Czechoslovak Mathematical Journal

Ivan Dobrakov

On integration in Banach spaces, III

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 3, 478–499

Persistent URL: `http://dml.cz/dmlcz/101628`

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

ON INTEGRATION IN BANACH SPACES, III

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(Received January 23, 1978)

INTRODUCTION

Let T and S be non empty sets and let $\mathscr{P}$ and $\mathscr{Q}$ be δ-rings of subsets of T and S, respectively. Let X, Y and Z be real or complex Banach spaces, and let $\boldsymbol{m} : \mathscr{P} \to L(X, Y)$ and $\boldsymbol{l} : \mathscr{Q} \to L(Y, Z)$ be two operator valued measures countably additive in the strong operator topologies with finite semivariations $\boldsymbol{m}^\wedge$ and $\boldsymbol{l}^\wedge$. In this part of our theory of integration we investigate the existence of the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$, countably additive in the strong operator topology, and the validity of a Fubini type theorem for $\mathscr{P} \otimes \mathscr{Q}$ – measurable functions $\boldsymbol{f} : T \times S \to X$. Here $\mathscr{P} \otimes \mathscr{Q}$ denotes the smallest δ-ring containing all rectangles $A \times B$, $A \in \mathscr{P}$, $B \in \mathscr{Q}$, and $(\boldsymbol{l} \otimes \boldsymbol{m})(A \times B) = \boldsymbol{l}(B)\,\boldsymbol{m}(A)$. The main results of the paper, namely Theorems 1 and 15, were announced in [9].

In Theorem 1 we prove that the most natural condition: "for each $E \in \mathscr{P} \otimes \mathscr{Q}$ and each $\boldsymbol{x} \in X$ the function $s \to \boldsymbol{m}(E^s)\,\boldsymbol{x}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$", is necessary and sufficient for the existence of the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$, and that if it is satisfied, then $(\boldsymbol{l} \otimes \boldsymbol{m})(E)\,\boldsymbol{x} = \int_S \boldsymbol{m}(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l}$ for each $E \in \mathscr{P} \otimes \mathscr{Q}$ and each $\boldsymbol{x} \in X$. As a consequence, in Theorem 3 we prove that the continuity of the semivariation $\boldsymbol{l}^\wedge$ on $\mathscr{Q}(B_n \in \mathscr{Q},\ B_n \searrow \emptyset \Rightarrow \boldsymbol{l}^\wedge(B_n) \searrow 0$, see the $*$-Theorem in Section 1.1 in [6]) is sufficient for the existence of the product measure $\boldsymbol{l} \otimes \boldsymbol{m}$ on $\mathscr{P} \otimes \mathscr{Q}$, and the continuity of $\boldsymbol{l}^\wedge$ on $\mathscr{Q}$ and $\boldsymbol{m}^\wedge$ on $\mathscr{P}$ imply the continuity of $\widehat{(\boldsymbol{l} \otimes \boldsymbol{m})}$ on $\mathscr{P} \otimes \mathscr{Q}$. Results similar to Theorem 3 were obtained by different approaches and in various settings by M. DUCHOŇ in [10]–[16] and CH. SWARTZ in [28], [29] and [30], see also [2], [4], [17], [18], [25], [28] and [32].

Using Theorem 1, in Theorems 4 and 5 we establish the validity of the Fubini theorem for functions which are uniform limits of $\mathscr{P} \otimes \mathscr{Q}$ – simple functions, particularly for elements of $C_0(T \times S, X)$.

Let the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$ exist and let the function $\boldsymbol{f} : T \times S \to X$ be integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$. Then, as the very simple example at the beginning of § 2 shows, the function $t \to \boldsymbol{f}(t, s)$, $t \in T$, need not be intcgrable with respect to $\boldsymbol{m}$ for any $s \in S$, even if the variations of both $\boldsymbol{m}$ and $\boldsymbol{l}$ are bounded. Hence in a general Fubini type theorem we must suppose that for each $s \in S$ the function $t \to f(t, s)$, $t \in T$, is integrable with respect to m. Adopting this assumption, our main task is to establish the $\mathcal{Q}$-measurability of the partial integral g_E, $g_E(s) = \int_{E^s} f(\cdot, s)\, dm$, $s \in S$, for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$. Although the author did not succeed in solving this problem in general, in § 2 we establish the $\mathcal{Q}$-measurability of g_E in the following important cases: 1) the semivariation $m^\wedge$ is continuous on $\mathcal{P}$ (Theorem 9), 2) Y is a separable Banach space (Theorem 10), and 3) $\mathcal{P}$ is generated by a countable family (Theorem 12). Further we prove the l-essential $\mathcal{Q}$-measurability of g_E, see Definition 2, which is also sufficient, in the following important cases: 4) Z is separable or is a dual of a separable Banach space, and 5) l is countably additive in the uniform operator topology on $\mathcal{Q}$, see Theorems 13 and 14. Note that case 5) includes the following important subcase 6): $l : \mathcal{Q} \to L(Y, Z)$ is given by an equality $l(B)\, y = u(y, \gamma(B))$, where $u : Y \times Z_1 \to Z$, Z_1 being a Banach space, is a separately continuous bilinear map and $\gamma : \mathcal{Q} \to Z_1$ is a countably additive vector measure. Indeed, by the Uniform Boundedness Principle u is bounded on $Y \times Z_1$, hence $l : \mathcal{Q} \to L(Y, Z)$ is countably additive in the uniform operator topology.

Assuming the integrability of $f(\cdot, s)$ with respect to m for each $s \in S$, and the l-essential $\mathcal{Q}$-measurability of g_E for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$, in § 3 we prove the Fubini theorem and an important special case of it. This special case includes the recent results of Theorems 8 and 9 from [16], where the integral of R. G. BARTLE [3] is used.

Let $\mathcal{D}$ be a δ-ring of subsets of S. We say that $g : S \to Y$ is $\mathcal{D}$-measurable, if there is a sequence g_n, $n = 1, 2, \ldots$ of $\mathcal{D}$-simple functions (on S with values in Y) such that $g_n(s) \to g(s)$ for each $s \in S$. In addition to the information about this measurability given in § 1 in Part I (from now on [6] will be referred to as Part I and [7] as Part II) see also [24]. If $g : S \to Y$ is integrable with respect to $l : \mathcal{Q} \to L(Y, Z)$, then by $\int_S g\, dl$ we understand the integral $\int_D g\, dl$, where $D = \{s \in S;\ g(s) \neq 0\} \in \mathfrak{S}(\mathcal{Q})$.

We note that a nice and deep Radon-Nikodym theorem for our integral was proved by H. B. MAYNARD in [26, Theorem 5].

As is well known, to each countably additive vector measure on a σ-ring there is a finite non negative countably additive measure on that σ-ring with the same zero sets; for a short proof see [20, Theorem 3.10]. Such a measure is called *a control measure* for the given vector measure.

Correction to Part I. In the definition of μ in the proof of Theorem 1 in Part I the vector measures $E \to \int_E f_n\, dm$, $E \in \mathfrak{S}(\mathcal{P})$, $n = 1, 2, \ldots$, must be replaced by their control measures.

1. PRODUCTS OF OPERATOR VALUED MEASURES

We shall use the notation and terminology introduced in Parts I and II and in Introduction. Let $\mathcal{P}_0$ and $\mathcal{Q}_0$ be δ-rings of subsets of T and S, respectively, and let $m : \mathcal{P}_0 \to L(X, Y)$ and $l : \mathcal{Q}_0 \to L(Y, Z)$ be operator valued measures countably additive in the strong operator topologies. Then $\mathscr{P}$ denotes the greatest δ-subring of $\mathscr{P}_0$ where the semivariation $\boldsymbol{m}^\wedge$ is finite. By $\mathscr{P}_2$ we denote the greatest δ-subring of $\mathscr{P}_0$ where $\boldsymbol{m}$ is countably additive in the uniform operator topology, and by $\mathscr{P}^\sim$ we denote the greatest δ-subring of $\mathscr{P}_0$ (equivalently, of $\mathscr{P}$, see Corollary of Theorem 5 in Part II), where the semivariation $\boldsymbol{m}^\wedge$ is continuous. Similarly we have $\mathscr{Q}$, $\mathscr{Q}_2$ and $\mathscr{Q}^\sim$.

For a class of sets $\mathscr{A}$, we denote by $\mathfrak{S}(\mathscr{A})$ the smallest σ-ring containing $\mathscr{A}$, which we call the σ-ring generated by $\mathscr{A}$. Similarly we have $\delta(\mathscr{A})$, the σ-ring generated by $\mathscr{A}$. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are δ-rings of subsets of T and S, respectively, then clearly $\mathfrak{S}(\mathscr{D}_1 \otimes \mathscr{D}_2) = \mathfrak{S}(\mathscr{D}_1) \otimes \mathfrak{S}(\mathscr{D}_2)$. Further, for each $E \in \delta(\mathscr{D}_1 \otimes \mathscr{D}_2)$ there are $A \in \mathscr{D}_1$ and $B \in \mathscr{D}_2$ such that $E \subset A \times B$. Finally, for $E \subset T \times S$ and $s \in S$ we put $E^s = \{t \in T; (t, s) \in E\}$.

Before proceeding to the next definition we note that the Hahn-Banach theorem and the uniqueness of the extension of a finite scalar measure from a ring to the generated σ-ring, see [21, § 13], imply that if $\boldsymbol{n}_1, \boldsymbol{n}_2 : \mathscr{P}_0 \otimes \mathscr{Q}_0 \to L(\boldsymbol{X}, \boldsymbol{Z})$ are two operator valued measures countably additive in the strong operator topologies such that $\boldsymbol{n}_1(A \times B) = \boldsymbol{n}_2(A \times B)$ for each $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}_0$, then they are identical on $\mathscr{P}_0 \otimes \mathscr{Q}_0$ (Theorem E in § 33 and Theorem D in § 13 in [21] are also used).

Definition 1. We say that the *product of measures* $\boldsymbol{m} : \mathscr{P}_0 \to L(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{l} : \mathscr{Q}_0 \to L(\boldsymbol{Y}, \boldsymbol{Z})$ exists on $\mathscr{P}_0 \otimes \mathscr{Q}_0$, if there is a necessarily unique $L(\boldsymbol{X}, \boldsymbol{Z})$ valued measure countably additive in the strong operator topology on $\mathscr{P}_0 \otimes \mathscr{Q}_0$, which we denote by $\boldsymbol{l} \otimes \boldsymbol{m}$, such that $(\boldsymbol{l} \otimes \boldsymbol{m})(A \times B) = \boldsymbol{l}(B)\, \boldsymbol{m}(A)$ for each $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}_0$.

Lemma 1. *For each $x \in X$ let there be a countably additive Z-valued vector measure $\boldsymbol{\mu}_x$ on $\mathscr{P}_0 \otimes \mathscr{Q}$ such that $\boldsymbol{\mu}_x(A \times B) = \boldsymbol{l}(B)\, \boldsymbol{m}(A)\, x$ for each $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}$. Then the product measure $\boldsymbol{l} \otimes \boldsymbol{m}$ exists on $\mathscr{P}_0 \otimes \mathscr{Q}$.*

Proof. For $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and $\boldsymbol{x} \in \boldsymbol{X}$ put $(\boldsymbol{l} \otimes \boldsymbol{m})(E)\, x = \boldsymbol{\mu}_x(E)$. We have to prove

(a) $\boldsymbol{\mu}_{\alpha x_1 + \beta x_2}(E) = \alpha \,.\, \boldsymbol{\mu}_{x_1}(E) + \beta \,.\, \boldsymbol{\mu}_{x_2}(E)$, and

(b): $\lim_{x \to 0} \boldsymbol{\mu}_x(E) = 0$, $x \in X$, for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$, all $x_1, x_2 \in X$ and all scalars α, β.

Denote by $\mathscr{R}$ the ring of all finite unions of pairwise disjoint rectangles $A \times B$, $A \in \mathscr{P}_0$, $B \in \mathscr{Q}$, see Theorem E in § 33 in [21]. We shall need the following fact:

(c): Let $z^* \in Z^*$ and let $E \in \mathscr{P}_0 \otimes \mathscr{Q}$. Then the obvious inequality $|z^* \boldsymbol{\mu}_x(E_1) - z^* \boldsymbol{\mu}_x(E_2)| \leqq v(z^* \boldsymbol{\mu}_x, E_1 \,\Delta\, E_2)$, $E_1, E_2 \in \mathscr{P}_0 \otimes \mathscr{Q}$, and Theorem D in § 13 in [21] imply that for each $\varepsilon > 0$ there is a set $F \in \mathscr{R}$ such that $|z^* \boldsymbol{\mu}_x(E) - z^* \boldsymbol{\mu}_x(F)| < \varepsilon$.

