s$. Moreover, if we set

$$v(t) = \begin{cases} U(t,r)U(r,s)x, & \forall t \in \left[r,r+\tau\left(r,U(r,s)x\right)\right), \\ U(t,s)x, & \forall t \in \left[s,r\right], \end{cases}$$

then

$$v(t) = x + A\int_s^t v(l)dl + \int_s^t F(l,v(l))dl,\ \forall t \in \left[s,r+\tau\left(r,U(r,s)x\right)\right].$$

Thus, by the definition of $\tau\left(s,x\right)$ we have $s+\tau\left(s,x\right) \geq r+\tau\left(r,U(r,s)x\right)$ and the result follows.

It remains to prove assertion (iv) of Definition 5.1. Assume that $\tau(s,x) < +\infty$ and that $\|U(t,s)x\| \nrightarrow +\infty$ as $t \nearrow s+\tau(s,x)$. Then we can find a constant $\xi > 0$ and a sequence $\{t_n\}_{n\geq 0} \subset [s, s+\tau(s,x))$, such that $t_n \to s+\tau(s,x)$ as $n \to +\infty$ and $\|U(t_n,s)x\| \leq \xi$, $\forall n \geq 0$. Using Lemma 5.4 with $\tau = [0, s+\tau(s,x)]$ and $\beta = 2$, we know that there exists $\gamma(\tau,\beta,\xi) \in (0,\tau_0]$ for each $n \geq 0$, $t_n + \tau(t_n,x) \geq t_n + \gamma(\tau,\beta,\xi)$. From the first part of the proof we have $s+\tau(s,x) \geq t_n + \gamma(\tau,\beta,\xi)$, and, when $n \to +\infty$, we obtain

$$s+\tau(s,x) \geq s+\tau(s,x)+\gamma(\tau,\beta,\xi),$$

which is impossible since $\gamma(\tau,\beta,\xi) > 0$. □

Lemma 5.6. *Let Assumption* 5.1 *be satisfied. Then the following are satisfied:*

(i) *The map* $(s,x) \to \tau(s,x)$ *is lower semi-continuous on* $[0,+\infty) \times X_0$.
(ii) *For each* $(s,x) \in [0,+\infty) \times X_0$, *each* $\tau \in (0,\tau(s,x))$, *and each sequence* $\{(s_n,x_n)\}_{n\geq 0} \subset [0,+\infty) \times X_0$ *such that* $(s_n,x_n) \to (s,x)$ *as* $n \to +\infty$, *one has*

$$\sup_{l\in[0,\tau]} \|U(l+s_n,s_n)x_n - U(l+s,s)x\| \to 0 \text{ as } n \to +\infty.$$

(iii) $D_\tau = \{(t,s,x) \in [0,+\infty)^2 \times X_0 : s \leq t < s+\tau(s,x)\}$ *is open in* $D = \{(t,s,x) \in [0,+\infty)^2 \times X_0 : t \geq s\}$.
(iv) *The map* $(t,s,x) \to U(t,s)x$ *is continuous from* D_τ *into* X_0.

Proof. Let $(s,x) \in [0,+\infty) \times X_0$ be fixed. Consider a sequence $\{(s_n,x_n)\}_{n\geq 0} \subset [0,+\infty) \times X_0$ satisfying $(s_n,x_n) \to (s,x)$ as $n \to +\infty$. Let $\widehat{\tau} \in (0,\tau(s,x))$ be fixed. Define

$$\xi = 2 \sup_{t\in[s,s+\widehat{\tau}]} \|U(t,s)x\| + 1 > 0$$

and

$$\widehat{\tau}_n = \sup\{t \in (0,\tau(s_n,x_n)) : \|U(l+s_n,s_n)x_n\| \leq 2\xi, \ \forall l \in [0,t]\}.$$

Let $\varepsilon \in (0,\tau_0]$ be such that

$$\xi_1 := \delta(\varepsilon) MK(\widehat{\tau}+\widehat{s}, 2\xi) < 1, \ \widehat{s} = \sup_{n\geq 0} s_n.$$

