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ON INTERMEDIATE RICCI CURVATURE AND FUNDAMENTAL GROUPS

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Synge's Theorem states that a closed Riemannian n-manifold with positive sectional curvature is orientable if n is odd and has fundamental group of order 1 or 2 if n is even. Products of real projective spaces show that Synge's Theorem is false for positive Ricci curvature. On the other hand, there is some evidence which suggests that large manifolds with positive Ricci curvature resemble large manifolds with positive sectional curvature ([Ander1], [Cold1,2], [CheCol1,2], [Perl1,2]). There is also some evidence to the contrary ([Ander2], [Otsu]).

It will be shown here that Synge's Theorem remains valid for any manifold M with positive Ricci curvature provided the first systole, $sys_1\ M$ (i.e., the length of the shortest closed noncontractible curve) is sufficiently large.

THEOREM 1. *Let M be a complete Riemannian n-manifold with $Ric\ M \geq n-1$ and $sys_1\ M > \pi\sqrt{\frac{n-2}{n-1}}$.*

(i) *If n is even and M is orientable, then M is simply connected.*
(ii) *If n is odd, then M is orientable.*

It is easy to see that a nonsimply connected, complete, Riemannian n-manifold with $Ric\ M \geq n-1$ has $sys_1\ M \leq \pi$ and that equality holds only if M is isometric to RP^n. Indeed if $sys_1\ M \geq r$, then the diameter of the universal cover $\tilde{M}$ is $\geq r$, so r must be $\leq \pi$ by the Bonnet-Myers Theorem. If $r = \pi$, then $\tilde{M}$ is isometric to S^n by [Cheng], and M is easily seen to be RP^n (cf. [Wil1]). By combining results in [Cold1,2], [CheCol2], and [FukYam] with an idea from [Wil1,2] it is also easy to see the following.

> *Given $n \in \mathbb{N}$ there is an $\varepsilon(n) > 0$ so that a complete, nonsimply connected, Riemannian n-manifold M with $Ric\ M \geq n-1$ and $sys_1\ M \geq \pi - \varepsilon$ is diffeomorphic to RP^n.*

(See the end of the paper for a sketch of the proof.)

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A careful analysis of the proofs of the results cited above might yield an explicit estimate for $\pi - \varepsilon(n)$; however, the answer would be considerably larger than the number $\pi\sqrt{\frac{n-2}{n-1}}$ in Theorem 1. In fact the pinching constant in Theorem 1 is optimal.

Example A. For each natural number $k \geq 2$, put the product metric on $S^k \times S^k$ with each factor having constant curvature 1. Let $\mathbb{Z}_2$ act as the antipodal map on both factors. Then the quotient $M = (S^k \times S^k)/\mathbb{Z}_2$ is orientable, has fundamental group isomorphic to $\mathbb{Z}_2$, Ricci curvature $\equiv k - 1$, and $sys_1\ M = \pi\sqrt{2}$. To rescale so that the Ricci curvature is $2k - 1 = \dim M - 1$, we must multiply all lengths by $\sqrt{\frac{k-1}{2k-1}}$. So sys_1 becomes $\pi\sqrt{2}\sqrt{\frac{k-1}{2k-1}} = \pi\sqrt{\frac{2k-2}{2k-1}}$.

Example B. In the odd dimensional case we also consider a product of spheres, $S^k \times S^{k+1}$ $(k \geq 2)$ with an Einstein product metric, and set $M = (S^k \times S^{k+1})/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is acting as the antipodal map on both factors. If the S^k factor has constant curvature $\frac{k}{k-1}$ and the S^{k+1} factor has constant curvature 1, then the metric is Einstein with Ricci curvature $\equiv k$, and $sys_1 M = \pi\sqrt{\frac{2k-1}{k}}$. To rescale so that M has Ricci curvature $\equiv 2k = \dim M - 1$ we must multiply all lengths by $\sqrt{\frac{1}{2}}$. The first systole then becomes $\pi\sqrt{\frac{2k-1}{2k}}$.

Nothing can be done about the fact that the pinching constant in Theorem 1 converges to π as n goes to ∞. (It is optimal.) As one might expect, the reason for this is that, in some sense, the hypothesis $Ric\ M \geq n - 1$ means less and less as n goes to infinity. This principle is exemplified in the proof as well as in the examples above. However, the method of proof is quite flexible. One can change both the curvature and the systole hypotheses and obtain additional information, and with these changes the pinching constant becomes independent of the dimension.