Let α, β and x_1, x_2 be given. Then (a) is true for $E \in \mathscr{R}$, since $\boldsymbol{\mu}_x(A \times B) = \boldsymbol{l}(B)\, \boldsymbol{m}(A)\, x$ for each $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}$, the values of $\boldsymbol{l}$ and $\boldsymbol{m}$ are linear operators and $\boldsymbol{\mu}_x$ is additive. Thus by (c) and the Hahn-Banach theorem (a) is true for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$.

To prove (b), let $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and take $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}$ so that $E \subset A \times B$. Let $F \in \mathscr{R} \cap (A \times B)$. Without loss of generality we may suppose that $F = \bigcup_{i=1}^{r} (A_i \times B_i)$, $A_i \in \mathscr{P}_0$, $B_i \in \mathscr{Q}$, $i = 1, \ldots, r$, with pairwise disjoint B_i. But then $|z^* \mu_x(F)| \leqq |\mu_x(F)| = |\sum_{i=1}^{r} \mu_x(A_i \times B_i)| = |\sum_{i=1}^{r} l(B_i)\, m(A_i)\, x| \leqq |x| \,.\, \|m\|\,(A)\,.\, l^{\wedge}(B)$ for each $z^* \in Z^*$ with $|z^*| \leqq 1$. Since $B \in \mathscr{Q}$, we have $l^{\wedge}(B) < +\infty$. By Uniform Boundedness Principle we conclude $\|m\|\,(A) = \sup_{|x| \leqq 1} \|m(\cdot)\, x\|\,(A) = \sup_{|x| \leqq 1} \sup_{|y^*| \leqq 1} v(y^* m(\cdot)\, x, A) < +\infty$. Thus $\lim_{x \to 0} |z^* \mu_x(F)| = 0$ uniformly for $F \in \mathscr{R} \cap (A \times B)$ and $z^* \in Z^*$ with $|z^*| \leqq 1$, hence using (c) we easily obtain (b) for each E.

Lemma 2. *Let $\mathscr{D}$ be a δ-ring of subsets of S. Then:*

1) *for each $E \in \mathscr{P}_0 \otimes \mathscr{D}$ and each $x \in X$ the function $s \to m(E^s)\, x$, $s \in S$, is bounded and $\mathscr{D}$-measurable,*

2) *for each $E \in \mathscr{P}_2 \otimes \mathscr{Q}$ the function $s \to \|m(E^s)\|$, $s \in S$, is bounded and $\mathscr{D}$-measurable, and*

3) *for each $E \in \mathscr{P}^{\sim} \otimes \mathscr{Q}$ the function $s \to m^{\wedge}(E^s)$, $s \in S$, is bounded and $\mathscr{D}$-measurable.*

Proof. 1) Let $E \in \mathscr{P}_0 \otimes \mathscr{D}$ and let $x \in X$. Take $A \in \mathscr{P}_0$ and $B \in \mathscr{D}$ so that $E \subset A \times B$, and denote by $\mathscr{M}$ the class of all sets $M \in \mathscr{P}_0 \otimes \mathscr{D} \cap (A \times B)$ for which 1) holds. Then clearly $\mathscr{M}$ contains the ring $\mathscr{R} \cap (A \times B)$, where $\mathscr{R}$ is the ring of all finite unions of pairwise disjoint rectangles $A_1 \times B_1$, $A_1 \in \mathscr{P}_0$, $B_1 \in \mathscr{D}$. Since $\sup_{s \in S} |m(M^s)\, x| \leqq \|m(\cdot)\, x\|\,(A) < +\infty$ for each $M \in \mathscr{M}$, and since the $\mathscr{D}$-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I and Lemma 1,2 in [24], the countable additivity of $m(\cdot)\, x$ on $\mathscr{P}_0$ implies that $\mathscr{M}$ is a monotone class. Thus $\mathscr{M} = \mathscr{P}_0 \otimes \mathscr{D} \cap (A \times B)$ by Theorem B in § 6 in [21], hence $E \in \mathscr{M}$.

2) and 3) may be proved similarly using the continuity and finiteness of the semivariations $\|m\|$ on $\mathscr{P}_2$ and $m^{\wedge}$ on $\mathscr{P}^{\sim}$, respectively.

Theorem 1. *The product measure $l \otimes m : \mathscr{P}_0 \otimes \mathscr{Q} \to L(X, Z)$ exists if and only if for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and each $x \in X$ the function $s \to m(E^s)\, x$, $s \in S$, is integrable with respect to l. In that case*

$$(1) \qquad (l \otimes m)(E)\, x = \int_S m(E^s)\, x \, \mathrm{d}l$$

for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and each $x \in X$.

Proof. Suppose that the product measure $l \otimes m : \mathscr{P}_0 \otimes \mathscr{Q} \to L(X, Z)$ exists and let $x \in X$. Denote by $\mathscr{D}$ the class of all sets $D \in \mathscr{P}_0 \otimes \mathscr{Q}$ for which the function

$s \to \boldsymbol{m}(\mathscr{D}^s)\, \boldsymbol{x}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$ and for which the equation (1) is valid. Then clearly $\mathscr{D}$ is a subring of $\mathscr{P}_0 \otimes \mathscr{Q}$ which contains all rectangles $A \times B$, $A \in \mathscr{P}_0$, $B \in \mathscr{Q}$, hence we have to prove that $\mathscr{D}$ is a δ-ring, see Theorem E in § 33 in [21]. Let $D_n \in \mathscr{D}$, $n = 1, 2, \ldots$, let $D_n \searrow D$, and let $F \in \mathfrak{S}(\mathscr{P}_0 \otimes \mathscr{Q})$. Then $\boldsymbol{m}(D_n^s)\, \boldsymbol{x} \to \boldsymbol{m}(D^s)\, \boldsymbol{x}$ for each $s \in S$ by the countable additivity of the vector measure $\boldsymbol{m}(\cdot)\, \boldsymbol{x} : \mathscr{P}_0 \to \boldsymbol{Y}$, hence the function $s \to \boldsymbol{m}(D^s)\, \boldsymbol{x}$, $s \in S$, is $\mathscr{Q}$-measurable, see Section 1.2 in Part I and Lemma 1.2 in [24]. Further, (1) and the countable additivity of the vector measure $(\boldsymbol{l} \otimes \boldsymbol{m})(\cdot)\, \boldsymbol{x} : \mathscr{P}_0 \otimes \mathscr{Q} \to \boldsymbol{Z}$ imply that $\int_F \boldsymbol{m}(D_n^s)\, \boldsymbol{x}\, \mathrm{d}\boldsymbol{l} \to (\boldsymbol{l} \otimes \boldsymbol{m})(D \cap F)\, \boldsymbol{x}$ for each $F \in \mathfrak{S}(\mathscr{P}_0 \otimes \mathscr{Q})$ ($F \cap D \in \mathscr{P}_0 \otimes \mathscr{Q}$ for each $F \in \mathfrak{S}(\mathscr{P}_0 \otimes \mathscr{Q})$). Thus by Theorem 16 in Part I the function $s \to \boldsymbol{m}(D^s)\, \boldsymbol{x}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$ and (1) is true for D. Hence $D \in \mathscr{D}$, so $\mathscr{D}$ is a δ-ring. Since $\boldsymbol{x} \in \boldsymbol{X}$ was arbitrary, the necessary part of the first assertion and the second assertion of the theorem are proved.

Suppose now that for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and each $\boldsymbol{x} \in \boldsymbol{X}$ the function $s \to \boldsymbol{m}(E^s)\, \boldsymbol{x}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$. For $\boldsymbol{x} \in \boldsymbol{X}$ and $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ put $\boldsymbol{\mu}_x(E) = \int_S \boldsymbol{m}(E^s)\, \boldsymbol{x}\, \mathrm{d}\boldsymbol{l}$. Since $\boldsymbol{\mu}_x(A \times B) = \boldsymbol{l}(B)\, \boldsymbol{m}(A)\, \boldsymbol{x}$ for each $A \in \mathscr{P}_0$, $B \in \mathscr{Q}$, and $\boldsymbol{x} \in \boldsymbol{X}$, according to Lemma 1 it suffices to prove that for each $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{\mu}_x : \mathscr{P}_0 \otimes \mathscr{Q} \to \boldsymbol{Z}$ is a countably additive vector measure. Let $\boldsymbol{x} \in \boldsymbol{X}$, and suppose that $E_n \in \mathscr{P}_0 \otimes \mathscr{Q}$, $n = 1, 2, \ldots$ are pairwise disjoint sets with $\bigcup_{n=1}^{\infty} E_n = E \in \mathscr{P}_0 \otimes \mathscr{Q}$. We have to show that $\boldsymbol{\mu}_x(E) = \sum_{n=1}^{\infty} \boldsymbol{\mu}_x(E_n)$, where the series converges unconditionally in $\boldsymbol{Z}$. Take $A \in \mathscr{P}_0$ and $B \in \mathscr{Q}$ so that $E \subset A \times B$, and consider the σ-ring $\mathscr{P}_0 \otimes \mathscr{Q} \cap (A \times B)$. Since $\boldsymbol{\mu}_x : \mathscr{P}_0 \otimes \mathscr{Q} \cap (A \times B) \to \boldsymbol{Z}$ is additive, by the Orlicz-Pettis theorem, see IV.10.1 in [19], it is sufficient to prove that $\boldsymbol{z}^* \boldsymbol{\mu}_x(E) = \sum_{n=1}^{\infty} \boldsymbol{z}^* \boldsymbol{\mu}_x(E_n)$ for each $\boldsymbol{z}^* \in \boldsymbol{Z}^*$, where the series converges unconditionally. Let E_n', $n = 1, 2, \ldots$ be any rearrangement of the sequence E_n, $n = 1, 2, \ldots$, and let $\boldsymbol{z}^* \in \boldsymbol{Z}^*$. Then for each $n = 1, 2, \ldots$ we have

$$\left|\boldsymbol{z}^* \boldsymbol{\mu}_x(E) - \sum_{i=1}^{n} \boldsymbol{z}^* \boldsymbol{\mu}_x(E_n')\right| = \left|\boldsymbol{z}^* \boldsymbol{\mu}_x\Big(\bigcup_{i=n+1}^{\infty} E_i'\Big)\right| =$$

$$= \left|\boldsymbol{z}^*\left(\int_S \boldsymbol{m}\Big(\Big[\bigcup_{i=n+1}^{\infty} E_i'\Big]^s\Big)\, \boldsymbol{x}\, \mathrm{d}\boldsymbol{l}\right)\right| = \left|\int_S \boldsymbol{m}\Big(\Big[\bigcup_{i=n+1}^{\infty} E_i'\Big]^s\Big)\, \boldsymbol{x}\, \mathrm{d}(\boldsymbol{z}^*\boldsymbol{l})\right| \leqq$$

$$\leqq \int_B \|\boldsymbol{m}(\cdot)\, \boldsymbol{x}\|\, \Big(\Big[\bigcup_{i=n+1}^{\infty} E_i'\Big]^s\Big)\, \mathrm{d}v(\boldsymbol{z}^*\boldsymbol{l}, \cdot)\,,$$

see the paragraph after Theorem 7 in Part I and Lemma 2.2. Since $\|\boldsymbol{m}(\cdot)\, \boldsymbol{x}\|\, \big(\big[\bigcup_{i=n+1}^{\infty} E_i'\big]^s\big) \searrow 0$ as $n \to +\infty$ for each $s \in S$ by the countable additivity of the vector measure $\boldsymbol{m}(\cdot)\, \boldsymbol{x} : \mathscr{P}_0 \to \boldsymbol{Y}$, since $\|\boldsymbol{m}(\cdot)\, \boldsymbol{x}\|\, \big(\big[\bigcup_{i=n+1}^{\infty} E_i'\big]^s\big) \leqq \|\boldsymbol{m}(\cdot)\, \boldsymbol{x}\|\, (B) < +\infty$ for each $s \in S$ and $n = 1, 2, \ldots$, and since $v(\boldsymbol{z}^*\boldsymbol{l}, B) = \widehat{\boldsymbol{z}^*\boldsymbol{l}}\,(B) \leqq |\boldsymbol{z}^*| \,.\, \boldsymbol{l}^{\wedge}(B) <$

$< +\infty$, see Example 5 in Section 1.1 in Part I, we conclude $\int_B \|m(\cdot)\, x\| \left([\bigcup_{i=n+1}^{\infty} E'_i]^s\right) dv(z^* l, \cdot) \to 0$ as $n \to +\infty$ by the Lebesgue dominated convergence theorem. Thus $\sum_{i=1}^{n} z^* \mu_x(E'_i) \to z^* \mu_x(E)$, which was to be shown. The theorem is proved.