Set

$$\xi_2^n = \delta(\varepsilon) M \sup_{h\in[0,\widehat{\tau}]} \|F(h+s_n, U(l+s,s)x) - F(h+s, U(l+s,s)x)\| \to 0$$

as $n \to +\infty$. Then, we have for each $l \in [0, \min(\widehat{\tau}_n, \widehat{\tau})]$ and each $r \in [0, l]$ with $l - r \leq \varepsilon$ that

$$\begin{aligned} U(l+s,s)x &= U(l+s,r+s)U(r+s,s)x \\ &= T_{A_0}(l-r)U(r+s,s)x + \frac{d}{dt}(S_A * F(\cdot + r + s, U(\cdot + r + s, s)x)(l-r). \end{aligned}$$

Hence,

$$\begin{aligned} &|U(l+s_n,s_n)x_n - U(l+s,s)x| \\ &\quad = |U(l+s_n,r+s_n)U(r+s_n,s_n)x_n - U(l+s,r+s)U(r+s,s)x| \\ &\quad \leq |T_{A_0}(l-r)\left[U(r+s_n,s_n)x_n - U(r+s,s)x\right]| \\ &\qquad + M\delta(\varepsilon) \sup_{h\in[r,l]} |F(h+s_n, U(h+s_n,s_n)x_n) - F(h+s, U(h+s,s)x)| \\ &\quad \leq |U(r+s_n,s_n)x_n - U(r+s,s)x| \\ &\qquad + \xi_1 \sup_{h\in[r,l]} |U(h+s_n,s_n)x_n - U(h+s,s)x| + \xi_2^n. \end{aligned}$$

Therefore, for each $l \in [0, \min(\widehat{\tau}_n, \widehat{\tau})]$ and each $r \in [0, l]$ with $l - r \leq \varepsilon$,

$$\begin{aligned} &\sup_{h\in[r,l]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \\ &\leq \frac{1}{1-\xi_1}\left[|U(r+s_n,s_n)x_n - U(r+s,s)x| + \xi_2^n\right]. \end{aligned}$$

From this we deduce for $r = 0$ that

$$\sup_{h\in[0,\min(\varepsilon,\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \to 0 \text{ as } n \to +\infty,$$

and by induction we have that

$$\sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \to 0 \text{ as } n \to +\infty. \tag{5.3}$$

It follows that

$$\begin{aligned} &\sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n| \\ &\leq \sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| + \xi. \end{aligned}$$

Since $\xi > 0$, there exists $n_0 \geq 0$ such that $\widehat{\tau}_n > \widehat{\tau}$, $\forall n \geq n_0$, and the result follows by using (5.3).

Now (iii) follows from (i). Moreover, if $(t_n, s_n, x_n) \to (t, s, x)$, then we have

$$\|U(t_n, s_n)x_n - U(t,s)x\| \le \|U((t_n - s_n) + s_n, s_n)x_n - U((t_n - s_n) + s, s)x\| \\ + \|U((t_n - s_n) + s, s)x - U((t - s) + s, s)x\|$$

and by using (ii),

$$\|U(t_n, s_n)x_n - U(t,s)x\| \to 0 \text{ as } n \to +\infty.$$

This proves (iv). □

6. Age-Structured Problems in L^p

In this section we consider the age-structured problems in L^p. Let $(Y, \|\cdot\|_Y)$ be a Banach space, $p \in [1, +\infty)$, and $a_0 \in (0, +\infty]$. We are now interested in solutions $v \in C([0, \tau_0], L^p((0, a_0), Y))$ of the following problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = A(a)v(t,a) + \mathcal{F}(t, v(t))(a), \ a \in (0, a_0), \\ v(t, 0) = \mathcal{K}(t, v(t)), \\ v(0, \cdot) = \psi \in L^p((0, a_0), Y), \end{cases} \tag{6.1}$$

where $\mathcal{K} : [0, \tau_0] \times L^p((0, a_0), Y) \to Y$ and $\mathcal{F} : [0, \tau_0] \times L^p((0, a_0), Y) \to L^p((0, a_0), Y)$ are continuous maps.

In order to apply the results obtained in Sections 2–5 to study the age-structured problem (6.1) in L^p, as in Thieme [43, 44], we assume that the family of linear operators $\{A(a)\}_{0 \le a \le a_0}$ generates an exponentially bounded evolution family $\{U(a,s)\}_{0 \le s \le a < a_0}$. We refer to Kato and Tanabe [27], Acquistapace and Terreni [2], Acquistapace [1], and the monograph of Chicone and Latushkin [15] for further information on evolution families. Then we introduce a closed, bounded operator B based on $\{U(a,s)\}_{0 \le s \le a < a_0}$. Next we rewrite system (6.1) as a Cauchy problem with the linear operator B and show that B generates an integrated semigroup $\{S_B(t)\}_{t \ge 0}$. Now the results in the previous sections can be applied to the problem. A similar argument applies to the general system (1.3).

Definition 6.1. A family of bounded linear operators $\{U(a,s)\}_{0 \le s \le a < a_0}$ on Y is called an *exponentiallyly bounded evolution family* if the following conditions are satisfied:

(a) $U(a, a) = Id_Y$ if $0 \le a < a_0$.
(b) $U(a, r)U(r, s) = U(a, s)$ if $0 \le s \le r \le a < a_0$.

(c) For each $y \in Y$, the map $(a,s) \to U(a,s)y$ is continuous from $\{(a,s) : 0 \leq s \leq a < a_0\}$ into Y.

(d) There exist two constants, $M \geq 1$ and $\omega \in \mathbb{R}$, such that $\|U(a,s)\| \leq Me^{\omega(a-s)}$ if $0 \leq s \leq a < a_0$.

From now on, set $X = Y \times L^p((0,a_0),Y)$ and $X_0 = \{0_Y\} \times L^p((0,a_0),Y)$ endowed with the product norm

$$\left\| \begin{pmatrix} y \\ \psi \end{pmatrix} \right\| = \|y\|_Y + \|\psi\|_{L^p((0,a_0),Y)}.$$

Define for each $\lambda > \omega$, $J_\lambda : X \to X_0$ a linear operator defined by

$$J_\lambda \begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \Leftrightarrow$$
$$\varphi(a) = e^{-\lambda a}U(a,0)y + \int_0^a e^{-\lambda(a-s)}U(a,s)\psi(s)ds,\ a \in (0,a_0).$$

Lemma 6.2. *Assume that* $\{A(a)\}_{0 \leq a \leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0 \leq s \leq a < a_0}$. *Then there exists a unique closed linear operator* $B : D(B) \subset X \to X$ *such that* $(\omega,+\infty) \subset \rho(B)$, $J_\lambda = (\lambda I - B)^{-1}$, $\forall \lambda > \omega$, *and* $\overline{D(B)} = X_0$.

Proof. It is straightforward to check that J_λ is a pseudo resolvent on $(\omega,+\infty)$ (i.e., $J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu,\ \forall \lambda, \mu \in (\omega,+\infty)$). By construction we have $R(J_\lambda) \subset X_0$. Moreover, let $x = \begin{pmatrix} y \\ \psi \end{pmatrix} \in X$ and assume that $J_\lambda x = 0$. Then, for $a \in (0,a_0)$

$$I_a := \frac{1}{a}\int_0^a \left\| e^{-\lambda\xi}U(\xi,0)y + \int_0^\xi e^{-\lambda(\xi-s)}U(\xi,s)\psi(s)ds \right\| d\xi = 0$$

and $\lim_{a\to 0^+} I_a = \|y\|$. So $y = 0$ and $N(J_\lambda) \subset X_0$. Moreover, using Young's inequality, we have for all $\lambda > \omega$ that

$$\begin{aligned} \left\| J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| &\leq M \left\| \left(e^{(-\lambda+\omega)\cdot} * \|\psi(\cdot)\| \right)(\cdot) \right\|_{L^p((0,a_0),\mathbb{R})} \\ &\leq M \left\| e^{(-\lambda+\omega)\cdot} \right\|_{L^1((0,a_0),\mathbb{R})} \|\psi\|_{L^p((0,a_0),Y)}, \end{aligned}$$

so

$$\left\| J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| \leq \frac{M}{\lambda - \omega} \|\psi\|_{L^p((0,a_0),Y)}.$$