Let $g : S \to Y$ be a $\mathcal{Q}$-merasurable function. In Definition 1 in Part II we defined its L_1-norm $l^\wedge(g, B)$ on a set $B \in \mathfrak{S}(\mathcal{Q})$ (actually, it is in general only a L_1-pseudonorm) by the equality $l^\wedge(g, B) = \sup\{|\int_B h\, dl|;\ h : S \to Y$ is $\mathcal{Q}$-simple and $|h(s)| \leqq |g(s)|$ for each $s \in S\}$. Obviously this definition is meaningful for any real valued function g on S. What is more important, Theorems 1, 2, 3, 5 and 6 remain valid in this case, and if the functions considered are $\mathcal{Q}$-measurable, then also the important Theorems 16 and 17 are valid. (We mean theorems from Part II.) In the following we shall use these facts freely.

From Theorem 1 and from the definitions we easily obtain

Theorem 2. *Let the product measure $l \otimes m : \mathcal{P}_0 \otimes \mathcal{Q} \to L(X, Z)$ exist, let $E \in \mathfrak{S}(\mathcal{P}_0 \otimes \mathcal{Q})$ and let $f : T \times S \to X$ be a $\mathcal{P}_0 \otimes \mathcal{Q}$-measurable function. Then*

$$\|l \otimes m\|\,(E) \leqq l^\wedge(\|m\|\,(E^s), S)$$

and

$$(\widehat{l \otimes m})(f, E) \leqq l^\wedge(m^\wedge(f(\cdot, s), E^s), S)\,.$$

Particularly, $\|l \otimes m\|\,(A \times B) \leqq \|m\|\,(A).\ l^\wedge(B) < +\infty$, and $(\widehat{l \otimes m})(A \times B) \leqq m^\wedge(A).\ l^\wedge(B)$ for each $A \in \mathcal{P}_0$ and $B \in \mathcal{Q}$. Hence $(\widehat{l \otimes m})$ is finite on $\mathcal{P} \otimes \mathcal{Q}$.

Theorem 3. *The product measure $l \otimes m$ exists on $\mathcal{P}_0 \otimes \mathcal{Q}^\sim$, on $\mathcal{P}_2 \otimes \mathcal{Q}^\sim$ it is countably additive in the uniform operator topology, and its semivariation $(\widehat{l \otimes m})$ is continuous on $\mathcal{P}^\sim \otimes \mathcal{Q}^\sim$.*

Proof. Let $E \in \mathcal{P}_0 \otimes \mathcal{Q}^\sim$ and let $x \in X$. By Lemma 2.1 the function $s \to m(E^s)\, x$, $s \in S$, is bounded and $\mathcal{Q}^\sim$-measurable. Since $\{s \in S,\ m(E^s)\, x \neq 0\} \in \mathcal{Q}^\sim$, and since the semivariation $l^\wedge$ is continuous on $\mathcal{Q}^\sim$, by Theorem 5 from Part I the function $s \to m(E^s)\, x$, $s \in S$, is integrable. Since $E \in \mathcal{P}_0 \otimes \mathcal{Q}^\sim$ and $x \in X$ were arbitrary, by Theorem 1 the product measure $l \otimes m$ exists on $\mathcal{P}_0 \otimes \mathcal{Q}^\sim$.

It is easy to see that the product measure $l \otimes m$ is countably additive in the uniform operator topology on $\mathcal{P}_2 \otimes \mathcal{Q}^\sim$ if and only if $E_n \in \mathcal{P}_2 \otimes \mathcal{Q}^\sim$, $n = 1, 2, \ldots$ and $E_n \searrow \emptyset$ imply that $\|l \otimes m\|\,(E_n) \searrow 0$. Let $E_n \in \mathcal{P}_2 \otimes \mathcal{Q}^\sim$, $n = 1, 2, \ldots$ and let $E_n \searrow \emptyset$. By Lemma 2.2 the functions $s \to \|m\|\,(E_n^s)$, $s \in S$, $n = 1, 2, \ldots$ are bounded and $\mathcal{Q}^\sim$-measurable. Since $\{s \in S;\ \|m\|\,(E_1^s) \neq 0\} \in \mathcal{Q}^\sim$, they belong to $\mathcal{L}_1(l)$, see Definition 4 and Theorem 1.c) in Part II. Since m is countably additive in the uniform operator topology on $\mathcal{P}_2$ and since $E_n^s \in \mathcal{P}_2$ for each $s \in S$ and $n = 1, 2, \ldots$, we obtain that $\|m\|\,(E_n^s) \searrow 0$ as $n \to +\infty$ for each $s \in S$. Thus by Theorem 17 in Part II and Theorem 2 we have $\|l \otimes m\|\,(E_n) \leqq l^\wedge(\|m\|\,(E_n^s), S) \searrow 0$, which was to be shown.

The last assertion of the theorem may be proved similarly as the second assertion.

Denote by $\bar{\mathfrak{I}}_s(\mathscr{P} \otimes \mathscr{Q})$ the closure of the set $\mathfrak{I}_s(\mathscr{P} \otimes \mathscr{Q})$ of all $\mathscr{P} \otimes \mathscr{Q}$-simple functions on $T \times S$ with values in X in the sup norm $\|\cdot\|_{T \times S}$, in the Banach space of all bounded X valued functions on $T \times S$. For elements of $\bar{\mathfrak{I}}_s(\mathscr{P} \otimes \mathscr{Q})$ we have the following Fubini type theorem.

Theorem 4. *Let the product measure $\boldsymbol{l} \otimes \boldsymbol{m}$ exist on $\mathscr{P} \otimes \mathscr{Q}$, let $f \in \bar{\mathfrak{I}}_s(\mathscr{P} \otimes \mathscr{Q})$ and let $F \in \mathscr{P} \otimes \mathscr{Q}$ (if $\boldsymbol{m}^\wedge(T) \,.\, \boldsymbol{l}^\wedge(S) < +\infty$, then let $F \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$). Then $\boldsymbol{f} \,.\, \chi_F$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, for each $s \in S$ the function $\boldsymbol{f}(\cdot, s) \,.\, \chi_F(\cdot, s)$ is integrable with respect to $\boldsymbol{m}$, for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ the function $s \to \int_{E^s} \boldsymbol{f}(\cdot, s) \,.$ $.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$, and $\int_E \boldsymbol{f} \,.\, \chi_F \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) =$ $= \int_S \int_{E^s} \boldsymbol{f}(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l}$ for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$.*

Proof. Let $f_n \in \mathfrak{I}_s(\mathscr{P} \otimes \mathscr{Q})$ be such that $\|f_n - f\|_{T \times S} \to 0$, $n = 1, 2, \ldots$, and take $A_0 \in \mathscr{P}$ and $B_0 \in \mathscr{Q}$ so that $F \subset A_0 \times B_0$. (If $\boldsymbol{m}^\wedge(T) \,.\, \boldsymbol{l}^\wedge(S) < +\infty$, we take such $A_0 \in \mathfrak{S}(\mathscr{P})$ and $B_0 \in \mathfrak{S}(\mathscr{Q})$.) Then $f_n(t, s) \to f(t, s)$ for each $(t, s) \in T \times S$. If $E \in$ $\in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, then $f_n \,.\, \chi_E \in \mathfrak{I}_s(\mathscr{P} \otimes \mathscr{Q})$ for each $n = 1, 2, \ldots$. Thus by the definition of the semivariation $(\widehat{\boldsymbol{l} \otimes \boldsymbol{m}})$ and Theorem 2 we have

$$\left| \int_E f_n \,.\, \chi_F \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) - \int_E f_k \,.\, \chi_F \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) \right| = \left| \int_{E \cap F} (f_n - f_k) \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) \right| \leqq$$
$$\leqq \|f_n - f_k\|_{T \times S} \,.\, (\widehat{\boldsymbol{l} \otimes \boldsymbol{m}})(F) \leqq \|f_n - f_k\|_{T \times S} \,.\, \boldsymbol{m}^\wedge(A_0) \,.\, \boldsymbol{l}^\wedge(B_0)$$
$$\text{for each} \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \quad \text{and each} \quad n, k = 1, 2, \ldots .$$

Since $\boldsymbol{m}^\wedge(A_0) \,.\, \boldsymbol{l}^\wedge(B_0) < +\infty$, we obtain by Theorem 7 from Part I that $\boldsymbol{f} \,.\, \chi_F$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, and

$$\int_E f_n \,.\, \chi_F \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) \to \int_E \boldsymbol{f} \,.\, \chi_F \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) \quad \text{for each} \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \,.$$

Let $s \in S$. Then

$$\left| \int_A f_n(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m} - \int_A f_k(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m} \right| \leqq$$
$$\leqq \|f_n - f_k\|_{T \times S} \,.\, \boldsymbol{m}^\wedge(A_0) \quad \text{for each} \quad A \in \mathfrak{S}(\mathscr{P}) \quad \text{and each} \quad n, k = 1, 2, \ldots .$$

Since $\boldsymbol{m}^\wedge(A_0) < +\infty$, by Theorem 7 from Part I the function $\boldsymbol{f}(\cdot, s) \,.\, \chi_F(\cdot, s)$ is integrable with respect to $\boldsymbol{m}$ and

$$\int_A f_n(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m} \to \int_A \boldsymbol{f}(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m}$$
$$\text{for each} \quad A \in \mathfrak{S}(\mathscr{P}) \,; \quad \text{particularly} \,,$$

(1)
$$\int_{E^s} f_n(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m} \to \int_{E^s} \boldsymbol{f}(\cdot, s) \,.\, \chi_F(\cdot, s)\, \mathrm{d}\boldsymbol{m}$$
$$\text{for each} \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \,.$$

Let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. Then using Theorem 14 from Part I we have

$$(2) \qquad \left| \int_B \int_{E^s} f_n(\cdot, s) \,.\, \chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l} - \int_B \int_{E^s} f_k(\cdot, s) \,.\, \chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l} \right| \leqq$$

$$\leqq \sup_{s \in B_0} \left| \int_{E^s} (f_n(\cdot, s) - f_k(\cdot, s)) \,\mathrm{d}\boldsymbol{m} \right| . \boldsymbol{l}^\wedge(B_0) \leqq$$

$\leqq \|f_n - f_k\|_{T \times S} \,.\, \boldsymbol{m}^\wedge(A_0) \,.\, \boldsymbol{l}^\wedge(B_0)$ for each $B \in \mathfrak{S}(\mathscr{Q})$ and each $n, k = 1, 2, \ldots$.

Since $\boldsymbol{m}^\wedge(A_0) \,.\, \boldsymbol{l}^\wedge(B_0) < +\infty$, the relations (1) and (2) imply according to Theorem 16 from Part I ($\|f_n - f_k\|_{T \times S} \to 0$ as $n, k \to +\infty$) that the function $s \to \int_{E^s} f(.,s) \,.$ $. \,\chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$ and that

$$\int_S \int_{E^s} f_n(\cdot \,.\, s) \,.\, \chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l} \to \int_S \int_{E^s} f(\cdot, s) \,.\, \chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l} \,.$$

It remains to observe that owing to Theorem 1

$$\int_E f_n \,.\, \chi_F \,\mathrm{d}\boldsymbol{l} \otimes \boldsymbol{m}) = \int_S \int_{E^s} f_n(\cdot, s) \,.\, \chi_F(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l}$$

for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ and each $n = 1, 2, \ldots$.

Let now T and S be locally compact Hausdorff topological spaces. By $\mathscr{B}_0(T)$, $\mathscr{B}_0(S)$ and $\mathscr{B}_0(T \times S)$ we denote the δ-rings of relatively compact Baire subsets of T, S and $T \times S$, respectively. According to Theorem E in § 51 in [21] we have $\mathscr{B}_0(T \times S) = \mathscr{B}_0(T) \otimes \mathscr{B}_0(S)$, and according to Theorem 8 in Part I we have $C_0(T \times S, \boldsymbol{X}) \subset \mathfrak{I}_s(\mathscr{B}_0(T \times S))$. Hence Theorem 4 yields immediately the following result:

Theorem 5. *Let T and S be locally compact Hausdorff topological spaces, let $\boldsymbol{m} : \mathscr{B}_0(T) \to L(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{l} : \mathscr{B}_0(S) \to L(\boldsymbol{Y}, \boldsymbol{Z})$ be Baire operator valued measures countably additive in the strong operator topologies with $\boldsymbol{m}^\wedge(T) \,.\, \boldsymbol{l}^\wedge(S) < +\infty$, let their product $\boldsymbol{l} \otimes \boldsymbol{m}$ exist on $\mathscr{B}_0(T) \otimes \mathscr{B}_0(S) = \mathscr{B}_0(T \times S)$ and let $\boldsymbol{f} \in C_0(T \times$ $\times\, S, \boldsymbol{X})$. Then $\boldsymbol{f}$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, $\boldsymbol{f}(\cdot, s)$ is integrable with respect to $\boldsymbol{m}$ for each $s \in S$, for each $E \in \mathfrak{S}(\mathscr{B}_0(T \times S))$ the function $s \to \int_{E^s} \boldsymbol{f}(\cdot, s) \,\mathrm{d}\boldsymbol{m}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$, and*

$$(1) \qquad \int_E \boldsymbol{f} \,\mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_S \int_{E^s} \boldsymbol{f}(\cdot, s) \,\mathrm{d}\boldsymbol{m} \,\mathrm{d}\boldsymbol{l}$$

for each $E \in \mathfrak{S}(\mathscr{B}_0(T \times S))$.

This theorem may be combined with results on representation of bounded linear operators on spaces of the type $C_0(T, \boldsymbol{X})$, see [4] and [8], to obtain results about bounded linear operators on $C_0(T \times S, X)$ which are of the form $Wf = U(Vf(\cdot, s))$, $f \in C_0(T \times S, X)$, where $V : C_0(T, X) \to Y$ and $U : C_0(S, Y) \to Z$. (The fact that $Vf(\cdot, s) \in C_0(S, Y)$ for $f \in C_0(T \times S, X)$ follows immediately from the boundedness of V and from the easily proved fact: Let $f \in C_0(T \times S, X)$, let $s \in S$ and $\varepsilon > 0$. Then there is an open neighbourhood $O(s)$ of s such that $|f(t, s) - f(t, s')| < \varepsilon$ for each $t \in T$ and each $s' \in O(s)$.)

We present one such result for illustration.

Corollary. *Let X be a reflexive Banach space and let $V : C_0(T, X) \to Y$ and $U : C_0(S, Y) \to Z$ be unconditionally converging bounded linear operators. Then $W : C_0(T \times S, X) \to Z$ defined by the equality $Wf = U(Vf(\cdot, s)), f \in C_0(T \times S, X)$, is weakly compact.*

Proof. According to Theorem 3 in [8], V and U have representations $Vg = \int_T g \, \mathrm{d}m$, $g \in C_0(T, X)$, and $Uh = \int_S h \, \mathrm{d}l$, $h \in C_0(S, Y)$, where $m : \mathfrak{S}(\mathscr{B}_0(T)) \to L(X, Y)$ and $l : \mathfrak{S}(\mathscr{B}_0(S)) \to L(Y, Z)$ are operator valued measures, and the semivariations $m^\wedge$ and $l^\wedge$ are continuous on $\mathfrak{S}(\mathscr{B}_0(T))$ and $\mathfrak{S}(\mathscr{B}_0(S))$, respectively. According to Theorem 3 the product measure $l \otimes m$ exists on $\mathfrak{S}(\mathscr{B}_0(T)) \otimes \mathfrak{S}(\mathscr{B}_0(S)) = \mathfrak{S}(\mathscr{B}_0(T \times S))$, and its semivariation $\widehat{(l \otimes m)}$ is continuous on $\mathfrak{S}(\mathscr{B}_0(T \times S))$. By Theorem 5 we have $Wf = \int_{T \times S} f \, \mathrm{d}(l \otimes m)$, $f \in C_0(T \times S, X)$. Since X is a reflexive Banach space, the continuity of the semivariation $\widehat{(l \otimes m)}$ on $\mathfrak{S}(\mathscr{B}_0(T \times S))$ is a necessary and sufficient for the weak compactness of W, see Remark 1 in [8]. The corollary is proved.

Some special cases. 1. Let Z contain no isomorphic copy of c_0. Then by the $*$-Theorem in Section 1.1 in Part I the semivariation $l^\wedge$ is continuous on $\mathscr{Q}$. Thus by Theorem 1 the product measure $l \otimes m$ exists on $\mathscr{P}_0 \otimes \mathscr{Q}$. By Theorem 2 the semivariation $\widehat{(l \otimes m)}$ is finite on $\mathscr{P} \otimes \mathscr{Q}$, hence by the $*$-Theorem it is continuous on $\mathscr{P} \otimes \mathscr{Q}$.

2. Let X be the space of scalars and let $Y = Z$ be a commutative Banach algebra, or let $X = Y = Z$ be a commutative Banach algebra, or let $X = Y = Z$ and let $l(B)\, m(A) = m(A)\, l(B)$ for each $A \in \mathscr{P}$ and $B \in \mathscr{Q}$. Suppose further that the product measure $l \otimes m$ exists on $\mathscr{P} \otimes \mathscr{Q}$. Then by Lemma 1 the product measure $m \otimes l$ exists on $\mathscr{Q} \otimes \mathscr{P} = \mathscr{P} \otimes \mathscr{Q}$ and is equal to $l \otimes m$. Thus in this case

$$\int_S \int_{E^s} f(\cdot, s) \,.\, \chi_F(\cdot, s) \, \mathrm{d}m \, \mathrm{d}l = \int_T \int_{E^t} f(t, \cdot) \,.\, \chi_F(t, \cdot) \, \mathrm{d}l \, \mathrm{d}m \,,$$

in Theorem 4 and similarly

$$\int_S \int_{E^s} f(\cdot, s) \, \mathrm{d}m \, \mathrm{d}l = \int_T \int_{E^t} f(t, \cdot) \, \mathrm{d}l \, \mathrm{d}m$$

in Theorem 5.

Results on the products of operator valued measures have applications in convolutions of vector measures, see for example [34], [23], [14].

2. MEASURABILITY OF THE PARTIAL INTEGRAL

Example. Let $T = S = \{1, 2, \ldots\}$, let $\mathscr{P} = \mathscr{Q} = 2^T$, let X be the space of real numbers, and let $\boldsymbol{Y} = \boldsymbol{Z} = c_0$. Let $\boldsymbol{m} : 2^T \to L(X, c_0) = c_0$ and $\boldsymbol{l} : 2^S \to L(c_0, c_0)$ be defined by the countable additivity from the following elementary values:

$$\boldsymbol{m}(\{k\}) = \begin{cases} 0 \text{ if } k \text{ is even}, \\ \Big(\overbrace{0, \ldots, 0, \frac{1}{k^2}}^{k}, 0, 0, \ldots\Big) \in c_0 \text{ if } k \text{ is odd}, \end{cases}$$

$$\boldsymbol{l}(\{k\}) = \begin{cases} 0 \text{ if } k \text{ is odd}, \\ \Big(\overbrace{0, \ldots, 0, \frac{1}{k^2}}^{k}, 0, 0, \ldots\Big) \in c_0 \text{ if } k \text{ is even}. \end{cases}$$

Then clearly $\boldsymbol{m}$ and $\boldsymbol{l}$ are operator valued measures with bounded countably additive variations and their product $\boldsymbol{l} \otimes \boldsymbol{m} = \boldsymbol{m} \otimes \boldsymbol{l}$ exists and is identically equal to zero. Thus every function $\boldsymbol{f} : T \times S \to X$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$. Now it is easy to see that the function $\boldsymbol{f}(\cdot, s), \boldsymbol{f}(t, s) = t^{s+1}$, $(t, s) \in T \times S$, is not integrable with respect to $\boldsymbol{m}$ for any $s \in S = \{1, 2, \ldots\}$.

From this example it is clear that in a general Fubini theorem we must suppose that for a $\mathscr{P} \otimes \mathscr{Q}$-measurable function $\boldsymbol{f} : T \times S \to X$, the function $t \to \boldsymbol{f}(t, s)$, $t \in T$, is integrable with respect to the measure $\boldsymbol{m}$ for each $s \in S$. Since a $\mathscr{P} \otimes \mathscr{Q}$-measurable function is, by definition, a pointwise limit of a sequence of $P \otimes \mathscr{Q}$-simple functions, we conclude from Theorem A in § 34 [21] and from the fact that the $\mathscr{P}$-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I and Lemma 1.2 in [24], that the function $\boldsymbol{f}(\cdot, s)$ is $\mathscr{P}$-measurable for each $s \in S$ provided $\boldsymbol{f} : T \times S \to X$ is $\mathscr{P} \otimes \mathscr{Q}$-measurable.

Let $\boldsymbol{f} : T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{Q}$-measurable function and let $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$. In this section we investigate the $\mathscr{Q}$-measurability and the essential $\boldsymbol{l} - \mathscr{Q}$-measurability of the partial integral $\boldsymbol{g}_E, \boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. In fact, $\mathscr{Q}$ is replaced in Theorems 6–12 by an arbitrary δ-ring $\mathscr{D}$ of subsets of S. Besides, we obtain results on the $\mathscr{D}$-measurability of the function h_E, $h_E(s) = \boldsymbol{m}^\wedge(\boldsymbol{f}(\cdot, s), E^s)$, $s \in S$, and important results which are needed for the proof of the Fubini theorem in § 3.

Theorem 6. *Let $\mathscr{D}$ be a δ-ring of subsets of S and let $\boldsymbol{f} : T \times S \to X$ be a $\mathscr{P}^\sim \otimes \mathscr{D}$-measurable function. Then for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the function h_E, $h_E(s) = \boldsymbol{m}^\wedge(\boldsymbol{f}(\cdot, s), E^s)$, $s \in S$, is $\mathscr{D}$-measurable.*

Proof. Let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ and let f_n, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P}^\sim \otimes \mathscr{D}$-simple functions such that $f_n(t, s) \to f(t, s)$ and $|f_n(t, s)| \nearrow |f(t, s)|$ for each $(t, s) \in T \times S$, see Section 1.2 in Part I. According to Theorem 4 in Part II we have $m^\wedge(f(\cdot, s), E^s) = \sup_{|y^*| \leq 1} \int_{E^s} |f(\cdot, s)| \, dv(y^*m, \cdot)$ for each $s \in S$. The same equality holds for each f_n, $n = 1, 2, \ldots$. Hence $m^\wedge(f(\cdot, s), E^s) = \lim_{n \to \infty} m^\wedge(f_n(\cdot, s), E^s)$ for each $s \in S$ by the Fatou lemma. Therefore it suffices to prove the theorem for each $\mathscr{P}^\sim \otimes \mathscr{D}$-simple function $f: T \times S \to X$.