Moreover, we can prove that $\forall \psi \in C_c^0\left((0,a_0),Y\right)$,

$$\lim_{\lambda\to+\infty} \lambda J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$

By the density of $C_c^0\left((0,a_0),Y\right)$ in $L^p\left((0,a_0),Y\right)$, we obtain that

$$\lim_{\lambda\to+\infty} \lambda J_\lambda x = x, \quad \forall x \in X_0,$$

and by using a standard argument (see Yosida [54], Section VIII.4), the result follows. $\square$

Define $F : [0,+\infty) \times X_0 \to X$ by

$$F\left(t, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} \mathcal{K}(t,\varphi) \\ \mathcal{F}(t,\varphi) \end{pmatrix}$$

and denote

$$u = \begin{pmatrix} x \\ v \end{pmatrix}.$$

Consider equation (6.1) as the following Cauchy problem:

$$\frac{du}{dt} = Bu(t) + F(t,u(t)), \quad t \geq 0, \quad u(0) = x \in X_0. \tag{6.2}$$

Lemma 6.3. *Assume that* $\{A(a)\}_{0\leq a\leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\leq s\leq a<a_0}$. *Then* B *satisfies Assumption* 2.1.

Proof. One can check that

$$\left\| (\lambda I - B)^{-1} \begin{pmatrix} y \\ 0 \end{pmatrix} \right\| \leq \frac{M}{p^{1/p}\left(\lambda-\omega\right)^{1/p}} \left\| y \right\|, \ \forall \lambda > \omega.$$

Using the Young inequality we have

$$\left\| (\lambda I - B)^{-k} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right\| \leq \frac{M}{\left(\lambda-\omega\right)^k} \left\| \varphi \right\|_{L^p((0,a_0),Y)}, \ \forall \lambda > \omega, \ \forall k \geq 1.$$

This completes the proof. $\square$

Now we can claim that B_0 (the part of B in X_0) generates a C_0-semigroup $\{T_{B_0}(t)\}_{t\geq 0}$ and B generates an integrated semigroup $S_B(t)$.

We obtain usual formula for $T_{B_0}(t)$ and $S_B(t)$ (see Thieme [43, 44]).

Lemma 6.4. *Assume that* $\{A(a)\}_{0\leq a\leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\leq s\leq a<a_0}$. *Then* $\{T_{B_0}(t)\}_{t\geq 0}$, *the* C_0*-semigroup generated by* B_0 *(the part of* B *in* X_0*), is defined by*

$$T_{B_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{B_0}(t)\varphi \end{pmatrix}$$

with

$$\widehat{T}_{B_0}(t)\left(\varphi\right)(a) = \begin{cases} U(a,a-t)\varphi(a-t) & \text{if } a \geq t, \\ 0 & \text{if } a\in[0,t]. \end{cases}$$

Moreover, $\{S_B(t)\}_{t\geq 0}$, *the integrated semigroup generated by* B, *is defined by*

$$S_B(t)\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ W(t)y + \int_0^t \widehat{T}_{B_0}(s)\varphi\, ds \end{pmatrix}$$

with

$$W(t)\left(y\right)(a) = \begin{cases} U(a,0)y & \text{if } a\leq t, \\ 0 & \text{if } a\geq t. \end{cases}$$

Proof. If $T_{B_0}(t)$ and $S_B(t)$ are defined by the above formulas, then it is readily checked that

$$\frac{d}{dt}\left(\lambda I-A\right)^{-1}T_{B_0}(t)x = \lambda\left(\lambda I - A\right)^{-1}T_{B_0}(t)x - T_{B_0}(t)x$$

and

$$\frac{d}{dt}\left(\lambda I-A\right)^{-1}S_B(t)x = \lambda\left(\lambda I-A\right)^{-1}S_B(t)x - S_B(t)x + \left(\lambda I - A\right)^{-1}x.$$