Let $f: T \times S \to X$ be a $\mathscr{P}^\sim \otimes \mathscr{D}$-simple function of the form $f = \sum_{i=1}^{r} x_i \cdot \chi_{E_i}$, $x_i \in X$, $E_i \in \mathscr{P}^\sim \otimes \mathscr{D}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r$, and let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$. Since $\mathscr{P}^\sim \otimes \mathscr{D} \cap \mathfrak{S}(\mathscr{P} \otimes \mathscr{D}) = \mathscr{P}^\sim \otimes \mathscr{D}$, and since $E_i \in \mathscr{P}^\sim \otimes \mathscr{D}$, $i = 1, \ldots, r$, we may suppose without loss of generality that $E \in \mathscr{P}^\sim \otimes \mathscr{D}$. Take $A \in \mathscr{P}^\sim$ and $B \in \mathscr{D}$ so that $E \subset A \times B$. Let $x \in X$ and $|x| = 1$, and let $d: T \to X$ be the $\mathscr{P}^\sim$-simple function defined by the equality $d = (\sum_{i=1}^{r} |x_i|) \cdot x \cdot \chi_A$. Then clearly $d \in \mathscr{L}_1(m)$, see Theorem 1c) and Definition 4 in Part II. Denote by $\mathscr{R}$ the ring of all finite unions of pairwise disjoint rectangles $C \times D$, $C \in \mathscr{P}^\sim$ and $D \in \mathscr{D}$, see Theorem E in § 33 [21]. If $F_i \in \mathscr{R} \cap (A \times B)$ for each $i = 1, \ldots, r$, then for $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$ the function $s \to m^\wedge(g(\cdot, s), A)$, $s \in S$, is clearly $\mathscr{D}$-measurable. Denote by $\mathscr{M}_1$ the class of all sets $F_1 \in \mathscr{P}^\sim \otimes \mathscr{D} \cap (A \times B)$ for which the function $s \to m^\wedge(g(\cdot, s), A)$, $s \in S$, is $\mathscr{D}$-measurable provided $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$ and $F_2, \ldots, F_r \in \mathscr{R} \cap (A \times B)$. Then $\mathscr{R} \cap (A \times B) \subset \mathscr{M}_1$, and since $|g(t, s)| \leq |g_0(t)|$ for each $(t, s) \in T \times S$, $\mathscr{M}_1$ is a monotone class by Theorem 17 from Part II. Thus $\mathscr{M}_1 = \mathscr{P}^\sim \otimes \mathscr{D} \cap (A \times B)$ by Theorem B in § 6 [21]. Similarly, if $\mathscr{M}_2$ is the class of all sets $F_2 \in \mathscr{P}^\sim \otimes \mathscr{D} \cap (A \times B)$ for which the function $s \to m^\wedge(g(\cdot, s), A)$, $s \in S$, is $\mathscr{D}$-measurable provided $g = \sum_{i=1}^{r} x_i \cdot \chi_{F_i}$, $F_1 \in \mathscr{M}_1$ and $F_3, \ldots, F_r \in \mathscr{R} \cap (A \times B)$, then $\mathscr{M}_2 = \mathscr{P}^\sim \otimes \mathscr{D} \cap (A \times B)$. Continuing in this way we obtain that $\mathscr{M}_r = \mathscr{P}^\sim \otimes \mathscr{D} \cap (A \times B)$, which was to be shown. The theorem is proved.

Let us remind that a subset $\Lambda \subset Y^*$ is called norming (or total) for Y if $|y| = \sup_{y^* \in \Lambda} |y^*y|$ for each $y \in Y$, see Definition 2.8.1 in [22]. It is well known, see Theorem 2.8.5 in [22], that separable Banach spaces and their duals have countable norming sets.

Theorem 7. *Let $\mathscr{D}$ be a δ-ring of subsets of S, let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{D}$-measurable function and let Y have a countable norming set. Then for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the function h_E, $h_E(s) = m^\wedge(f(\cdot, s), E^s)$, $s \in S$, is $\mathscr{D}$-measurable.*

Proof. Let $y_n^* \in Y^*$, $n = 1, 2, \ldots$ be a countable norming set for Y and let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$. Then by Theorem 4 from Part II, $h_E(s) = m^\wedge(f(\cdot, s), E^s) = \sup_n$

$\int_{E^s}|f(\cdot, s)|\, dv(y_n^* m, \cdot)$ for each $s \in S$. Hence by Theorem A in § 20 [21] it suffices to prove the $\mathscr{D}$-measurability of the function $s \to \int_{E^s}|f(\cdot, s)|\, dv(y_n^* m, \cdot)$ $s \in S$, for each $n = 1, 2, \ldots$. But this follows immediately from Theorem 6, since by assumption the function f is $\mathscr{P} \otimes \mathscr{D}$-measurable, and since $v(y_n^* m, \cdot)$ is a countably additive finite non negative measure on $\mathscr{P}$ for each $n = 1, 2, \ldots$, see Example 5 in Part I.

Theorem 8. *Let $\mathscr{D}$ be a δ-ring of subsets of S, let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{D}$-measurable function and let $f(\cdot, s) \in \mathscr{L}_1(m)$ for each $s \in S$ (see* Part II). *Then for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the functions g_E, $g_E(s) = \int_{E^s} f(\cdot, s)\, dm$, $s \in S$, and h_E, $h_E(s) = m^\wedge(f(\cdot, s), E^s)$, $s \in S$, are $\mathscr{D}$-measurable. If $\mathscr{D} = \mathscr{Q}$, if the product measure $l \otimes m$ exists on $\mathscr{P} \otimes \mathscr{Q}$, and if $h_{T \times S} \in \mathscr{L}_1(l)$, then $f \in \mathscr{L}_1(l \otimes m)$.*

Proof. Let f_n, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P} \otimes \mathscr{D}$-simple functions on $T \times S$ such that $f_n(t, s) \to f(t, s)$ and $|f_n(t, s)| \nearrow |f(t, s)|$ for each $(t, s) \in T \times S$, see Section 1.2 in Part I. Then clearly $f_n(\cdot, s) \in \mathscr{L}_1(m)$ for each $n = 1, 2, \ldots$ and each $s \in S$, hence f is $\mathscr{P}^\sim \otimes \mathscr{D}$-measurable. Thus by Theorem 6 the function h_E is $\mathscr{D}$-measurable for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$. Further, according to Theorem 17 in Part II we have $m^\wedge(f(\cdot, s) - f_n(\cdot, s), T) \to 0$ for each $s \in S$. Let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ and put $g_{n,E}(s) = \int_{E^s} f_n(\cdot, s)\, dm$, $s \in S$, $n = 1, 2, \ldots$. Then according to Lemma 2.1 the functions $g_{n,E}$, $n = 1, 2, \ldots$ are $\mathscr{D}$-measurable. Applying Corollary of Theorem 2 from Part II we obtain that $|g_{n,E}(s) - g_E(s)| \leqq m^\wedge(f(\cdot, s) - f_n(\cdot, s), T) \to 0$ as $n \to \infty$. Thus $g_{n,E}(s) \to g_E(s)$ for each $s \in S$ which proves the $\mathscr{D}$-measurability of g_E since the $\mathscr{D}$-measurable functions are closed under the formation of pointwise limits of sequences, see Section 1.2 in Part I of Lemma 1.2 in [24].

Concerning the second assertion of the theorem we have to show that the L_1-pseudonorm $(\widehat{l \otimes m})(f, \cdot)$ is continuous on $\mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. Let $E_k \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, $k = 1, 2, \ldots$, and let $E_k \searrow \emptyset$. Since by assumption $f(\cdot, s) \in \mathscr{L}_1(m)$ for each $s \in S$, we have $h_{E_k}(s) \to 0$ for each $s \in S$ by Theorem 17 in Part II. By assumption $h_{T \times S} \in \mathscr{L}_1(l)$, hence $l^\wedge(h_{E_k}, S) \to 0$ again by Theorem 17 in Part II. Thus by Theorem 2 we have $(\widehat{l \otimes m})(f, E_k) \leqq l^\wedge(h_{E_k}, S) \to 0$, which completes the proof of the theorem.

Theorem 9. *Let $\mathscr{D}$ be a δ-ring of subsets of S, let $f: T \times S \to X$ be a $\mathscr{P}^\sim \otimes \mathscr{D}$-measurable function and let for each $s \in S$ the function $t \to f(t, s)$, $t \in T$, be integrable with respect to m. Then for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the function g_E, $g_E(s) = \int_{E^s} f(\cdot, s)\, dm$, $s \in S$, is $\mathscr{D}$-measurable.*

Proof. Put $F = \{(t, s) \in T \times S, f(t, s) \neq 0\}$. Then $F \in \mathfrak{S}(\mathscr{P}^\sim \otimes \mathscr{D})$, hence there are $A \in \mathfrak{S}(\mathscr{P}^\sim)$ and $B \in \mathfrak{S}(\mathscr{D})$ such that $F \subset A \times B$. Take $A_n \in \mathscr{P}^\sim$, $n = 1, 2, \ldots$ so that $A_n \nearrow A$. Clearly $F_n = \{(t, s) \in T \times S, |f(t, s)| < n\} \in \mathfrak{S}(\mathscr{P}^\sim \otimes \mathscr{D})$ and $F_n \nearrow F$, $n = 1, 2, \ldots$. Now it is easy to see that $H_n = (A_n \times B) \cap F_n \in \mathscr{P}^\sim \otimes \mathfrak{S}(\mathscr{D})$, $H_n \nearrow F$ and $f(\cdot, s) . \chi_{H_n}(\cdot, s) \in \mathscr{L}_1(m)$ for each $n = 1, 2, \ldots$ and each $s \in S$. Thus by Theorem 8 the functions $g_{n,E}$, $g_{n,E}(s) = \int_{E^s} f(\cdot, s) . \chi_{H_n}(\cdot, s)\, dm$, $s \in S$, $n = 1, 2, \ldots$ and

$E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$, are $\mathcal{D}$-measurable. Since the integrability of the function $t \to \boldsymbol{f}(t, s)$, $t \in T$, for each $s \in S$ implies that $\boldsymbol{g}_E(s) = \lim_{n\to\infty} \boldsymbol{g}_{n,E}(s)$ for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$ and each $s \in S$, the theorem is proved.

Theorem 10. *Let $\mathcal{D}$ be a δ-ring of subsets of S, let $\boldsymbol{f}: T \times S \to \boldsymbol{X}$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function and let the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$. Then for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$ the function $\boldsymbol{g}_E$, $\boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, is weakly $\mathcal{D}$-measurable. Hence, if $\boldsymbol{Y}$ is a separable Banach space, then $\boldsymbol{g}_E$ is $\mathcal{D}$-measurable for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$.*

Proof. Let $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$ and let $\boldsymbol{y}^* \in \boldsymbol{Y}$. Then $\boldsymbol{y}^* \boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{y}^*\boldsymbol{m}$ for each $s \in S$, see the paragraph after Theorem 7 in Part I. According to Example 5 in § 1 in Part I we have $v(\boldsymbol{y}^*\boldsymbol{m}, A) = \widehat{\boldsymbol{y}^* \boldsymbol{m}}(A) \leq |\boldsymbol{y}^*| \, . \, \boldsymbol{m}^\wedge(A) < +\infty$ for each $A \in \mathcal{P}$, hence $\widehat{\boldsymbol{y}^*\boldsymbol{m}}$ is continuous on $\mathcal{P}$. Thus the $\mathcal{D}$-measurability of $\boldsymbol{y}^*\boldsymbol{g}_E$ follows from Theorem 9. For the second assertion of the theorem see Theorem 3.5.3 in [22].

Theorem 11. *Let $\mathcal{D}$ be a δ-ring of subsets of S, let $\boldsymbol{f}: T \times S \to \boldsymbol{X}$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function and let $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$. Let further*

$$\boldsymbol{f}_n = \sum_{i=1}^{r_n} \boldsymbol{x}_{n,i} \cdot \chi_{E_{n,i}}, \quad \boldsymbol{x}_{n,i} \in \boldsymbol{X}, \quad E_{n,i} \in \mathcal{P} \otimes \mathcal{D}, \quad n = 1, 2, \ldots, \quad i = 1, \ldots, r_n,$$

be a sequence of $\mathcal{P} \otimes \mathcal{D}$-simple functions such that $\boldsymbol{f}_n(t, s) \to \boldsymbol{f}(t, s)$ for each $(t, s) \in T \times S$, and let $\boldsymbol{X}_1$ be the closed linear span of $\boldsymbol{X}_0 = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{r_n} \boldsymbol{x}_{n,i}$ in $\boldsymbol{X}$. Then for each $s \in S$ the function $\boldsymbol{f}(\cdot, s)$ is integrable with respect to the restricted measure $\boldsymbol{m} : \mathcal{P} \to L(\boldsymbol{X}_1, \boldsymbol{Y})$ and the set of all finite sums of the form $\sum_{j=1}^{r} \boldsymbol{m}(A_j)\, \boldsymbol{x}_j$, $A_j \in \mathcal{P}$, $\boldsymbol{x}_j \in \boldsymbol{X}_0$, $j = 1, \ldots, r$ is dense in the subset $\{\int_A \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m};\ A \in \mathfrak{S}(\mathcal{P}),\ s \in S\}$ of $\boldsymbol{Y}$.