Assertion (i) of Lemma 2.10 is satisfied, and the result follows. □

Remark 6.5. If we choose $U(a,s)=e^{\omega(a-s)}Id_Y$, $\forall a,s\in[0,a_0)$ with $a\geq s$, then we have for $a,s\in[0,a_0)$ with $a\geq s$ that

$$\left\| S_B(a)\begin{pmatrix} y \\ 0 \end{pmatrix} - S_B(s)\begin{pmatrix} y \\ 0 \end{pmatrix}\right\| = \left(\int_s^a e^{p\omega l}dl\right)^{1/p}\|y\|.$$

This example shows that the dual approach is necessary in this context (see Remark 4.8(d) following Theorem 4.7).

Define $P: X\to X$ by

$$Px = \begin{pmatrix} y \\ 0 \end{pmatrix}, \ \forall x = \begin{pmatrix} y \\ \varphi \end{pmatrix}\in X$$

and set $X_1 = Y\times\left\{0_{L^p((0,a_0),Y)}\right\}$. We obtain the following theorem.

Theorem 6.6. *Assume that* $\{A(a)\}_{0\leq a\leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\leq s\leq a<a_0}$. *Then for each* $f\in L^p((0,\tau_0),X_1)\oplus L^1((0,\tau_0),X_0)$ *and each* $x\in\overline{D(B)}$, *there exists* $u\in C([0,\tau_0],\overline{D(B)})$, *a unique integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt}=Bu(t)+f(t),\ \ t\in[0,\tau_0]\,,\ u(0)=x, \tag{6.3}$$

given by

$$u(t)=T_{B_0}(t)x+\frac{d}{dt}\left(S_B*f\right)(t),\ \forall t\in[0,\tau_0]\,, \tag{6.4}$$

which satisfies for a certain $\widehat{M}>0$ *that is independent of* τ_0,

$$\begin{aligned}\|u(t)\| &\leq Me^{\omega t}\|x\|+\widehat{M}\left(\int_0^t\left(e^{\omega(t-s)}\|Pf(s)\|\right)^p ds\right)^{1/p}\\ &\quad +M\int_0^t e^{\omega(t-s)}\|(I-P)f(s)\|\,ds,\ \ \forall t\in[0,\tau_0]\,.\end{aligned}$$

Moreover,

$$u(t)=T_{B_0}(t)x+\begin{pmatrix}0\\ w(t)\end{pmatrix},\ \forall t\in[0,\tau_0] \tag{6.5}$$

with

$$w(t)(a)=\begin{cases}U(a,0)Pf(t-a)+\left(\int_0^t\widehat{T}_0(t-s)(I-P)f(s)ds\right)(a) & \text{if } a\leq t,\\ \left(\int_0^t\widehat{T}_0(t-s)(I-P)f(s)ds\right)(a) & \text{if } a\geq t.\end{cases}$$

Proof. Let $\psi\in C_c^\infty\left((0,a_0),Y^*\right)$ be fixed. We defined $x^*\in X_0^*$ by

$$x^*\begin{pmatrix}0\\ \varphi\end{pmatrix}=\int_0^{a_0}\psi(s)(\varphi(s))ds.$$

Let $x=\begin{pmatrix}y\\ \varphi\end{pmatrix}\in X$; we have

$$x^*\left((\lambda I-B)^{-1}Px\right)+x^*\left((\lambda I-B)^{-1}(I-P)x\right)$$

and

$$x^*\left((\lambda I-B)^{-1}(I-P)x\right)=\int_0^{+\infty}e^{(-\lambda+\omega)t}x^*\left(e^{-\omega t}T_{B_0}(t)\,(I-P)\,x\right)dt,$$

and for each $\lambda>\omega$ that

$$x^*\left((\lambda I-B)^{-1}P\begin{pmatrix}y\\ \varphi\end{pmatrix}\right)\ =\ \int_0^{a_0}e^{-\lambda a}\psi(a)\left(U(a,0)y\right)da$$

$$= \int_0^{+\infty} e^{(-\lambda+\omega)t} W_{x^*}(t)(y)\,dt$$

with

$$W_{x^*}(t)(y) = \begin{cases} e^{-\omega t}\psi(t)U(t,0)y & \text{if } 0 \le t < a_0, \\ 0 & \text{if } t \ge a_0. \end{cases}$$

$$x^*\left((\lambda I - B)^{-n} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right) = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}} x^*\left((\lambda I - B)^{-1} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right)$$
$$= \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1} e^{(-\lambda+\omega)t} W_{x^*}(t)(y)\,dt.$$

So

$$\left| x^*\left((\lambda I - B)^{-n} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right)\right| \le \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1} e^{-\lambda t} \chi_{x^*}(t)\,dt \, \|y\|_Y ,$$

where

$$\chi_{x^*}(t) = \begin{cases} M\,\|\psi(t)\|_{Y^*} & \text{if } t \in (0, a_0) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Lemma 6.3, Proposition 4.1-ii), Theorem 4.7, and Remark 4.8-a), we can define $u(t)$ by (6.4), which is an integrated solution of (6.3), and by using Lemma 2.10, we deduce that $u(t)$ satisfies (6.5). □

Assumption 6.1. The maps $\mathcal{K} : [0,+\infty) \times L^p((0,a_0), Y) \to Y$ and $\mathcal{F} : [0,+\infty) \times L^p((0,a_0), Y) \to L^p((0,a_0), Y)$ are continuous, and for each $\tau > 0$ and each $\xi > 0$, there exists $K(\tau,\xi) > 0$ such that

$$\|\mathcal{K}(t,\varphi) - \mathcal{K}(t,\psi)\| \le K(\tau,\xi)\,\|\varphi - \psi\|,$$
$$\|\mathcal{F}(t,\varphi) - \mathcal{F}(t,\psi)\| \le K(\tau,\xi)\,\|\varphi - \psi\|$$

whenever $t \in [0,\tau]$, $\varphi, \psi \in L^p((0,a_0), Y)$, $\|\varphi\| \le \xi$, $\|\psi\| \le \xi$.

From the above assumption, it follows that F satisfies the second part of Assumption 5.1, and we obtain the following theorem.

Theorem 6.7. *Let Assumption* 6.1 *be satisfied and assume that* $\{A(a)\}_{0\le a\le a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\le s\le a<a_0}$. *Then there exist a map* $\tau : [0,+\infty)\times X_0 \to (0,+\infty]$ *and a maximal non-autonomous semiflow* $U : D_\tau \to X_0$ *on* X_0, *such that for each* $x \in X_0$ *and each* $s \ge 0$, $U(\cdot,s)x \in C([s, s+\tau(s,x)), X_0)$ *is a unique maximal solution of*

$$U(t,s)x = x + B\int_s^t U(l,s)x\,dl + \int_s^t F(l, U(l,s)x)\,dl,\ \forall t \in [s, s+\tau(s,x)),$$

or equivalently of

$$U(t,s)x = T_{B_0}(t-s)x + \frac{d}{dt}\left(S_B * \left(F\left(\cdot + s, U(\cdot + s, s)x\right)\right)\right)(t-s),$$

$\forall t \in [s, s+\tau(s,x))$. *Moreover,* D_τ *is open in* D *and the map* $(t,s,x) \to U(t,s)x$ *is continuous from* D_τ *into* X_0.

Let Z be a Banach space and $H : D(H) \subset Z \to Z$ be a Hille-Yosida operator. Then equation (1.3) can be rewritten as

$$\frac{du}{dt} = Au(t) + F(t, u(t)),$$

where $A : D(A) \subset X \times Z \to X \times Z$, $F : [0, +\infty) \times X \times Z \to X \times Z$, and

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} Bu_1 \\ Hu_2 \end{pmatrix},\ \forall \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D(A) = D(B) \times D(H).$$

The problem is similar to the one we just studied.