Proof. In the proof of Theorem 15 in Part I we found, under the assumptions of the theorem and for each $s \in S$, a set $N(s) \in \mathfrak{S}(\mathcal{P})$, a sequence $F_k(s) \in \mathcal{P}$ and a subsequence $n_k(s)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} \int_A \boldsymbol{f}_{n_k(s)}(\cdot, s) \cdot \chi_{F_k(s) \cup N(s)}(\cdot, s)\, \mathrm{d}\boldsymbol{m} = \int_A \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$ uniformly with respect to $A \in \mathfrak{S}(\mathcal{P})$. It remains to observe that for each $s \in S$ the integrals on the left hand side of the last equality are of the form $\sum_{j=1}^{r} \boldsymbol{m}(A_j)\, \boldsymbol{x}_j$ with $A_j \in \mathcal{P}$, $\boldsymbol{x}_j \in \boldsymbol{X}_0$, $j = 1, \ldots, r$. Note that the semivariation of the restricted measure $\boldsymbol{m} : \mathcal{P} \to L(\boldsymbol{X}_1, \boldsymbol{Y})$ is less than or equal to the semivariation of $\boldsymbol{m} : \mathcal{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$, hence it is finite on $\mathcal{P}$.

Using Theorem 10 we immediately have

Corollary. *Let $\mathcal{D}$ be a δ-ring of subsets of S, let $\boldsymbol{f}: T \times S \to \boldsymbol{X}$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function, let the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each*

$s \in S$ the and let $\{\boldsymbol{m}(A)\, x;\ A \in \mathscr{P}\}$ be a separable subset of $\boldsymbol{Y}$ for each $x \in X$. Then

1) *$\{\int_A \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m};\ A \in \mathfrak{S}(\mathscr{P}),\ s \in S\}$ is a separable subset of $\boldsymbol{Y}$, and*
2) *for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the function $\boldsymbol{g}_E, \boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, is $\mathscr{D}$-measurable.*

Theorem 12. *Let $\mathscr{P}$ be generated by a countable family of subsets of T, let $\mathscr{D}$ be a δ-ring of subsets of S, let $\boldsymbol{f}: T \times S \to X$ be a $\mathscr{P} \times \mathscr{D}$-measurable function and let the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$. Then*

1) *$\{\int_A \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m};\ A \in \mathfrak{S}(\mathscr{P}),\ s \in S\}$ is a separable subset of $\boldsymbol{Y}$,*
2) *for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{D})$ the function $\boldsymbol{g}_E, \boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, is $\mathscr{D}$-measurable, and*
3) *the function v, $v(s) = \sup_{A \in \mathfrak{S}(\mathscr{P})} |\int_A \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}|$, $s \in S$, is finite valued and $\mathscr{D}$-measurable.*

Proof. Without loss of generality we may suppose that $\mathscr{P}$ is generated by a countable ring $\mathscr{R} = \{R_n,\ n = 1, 2, \ldots\}$, see Theorem C in § 5 [21].

1) and 2). According to Corollary of Theorem 11 it suffices to show that $\boldsymbol{Y}_x = \{\boldsymbol{m}(A)\, x;\ A \in \mathscr{P}\}$ is a separable subset of $\boldsymbol{Y}$ for each $x \in X$.

Let $x \in X$. Put $\mathscr{R}_n = (R_1 \cup \ldots \cup R_n) \cap \mathscr{R}$ and $\mathscr{S}_n = \mathfrak{S}(\mathscr{R}_n)$, $n = 1, 2, \ldots$. Then clearly $\mathscr{P} = \delta(\mathscr{R}) = \bigcup_{n=1}^{\infty} \mathscr{S}_n$. We will show that the set $\boldsymbol{Y}_0$ of all finite sums of the form $\sum_{i=1}^{r} \boldsymbol{m}(R_{n_i})\, x$ is dense in $\boldsymbol{Y}_x$ ($\boldsymbol{Y}_0$ is clearly countable). Let $A \in \mathscr{P}$. Then there is an n_A such that $A \in \mathscr{S}_{n_A}$. Let $\lambda_{n_A} : \mathscr{S}_{n_A} \to \langle 0, +\infty)$ be a control measure for the vector measure $\boldsymbol{m}(\cdot)\, x : \mathscr{S}_{n_A} \to \boldsymbol{Y}$. Then the desired assertion immediately follows from Theorem D in § 13 [21] applied to λ_{n_A} and from the simple inequality $|\boldsymbol{m}(A_1)\, x - \boldsymbol{m}(A_2)\, x| \leqq |\boldsymbol{m}(A_1 - A_2)\, x| + |\boldsymbol{m}(A_2 - A_1)\, x| \leqq 2\|\boldsymbol{m}(\cdot)\, x\|\, (A_1 \Delta A_2)$, $A_1, A_2 \in \mathscr{S}_{n_A}$.

3) Since $A \to \int_A \boldsymbol{f}(\cdot\ s)\, \mathrm{d}\boldsymbol{m}$, $A \in \mathfrak{S}(\mathscr{P})$ is a countably additive vector measure on a σ-ring, v is finite valued, see IV.10.4 in [19]. By Theorem IV.10.5 in [19] and Theorem D in § 13 [21] we have $v(s) = \sup_n |\int_{R_n} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}|$ for each $s \in S$, hence 2) and Theorem A in § 20 [21] imply the $\mathscr{D}$-measurability of v.

Theorem 13. *In the following cases: 1) X is separable, 2) $\boldsymbol{Y}$ has a countable norming set, and 3) $\mathfrak{S}(\mathscr{P}_2) \supset \mathscr{P}$; for each $A \in \mathfrak{S}(\mathscr{P})$ there is a countably additive measure $\lambda_A : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ such that $C \in \mathfrak{S}(\mathscr{P})$, $\lambda_A(A \cap C) = 0 \Rightarrow \boldsymbol{m}^\wedge(A \cap C) = 0$.*

Proof. Let $A \in \mathfrak{S}(\mathscr{P})$ and take $A_n \in \mathscr{P}$, $n = 1, 2, \ldots$ so that $A_n \nearrow A$. Since $\boldsymbol{m}^\wedge(C) = \sup_{|y^*| \leqq 1} v(y^* \boldsymbol{m}, C)$ for each $C \in \mathfrak{S}(\mathscr{P})$, see Lemma 1 in [8], we have $\boldsymbol{m}^\wedge(A \cap C) =$

$= \lim_{n \to \infty} \boldsymbol{m}^\wedge(A_n \cap C)$ for each $C \in \mathfrak{S}(\mathscr{P})$. Suppose that the theorem is proved for each $A \in \mathscr{P}$, take countably additive measures $\lambda_n : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ so that $C \in \mathfrak{S}(\mathscr{P})$, $\lambda_n(A_n \cap C) = 0 \Rightarrow \boldsymbol{m}^\wedge(A_n \cap C) = 0$, $n = 1, 2, \ldots$, and put

$$\lambda_A(C) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(A_n \cap C)}{1 + \lambda_n(T)}, \quad C \in \mathfrak{S}(\mathscr{P}).$$

Then clearly λ_A has the required properties. Consequently, it is sufficient to prove the theorem for each $A \in \mathscr{P}$.

1) Let $A \in \mathscr{P}$ and let $x_k \in X$, $k = 1, 2, \ldots$, be a dense subset of X. For each $k = 1, 2, \ldots$ let $\lambda_k : A \cap \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ be a control measure for the vector measure $\boldsymbol{m}(\cdot)\, x_k : A \cap \mathfrak{S}(\mathscr{P}) \to Y$. Then clearly

$$\lambda_A(C) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\lambda_k(A \cap C)}{1 + \lambda_k(A)},$$

$C \in \mathfrak{S}(\mathscr{P})$, has the required properties.

2) Let $A \in \mathscr{P}$ and let $y_h^* \in Y^*$, $k = 1, 2, \ldots$ be a countable norming set for Y. Then $\boldsymbol{m}^\wedge(A \cap C) = \sup_k v(y_k^* \boldsymbol{m}, A \cap C)$ for each $C \in \mathfrak{S}(\mathscr{P})$, see Lemma 1 in [8]. Now clearly it suffices to put

$$\lambda_A(C) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{v(y_k^* \boldsymbol{m}, A \cap C)}{1 + v(y_k^* \boldsymbol{m}, A)}, \quad C \in \mathfrak{S}(\mathscr{P}).$$

3) Similarly as at the beginning of the proof we may suppose that $A \in \mathscr{P}_2$. But then $\boldsymbol{m} : A \cap (\mathscr{P}) \to L(X, Y)$ is countably additive, hence a control measure for it has the required properties.

Definition 2. A function $\boldsymbol{u} : T \to X$ is called *$\boldsymbol{m}$-null* if there is an $N \in \mathfrak{S}(\mathscr{P})$ *with* $\boldsymbol{m}^\wedge(N) = 0$ such that $\{t \in T;\ \boldsymbol{u}(t) \neq 0\} \subset N$. A function $\boldsymbol{f} : T \to X$ is called *$\boldsymbol{m}$-essentially $\mathscr{P}$-measurable (integrable)* if it can be written in the form $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{u}$, where $\boldsymbol{g}$ is $\mathscr{P}$-measurable (integrable) and $\boldsymbol{u}$ is $\boldsymbol{m}$-null. In the case $\boldsymbol{f}$ is $\boldsymbol{m}$-essentially integrable we extend the integral defining $\int_A \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} = \int_A \boldsymbol{g}\, \mathrm{d}\boldsymbol{m}$ for each $A \in \mathfrak{S}(\mathscr{P})$.

Clearly our theory of integration extends with obvious modifications to $\boldsymbol{m}$-essentially measurable (integrable) functions. Particularly, if $\boldsymbol{f}_n : T \to X$, $n = 1, 2, \ldots$ are $\boldsymbol{m}$-essentially $\mathscr{P}$-measurable and $\lim_{n \to \infty} \boldsymbol{f}_n(t) = \boldsymbol{f}(t) \in X$ a.e. m, then $\boldsymbol{f}$ is also $\boldsymbol{f}$-essentially $\mathscr{P}$-measurable. Hence in the theorems of our extended theory the limit function is automatically $\boldsymbol{m}$-essentially $\mathscr{P}$-measurable. Note also that the range of an $\boldsymbol{m}$-null, hence also of an $\boldsymbol{m}$-essentially $\mathscr{P}$-measurable function, need not be separable.

Theorem 14. *Let $\boldsymbol{f} : T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{Q}$-measurable function, let the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$, and for each $B \in \mathfrak{S}(\mathscr{Q})$ let there* *be a countably additive measure* $\lambda_B : \mathfrak{S}(\mathcal{Q}) \to \langle 0, +\infty)$ *such that* $D \in \mathfrak{S}(\mathcal{Q})$, $\lambda_B(B \cap D) = 0 \Rightarrow \boldsymbol{l}^\wedge(B \cap D) \approx 0$, *see Theorem* 13. *Then for each set* $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$ *the function* $\boldsymbol{g}_E$, $\boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)$ $s \in S$, *is* $\boldsymbol{l}$*-essentially* $\mathcal{Q}$*-measurable.*

Proof. Let $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$. Take $A \in \mathfrak{S}(\mathcal{P})$ and $B \in \mathfrak{S}(\mathcal{Q})$ so that $E \subset A \times B$, and take the corresponding measure $\lambda_B : \mathfrak{S}(\mathcal{Q}) \to \langle 0, +\infty)$. Let $\boldsymbol{f}_n : T \to \boldsymbol{X}$, $n = 1, 2, \ldots$ be a sequence of $\mathcal{P} \otimes \mathcal{Q}$-simple functions such that $\boldsymbol{f}_n(t, s) \to \boldsymbol{f}(t, s)$ for each $(t, s) \in T \times S$, and let $\boldsymbol{X}_1$ be the closed linear span of the union of their ranges in $\boldsymbol{X}$. Then according to Theorem 11 we may replace $\boldsymbol{X}$ by the separable space $\boldsymbol{X}_1$. But then by Theorem 13-1), there is a countably additive measure $\mu_A : \mathfrak{S}(\mathcal{P}) \to \langle 0, +\infty)$ such that $C \in \mathfrak{S}(\mathcal{P})$ and $\mu_A(A \cap C) = 0 \Rightarrow \boldsymbol{m}^\wedge{}_1(A \cap C) = 0$, where $\boldsymbol{m}^\wedge{}_1$ is the semivariation of the restricted measure $\boldsymbol{m} : \mathcal{P} \to L(\boldsymbol{X}_1, \boldsymbol{Y})$ (clearly $\boldsymbol{m}^\wedge{}_1(C) \leqq \boldsymbol{m}^\wedge(C)$ for each $C \in \mathfrak{S}(\mathcal{P})$). Obviously $F = \bigcup_{n=0}^{\infty} \{(t, s) \in T \times S;\ \boldsymbol{f}_n(t, s) \neq 0\} \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) = \mathfrak{S}(\mathcal{P}) \otimes \mathfrak{S}(\mathcal{Q})$, where $\boldsymbol{f}_0 = \boldsymbol{f}$. Since $\lambda_B \otimes \mu_A : \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q}) \to \langle 0, +\infty)$ is a countably additive measure, according to the Egoroff--Lusin theorem, see Section 1.4 in Part I, there is a set $N \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$, $N \subset F$, and a sequence $F_k \in \mathcal{P} \otimes \mathcal{Q}$, $k = 1, 2, \ldots$ such that $(\lambda_B \otimes \mu_A)(N) = 0$, $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$ the sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$ converges uniformly to $\boldsymbol{f}$. Clearly $\boldsymbol{g}_E(s) = \boldsymbol{g}_{E \cap (F-N)}(s) + \boldsymbol{g}_{E \cap N}(s) = \lim_{k \to \infty} \boldsymbol{g}_{E \cap F_k}(s) + \boldsymbol{g}_{E \cap N}(s)$ for each $s \in S$. Owing to Theorem 4 each function $\boldsymbol{g}_{E \cap F_k}$, $k = 1, 2, \ldots$ is $\mathcal{Q}$-measurable. Thus to prove the theorem it is now sufficient to prove that the function $\boldsymbol{g}_{E \cap N}$ is $\boldsymbol{l}$-null. Obviously $\{s \in S;\ \boldsymbol{g}_{E \cap N}(s) \neq 0\} \subset B$. Since $0 = (\lambda_B \otimes \mu_A)(A \times B \cap N) = \int_B \mu_A(A \cap N^s)\, d\lambda_B$, there is a set $D \in \mathfrak{S}(\mathcal{Q})$ with $\lambda_B(B \cap D) = 0$ such that $\mu_A(A \cap N^s) = 0$ for each $s \in B - D$, see Theorem A in § 36 [21]. But then $\boldsymbol{m}^\wedge{}_1(A \cap N^s) = 0$, hence $\boldsymbol{g}_{E \cap N}(s) = 0$ for each $s \in B - D$. Thus $\{s \in S, \boldsymbol{g}_{E \cap N}(s) \neq 0\} \subset B \cap D$. However $\boldsymbol{l}^\wedge(B \cap D) = 0$, hence $\boldsymbol{g}_{E \cap N}$ is $\boldsymbol{l}$-null, which proves the theorem.

Remark 1. Let $\mathcal{D}$ be a δ-ring of subsets of S, let $\boldsymbol{f}: T \times S \to \boldsymbol{X}$ be a $\mathcal{P} \otimes \mathcal{D}$-measurable function and let for each $s \in S$ the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$. Then the $\mathcal{D}$-measurability of the function $\boldsymbol{g}_E$, $\boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, d\boldsymbol{m}$, $s \in S$, for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$, depends of course on the function $\boldsymbol{f}$. Particularly, if the range of $\boldsymbol{f}$ is relatively σ-compact in X, then Theorem 4 and Theorem 16 from Part I immediately imply the $\mathcal{D}$-measurability of $\boldsymbol{g}_E$ for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{D})$.

3. THE FUBINI THEOREM

For the proof of the general Fubini theorem we shall need also the following lemmas:

Lemma 3. *Let* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be* δ*-rings of subsets of* T *and* S, *respectively, and let* $\boldsymbol{f}: T \times S \to \boldsymbol{X}$ *be a* $\mathcal{D}_1 \otimes \mathcal{D}_2$*-measurable function. Then there are sequences* $A_n \in \mathcal{D}_1$, $B_n \in \mathcal{D}_2$, $n = 1, 2, \ldots$ *such that* $\boldsymbol{f}$ *is* $\delta(\{A_n \times B_n\}_{n=1}^{\infty})$*-measurable.*

Proof. According to the definition of a $\mathcal{D}_1 \otimes \mathcal{D}_2$-measurable function, see Section 1.2 in Part I, there is a sequence f_k, $k = 1, 2, \ldots$ of $\mathcal{D}_1 \otimes \mathcal{D}_2$-simple functions such that $f_k(t, s) \to f(t, s)$ for each $(t, s) \in T \times S$. Each f_k is of the form $f_k = \sum_{i=1}^{r_k} x_{k,i} \cdot \chi_{E_{k,i}}$ with $x_{k,i} \in X$, $E_{k,i} \in \mathcal{D}_1 \otimes \mathcal{D}_2$, $E_{k,i} \cap E_{k,j} = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r_k$. Since $\mathcal{D}_1 \otimes \mathcal{D}_2$ is the smallest δ-ring over all rectangles $A \times B$, $A \in \mathcal{D}_1$, $B \in \mathcal{D}_2$, the obviously valid δ-version of Theorem D in § 5 [21] implies that for each couple (k, i), $k = 1, 2, \ldots$, $i = 1, \ldots, r_k$, there are sequences $A_{k,i,j} \in \mathcal{D}_1$, $B_{k,i,j} \in \mathcal{D}_2$, $j = 1, 2, \ldots$, such that $E_{k,i} \in \delta(\{A_{k,i,j} \times B_{k,i,j}\}_{j=1}^{\infty})$. By a suitable enumeration of the countable set $\{(k, i, j);\ k = 1, 2, \ldots,\ i = 1, \ldots, r_k,\ j = 1, 2, \ldots\}$ we immediately obtain the required sequences $A_n \in \mathcal{D}_1$, $B_n \in \mathcal{D}_2$, $n = 1, 2, \ldots$.

The following lemma is an immediate consequence of the Orlicz-Pettis theorem, see Theorem 3.2.3 in [22] and Theorem IV.10.1 in [19].

Lemma 4. *Let $z_{n,k} \in \boldsymbol{Z}$, $k, n = 1, 2, \ldots$, let the series $\sum_{k=1}^{\infty} z_{n,k}$ be unconditionally convergent in $\boldsymbol{Z}$ for each $n = 1, 2, \ldots$ and let for each $I_n \subset \{1, 2, \ldots\}$ the series $\sum_{n=1}^{\infty} \sum_{k \in I_n} z_{n,k}$ be unconditionally convergent in $\boldsymbol{Z}$. Then the series $\sum_{k,n=1}^{\infty} z_{n,k}$ is unconditionally convergent in $\boldsymbol{Z}$.*

Using these lemmas we prove

Lemma 5. *Let $f: T \times S \to \boldsymbol{X}$ be a $\mathcal{P} \otimes \mathcal{Q}$-measurable function, let the function $f(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$, and let the function g_E, $g_E(s) = \int_{E^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, be integrable with respect to $\boldsymbol{l}$ for each $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$. Then the set function $E \to \int_S \int_{E^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l}$, $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$, is a countably additive $\boldsymbol{Z}$-valued vector measure on $\mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$.*

Proof. Let $E_k \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$, $k = 1, 2, \ldots$, be pairwise disjoint and let $E_0 = \bigcup_{k=1}^{\infty} E_k$. We have to show that $\int_S \int_{E_0^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l} = \sum_{k=1}^{\infty} \int_S \int_{E_k^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l}$ in the sense of unconditional convergence. According to Theorem 16 in Part I it suffices to show that the series on the right hand side is unconditionally convergent in $\boldsymbol{Z}$.

According to Lemma 3 there is a countable family $\mathcal{A} \subset \mathcal{P}$ such that $E_k \in \mathfrak{S}(\mathcal{A}) \otimes \mathfrak{S}(\mathcal{Q})$ for each $k = 0, 1, 2, \ldots$. Take $A \in \mathfrak{S}(\mathcal{A})$ and $B \in \mathfrak{S}(\mathcal{Q})$ so that $E_0 \subset A \times B$, and a sequence $B_n \in \mathcal{Q}$, $n = 0, 1, \ldots$ such that $B_n \nearrow B$ and $B_0 = \emptyset$. According to Theorem 12-3), the function v, $v(s) = \sup_{A_1 \in \mathfrak{S}(\mathcal{A})} \left|\int_{A_1 \cap E_0^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}\right|$, $s \in S$, is finite valued and $\mathcal{Q}$-measurable. Therefore $F_n = \{s \in S;\ 0 \leqq v(s) < n\} \in \mathfrak{S}(\mathcal{Q})$ for each $n = 0, 1, \ldots$, and $F_n \nearrow$. Put $G_n = B_n \cap F_n - B_{n-1} \cap F_{n-1}$, $n = 1, 2, \ldots$. Then G_n, $n = 1, 2, \ldots$ are pairwise disjoint elements of $\mathcal{Q}$ and $\bigcup_{n=1}^{\infty} G_n \subset B$. Put $z_{n,k} = \int_{G_n} \int_{E_k^s} f(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l}$, $n, k = 1, 2, \ldots$. Using Lemma 4 we shall show that the series $\sum_{n,k=1}^{\infty} z_{n,k}$ is unconditionally convergent in Z, and this will prove the lemma, since then by Theorem 16 from Part I we have $\sum_{n,k=1}^{\infty} z_{n,k} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \int_{G_n} \int_{E_k{}^s} f(\cdot, s)$ $\mathrm{d}\boldsymbol{m}\ \mathrm{d}\boldsymbol{l} = \sum_{k=1}^{\infty} \int_S \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}\boldsymbol{l}$. Hence it remains to verify the validity of the assumptions of Lemma 4.

Let n be fixed. We shall show that for each $z^* \in Z^*$ the equality $z^* \int_{G_n} \int_{E_0{}^s} f(\cdot, s)$ $\mathrm{d}\boldsymbol{m}\ \mathrm{d}\boldsymbol{l} = \sum_{k=1}^{\infty} z^* \int_{G_n} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}\boldsymbol{l} = \sum_{k=1}^{\infty} z^* z_{n,k}$ holds in the sense of unconditional convergence, and this by the Orlicz-Pettis theorem will prove the unconditional convergence of the series $\sum_{k=1}^{\infty} z_{n,k}$ in Z.

Since by assumption $f(\cdot, s)$ is integrable with respect to $\boldsymbol{m}$ for each $s \in S$, Theorem 16 from Part I immediately yields that $\int_{E_0{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m} = \sum_{k=1}^{\infty} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}$ in the sense of unconditional convergence in Z, for each $s \in S$.

From the definition of the function v it is clear that $\left|\sum_{k\in K} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\right| \leqq v(s)$ for each $s \in S$ and each $K \subset \{1, 2, \ldots\}$. Thus for any finite $K \subset \{1, 2, \ldots\}$ we have, see Theorem 14 in Part I, that $\left|\sum_{k\in K} z^* \int_{G_n} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}\boldsymbol{l}\right| \leqq |z^*| \,.\, \left|\int_{G_n}\left(\sum_{k\in K} \int_{E_k{}^s} f(\cdot, s)\right.\right.$ $\left.\left.\mathrm{d}\boldsymbol{m}\right)\,\mathrm{d}\boldsymbol{l}\right| \leqq |z^*| \,.\, \sup_{s\in G_n} \left|\sum_{k\in K} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\right| \,.\, \boldsymbol{l}^{\wedge}(G_n) \leqq |z^*| \,.\, \sup_{s\in G_n} v(s) \,.\, \boldsymbol{l}^{\wedge}(B_n) \leqq |z^*| \,.\, n \,.$ $.\, \boldsymbol{l}^{\wedge}(B_n) < +\infty$. Hence the series $\sum_{k=1}^{\infty} z^* \int_{G_n} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}\boldsymbol{l} = \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k{}^s} f(\cdot, s)$ $\mathrm{d}\boldsymbol{m}\ \mathrm{d}(z^*\boldsymbol{l})$ is unconditionally convergent in Z, hence by Theorem 16 from Part I $\sum_{k=1}^{\infty} z^* z_{n,k} = \sum_{k=1}^{\infty} \int_{G_n} \int_{E_k{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}(z^*\boldsymbol{l}) = \int_{G_n} \int_{E_0{}^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}(z^*\boldsymbol{l}) = z^* \int_{G_n} \int_{E_0{}^s} f(\cdot, s)$ $\mathrm{d}\boldsymbol{m}\ \mathrm{d}\boldsymbol{l}$, which was to be shown.