7. Neutral Delay Differential Equations in L^p

In this section, we show how to treat neutral delay differential equations in L^p as a special case of the age-structured models in L^p spaces. Early work on delay differential equations in L^p spaces using semigroup methods was due to Hale [24] and Webb [47, 48]. We refer to Wu [52] and Batkai and Piazzera [10] for more results and references on this subject.

Consider the neutral delay differential equation

$$\begin{cases} \frac{d}{dt}\left(x(t) - G_1(t, x_t)\right) = H\left(x(t) - G_1(t, x_t)\right) + G_2(t, x_t), t \geq 0, \\ x(0) = \widehat{x} \in Z,\ x_0 = \varphi \in L^p\left((-\tau, 0), Z\right), \\ y_0 := \widehat{x} - G_1(0, \phi) \in \overline{D(H)}. \end{cases} \tag{7.1}$$

This type of neutral delay differential equation in the space of continuous maps $C\left([-\tau, 0], Z\right)$ has been considered by some researchers; see, for example, Adimy and Ezzinbi [3]. As usual in the context of delay differential equations, the map $x_t \in L^p\left((-\tau, 0), Z\right)$ is defined as

$$x_t(\theta) = x(t+\theta) \text{ for almost every } \theta \in (-\tau, 0).$$

Then we can consider the solution of (7.1) as

$$\left(x(t) - G_1(t, x_t)\right) = T_{H_0}(t)y_0 + \frac{d}{dt}\int_0^t S_c(t-s)G_2(s, x_s)ds, \quad t \geq 0,$$

where $\{T_{H_0}(t)\}_{t\geq 0}$ is a linear semigroup generated by H_0, the part of H in $Z_0 := \overline{D(H)}$, and $\{S_H(t)\}_{t\geq 0}$ is the integrated semigroup generated by H. Set

$$y(t) = T_{H_0}(t)y_0 + \frac{d}{dt}\int_0^t S_c(t-s)G_2(s,x_s)ds, \quad t \geq 0.$$

Then we obtain $x(t) = G_1(t, x_t) + y(t), \quad t \geq 0$.

Now transform this problem into an age-structured problem. Define $J : L^p((-\tau, 0), Z) \to L^p((0,\tau), Z)$ by $J(\varphi)(a) = \varphi(-a)$. Clearly, J is invertible and $J^{-1}(\varphi)(-a) = \varphi(a)$. Set $v(t) = Jx_t$ for $t \geq 0$. The neutral delay differential equation becomes

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = 0 \text{ for almost every } a \in (0,\tau), \\ v(t,0) = K_1(t, v(t)) + y(t), \\ \dfrac{dy(t)}{dt} = Hy(t) + K_2(t, v(t)), \\ y(0) = y_0 \in \overline{D(H)}, \quad v(0,\cdot) = \psi = J\varphi \in L^p((0,\tau), Z), \end{cases} \tag{7.2}$$

where $K_i(t,\psi) = G_i\left(t, J^{-1}\psi\right), \; i = 1, 2$.

We can see that the class of neutral delay differential equations described by (7.1) corresponds to a special case of the age-structured model. Moreover, when $K_1 = 0$ the problem is similar to the one considered by Batkai and Piazzera [10]. The problem here is completely different compared with [10] when $K_1 \neq 0$, because we must consider the operator

$$A\begin{pmatrix} 0 \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' \\ Hy \end{pmatrix}, \quad L\begin{pmatrix} 0 \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix},$$

where $D(A) = \{0_Z\} \times W^{1,p}((0,\tau), Z) \times D(H)$, $D(L) = X_0$, and $X = Z \times L^p((0,\tau), Z) \times Z$. When $K_1 = 0$ it is sufficient to consider $(A+L)_0$, the part of $(A+L)$ in $\overline{D(A)}$. In fact, here $(A+L)_0$ is a Hille-Yosida operator, so the problem can be studied by using classical semigroup theory. When $p > 1$, A is not a Hille-Yosida operator; we need to investigate the following Cauchy problem:

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad t \in [0,T], \; u(0) = x \in \overline{D(A)}. \tag{7.3}$$

When $p > 1$, this problem has a unique integrated solution whenever $f \in L^p((0,T), X)$.

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