Let now $I_n \subset \{1, 2, \ldots\}$, $n = 1, 2, \ldots$, and put $E = \bigcup_{n=1}^{\infty} (T \times G_n) \cap (\bigcup_{k\in I_n} E_k)$. Since G_n, $n = 1, 2, \ldots$, are pairwise disjoint, the integrability of $\boldsymbol{g}_E$ with respect to $\boldsymbol{l}$ implies that the series $\sum_{n=1}^{\infty} \int_{G_n} \int_{(\bigcup_{k\in I_n} E_k)^s} f(\cdot, s)\,\mathrm{d}\boldsymbol{m}\,\mathrm{d}\boldsymbol{l} = \sum_{n=1}^{\infty} \left(\sum_{k\in I_n} z_{n,k}\right)$ is unconditionally convergent in Z. Thus the assumptions of Lemma 4 are satisfied, which was to be shown.

Lemma 6. *Let $f: T \to X$ be a $\mathscr{P}$-measurable function. Then there is a countably additive measure $\lambda : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ such that $N \in \mathfrak{S}(\mathscr{P})$, $\lambda(N) = 0 \Rightarrow f \,.\, \chi_N$ is integrable with respect to $\boldsymbol{m}$ and $\int_N f\,\mathrm{d}\boldsymbol{m} = 0$.*

Proof. Let $f_n : T \to X$, $n = 1, 2, \ldots$, be a sequence of $\mathscr{P}$-simple functions such that $f_n(t) \to f(t)$ for each $t \in T$. To each vector measure $A \to \int_A f_n\,\mathrm{d}\boldsymbol{m}$, $A \in \mathfrak{S}(\mathscr{P})$, $n =$

$= 1, 2, \ldots$, take a control measure $\lambda_n : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$. Now it suffices to put

$$\lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(A)}{1 + \lambda_n(T)}, \quad A \in \mathfrak{S}(\mathscr{P}).$$

Theorem 15. (The Fubini theorem.) *Let the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathscr{P} \otimes \mathscr{Q} \to \to L(X, Z)$ exist and let $\boldsymbol{f}: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{Q}$-measurable function. Let further the function $\boldsymbol{f}(\cdot, s)$ be integrable with respect to $\boldsymbol{m}$ for each $s \in S$, and let for each set $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ the function $\boldsymbol{g}_E$, $\boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, be $\boldsymbol{l}$-essentially $\mathscr{Q}$-measurable. Then the following conditions are equivalent:*

a) *$\boldsymbol{f}$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, and*

b) *$\boldsymbol{g}_E$ is essentially integrable with respect to $\boldsymbol{l}$ for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$,*

and if they hold, then

(F) $\int_E \boldsymbol{f}\, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_S \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}\, \mathrm{d}\boldsymbol{l}$ *for each* $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$.

Proof. Without loss of generality we may suppose that $\boldsymbol{g}_E$ is $\mathscr{Q}$-measurable for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. Let $\boldsymbol{f}_n : T \to X$, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P} \otimes \mathscr{Q}$-simple functions such that $\boldsymbol{f}_n(t, s) \to \boldsymbol{f}(t, s)$ and $|\boldsymbol{f}_n(t, s)| \nearrow |\boldsymbol{f}(t, s)|$ for each $(t, s) \in T \times S$. For each vector measure $E \to \int_E \boldsymbol{f}_n\, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m})$, $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, $n = 1, 2, \ldots$, take a control measure $\lambda_n : \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \to \langle 0, +\infty)$ and put

$$\lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(E)}{1 + \lambda_n(T)}, \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}).$$

Let X_1 be the closed linear span of the set $\{\boldsymbol{f}_n(t, s);\ (t, s) \in T \times S,\ n = 1, 2, \ldots\}$. Then X_1 is a separable Banach space, and according to Theorem 11 we may replace X by X_1, hence we may suppose that X is a separable Banach space.

Take $A_0 \in \mathfrak{S}(\mathscr{P})$ and $B_0 \in \mathfrak{S}(\mathscr{Q})$ so that $F = \{(t, s) \in T \times S;\ \boldsymbol{f}(t, s) \neq 0\} \subset \subset A_0 \times B_0$. Then by Theorem 13-1) there is a countably additive measure $\gamma_{A_0} : : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ such that $C \in \mathfrak{S}(\mathscr{P})$, $\gamma_{A_0}(A_0 \cap C) = 0 \Rightarrow \boldsymbol{m}^\wedge(A_0 \cap C) = 0$.

Let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. By assumption the function $\boldsymbol{g}_E$, $\boldsymbol{g}_E(s) = \int_{E^s} \boldsymbol{f}(\cdot, s)\, \mathrm{d}\boldsymbol{m}$, $s \in S$, is $\mathscr{Q}$-measurable. Hence by Lemma 6 there is a countably additive $\omega_E : \mathfrak{S}(\mathscr{Q}) \to \to \langle 0, +\infty)$ such that $D \in \mathfrak{S}(\mathscr{Q})$, $\omega_E(D) = 0$ implies that $\boldsymbol{g}_E \cdot \chi_D$ is integrable with respect to $\boldsymbol{l}$ and $\int_D \boldsymbol{g}_E\, \mathrm{d}\boldsymbol{l} = 0$.

Put $\mu_E(G) = \lambda(G) + (\omega_E \otimes \gamma_{A_0})(G)$, $G \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. Then we conclude from the above and from Theorem A in § 36 [21] that if $N \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ and $\mu_E(N) = 0$, then the function $\boldsymbol{f} \cdot \chi_{N \cap E}$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, the function $\boldsymbol{g}_{E \cap N}$ is integrable with respect to $\boldsymbol{l}$, and $\int_{E \cap N} \boldsymbol{f}\, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_S \boldsymbol{g}_{E \cap N}\, \mathrm{d}\boldsymbol{l} = 0$.

According to the Egoroff-Lusin theorem, see Section 1.4 in Part I, there is an $N \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ with $\mu_E(N) = 0$ and a sequence $F_k \in \mathscr{P} \otimes \mathscr{Q}$, $k = 1, 2, \ldots$, such that $F_k \nearrow F - N$ and on each F_k, $k = 1, 2, \ldots$, the sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$. Thus by Theorem 4 the function $\boldsymbol{f} \cdot \chi_{E \cap F_k}$ is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$ for each $k = 1, 2, \ldots$, the function $\boldsymbol{g}_{E \cap F_k}$ is integrable with respect to $\boldsymbol{l}$, and

(1)
$$\int_{G \cap E \cap F_k} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_S \boldsymbol{g}_{E \cap F_k \cap G} \, \mathrm{d}\boldsymbol{l} =$$
$$= \int_S \int_{(E \cap F_k \cap G)^s} f(\cdot, s) \, \mathrm{d}\boldsymbol{m} \, \mathrm{d}\boldsymbol{l} \quad \text{for each} \quad G \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) .$$

Since by assumption, the function $f(\cdot, s)$ is integrable with respect to $\boldsymbol{m}$ for each $s \in S$, we have

(2)
$$\boldsymbol{g}_{E \cap F_k}(s) = \int_{(E \cap F_k)^s} f(\cdot, s) \, \mathrm{d}\boldsymbol{m} \to \int_{[E \cap (F - N)]^s} f(\cdot, s) \, \mathrm{d}\boldsymbol{m} =$$
$$= \boldsymbol{g}_{E \cap (F \cap N)}(s) = \boldsymbol{g}_{E - N}(s) \quad \text{for each} \quad s \in S .$$

a) $\Rightarrow$ b) and (F). Suppose that f is integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$, and let $B \in \mathfrak{S}(\mathscr{Q})$. Then

(3)
$$\int_B \boldsymbol{g}_{E \cap F_k} \, \mathrm{d}\boldsymbol{l} = \int_{(A_0 \times B) \cap E \cap F_k} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) \to$$
$$\to \int_{(A_0 \times B) \cap (F - N) \cap E} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_{(A_0 \times B) \cap E} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) .$$

Thus by Theorem 16 from Part I, (2) and (3) imply that the function $\boldsymbol{g}_{E-N}$, hence also $\boldsymbol{g}_E$, is integrable with respect to $\boldsymbol{l}$ and that $\int_B \boldsymbol{g}_E \, \mathrm{d}\boldsymbol{l} = \int_B \boldsymbol{g}_{E-N} \, \mathrm{d}\boldsymbol{l} = \int_{(A_0 \times B) \cap E} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m})$ for each $B \in \mathfrak{S}(\mathscr{Q})$. Taking $B = B_0$ we have also the equality (F).

b) $\Rightarrow$ a) and (F). Suppose now that $\boldsymbol{g}_E$ is integrable with respect to $\boldsymbol{l}$ for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. Take $E = A_0 \times B_0$ in the proof of a) $\Rightarrow$ b) and (F) above. Then $f \cdot \chi_{F_k} = f \cdot \chi_{(A_0 \times B_0) \cap F_k}$ in integrable with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$ for each $k = 1, 2, \ldots$, and

(4)
$$(f \cdot \chi_{F_k})(t, s) \to (f \cdot \chi_{F-N})(t, s) \quad \text{for each} \quad (t, s) \in T \times S .$$

Since by Lemma 5 the set function $G \to \int_S \boldsymbol{g}_G \, \mathrm{d}\boldsymbol{l}$. $G \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ is a countably additive vector measure, by (1) we have

(5)
$$\int_G f \cdot \chi_{F_k} \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) = \int_{G \cap (A_0 \times B_0) \cap F_k} f \, \mathrm{d}(\boldsymbol{l} \otimes \boldsymbol{m}) =$$
$$= \int_S \boldsymbol{g}_{(A_0 \times B_0) \cap F_k \cap G} \, \mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{g}_{F_k \cap G} \, \mathrm{d}\boldsymbol{l} \to \int_S \boldsymbol{g}_{G \cap (F-N)} \, \mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{g}_G \, \mathrm{d}\boldsymbol{l} .$$

According to Theorem 16 from Part I, (4) and (5) imply the integrability of f with respect to $\boldsymbol{l} \otimes \boldsymbol{m}$ and the equality (F). The theorem is proved.

From Theorems 3, 13-3), 14, 15, and from Theorems 5 and 14 from part I we immediately obtain

Theorem 16. *Let $f : T \times S \to X$ be a bounded $\mathscr{P} \otimes \mathscr{Q}$-measurable function, let $m^\wedge(T) < +\infty$, let the function $f(\cdot, s)$ be integrable with respect to m for each $s \in S$ (if $\mathscr{P}^\sim = \mathscr{P} = \mathfrak{S}(\mathscr{P})$, then by Theorem 5 from Part I this is always true), and let $\mathscr{Q}^\sim = \mathscr{Q} = \mathfrak{S}(\mathscr{Q})$. Then the product measure $l \otimes m : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$ exists, the function g_E, $g_E(s) = \int_{E^s} f(\cdot, s)\, \mathrm{d}m$, $s \in S$, is essentially integrable with respect to l for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, the function f is integrable with respect to $l \otimes m$, and $\int_E f\, \mathrm{d}(l \otimes m) = \int_S \int_{E^s} f(\cdot, s)\, \mathrm{d}m\, \mathrm{d}l$ for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$.*

Remark 2. Let the product measure $l \otimes m : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$ exist, let $f : T \times S \to X$ be integrable with respect to $l \otimes m$, and let the function $f(\cdot, s)$ be integrable with respect to m for each $s \in S$. Then it is clear from the proof of Theorem 15, that if μ_E is replaced in this proof by the measure λ defined there, then there is a set $N \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ such that g_{E-N} is integrable with respect to l for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ and $\int_E f\, \mathrm{d}(l \otimes m) = \int_{E-N} f\, \mathrm{d}(l \otimes m) = \int_S g_{E-N}\, \mathrm{d}l$ for each $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$. (Using Theorem 13-1) we may take $N \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ such that $(l \otimes m)(N) = 0$.) However, as Example at the beginning of § 2 shows, it may happen that $N = T \times S$.

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