To be concrete, we recall ([Wu], [Shen]) that a Riemannian manifold M is said to have k^{th}-Ricci curvature $\geq \iota$ provided that for any choice $\{v, w_1, w_2, \dots, w_k\}$ of an orthonormal $(k+1)$-frame the sum of sectional curvatures $\sum_{i=1}^{k} sec(v, w_i)$ is $\geq \iota$. In short hand this is written as $Ric_k\ M \geq \iota$. Clearly $Ric_k\ M \geq \iota k$ implies $Ric_{k+1}\ M \geq \iota(k+1)$. $Ric_1\ M \geq \iota$ is the same as $sec\ M \geq \iota$, and $Ric_{n-1}\ M \geq \iota$ is the same as $Ric\ M \geq \iota$. Theorem 1 is a special case of our

MAIN THEOREM. *Let M be a complete Riemannian n-manifold with $Ric_k\ M \geq k$ and $sys_1\ M > \pi\sqrt{\frac{k-1}{k}}$.*

(i) *If n is even and M is orientable, then M is simply connected.*
(ii) *If n is odd, then M is orientable.*

This generalizes Synge's Theorem because in case $k = 1$ the pinching constant is 0.

Remark. There have apparently been no systematic attempts to find "legitimate" examples of manifolds with positive intermediate Ricci curvatures. (An "illegitimate" example is one that has positive sectional curvature.) It seems to the author that the techniques of [Berger], [Wal], [AlWal], [Esch1,2], and [Baza] should also yield some "legitimate" examples. As a starting point one should ask:

> Which n-dimensional, normal homogeneous spaces have $Ric_k > 0$ for some k which is small compared to n?

Proof of the main theorem. Part (i) would follow if we could show that a complete, even dimensional, orientable, nonsimply connected Riemannian manifold M with $Ric_k\ M \geq k$ has $sys_1\ M \leq \pi\sqrt{\frac{k-1}{k}}$. Similarly part (ii) follows if a complete, odd dimensional, nonorientable, Riemannian manifold M with $Ric_k\ M \geq k$ has $sys_1\ M \leq \pi\sqrt{\frac{k-1}{k}}$.

The two statements are proven together.

Let $\gamma\colon [0, l] \longrightarrow M$ be a noncontractable, normal, geodesic loop which is of minimal length in its free homotopy class. In case M is nonorientable and odd dimensional we also assume that γ induces an orientation reversing deck transformation on the universal cover $\tilde{M}$. To prove the main theorem it suffices to show that $l \leq \pi\sqrt{\frac{k-1}{k}}$.

The hypotheses on orientability and dimension parity are used to appeal to the following standard lemma of linear algebra [doCar].

Lemma 2. *Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$, and suppose* $\det A = (-1)^n$. *Then A fixes some nonzero vector of $\mathbb{R}^{n-1}$.*

Our hypotheses on γ combined with Lemma 2 imply that the map $P\colon T_{\gamma(0)}M \longrightarrow T_{\gamma(0)}M$ given by parallel transport around γ fixes a nonzero vector perpendicular to $\dot{\gamma}(0)$. We can therefore find a unit, periodic, parallel field E along γ with period l. Let $V\colon (-\varepsilon, \varepsilon) \times [0, l] \longrightarrow M$ be the variation

$$V(s, t) = \exp_{\gamma(t)} sE(t),$$

and let $\mathrm{Len}(s)$ denote the length of the curve $t \mapsto V(s, t)$. Then from the second variation formula and the fact that γ is a loop with minimal length in its free homotopy class we get

$$\begin{aligned} 0 \leq \frac{d^2}{ds^2}\mathrm{Len}(s)|_{s=0} &= \int_0^l \langle E', E'\rangle - \langle R(E, \dot{\gamma})\dot{\gamma}, E\rangle dt \\ &= -\int_0^l \langle R(E, \dot{\gamma})\dot{\gamma}, E\rangle dt \end{aligned} \tag{3}$$

Let $\{F_i\}_{i=1}^{k-1}$ be a parallel orthonormal $(k-1)$-frame along γ that is perpendicular to E and $\dot{\gamma}$. Since $Ric_k\ M \geq k$ it follows from equation (3) that

$$\frac{1}{l}\int_0^l \sum_{i=1}^{k-1} \langle R(F_i, \dot{\gamma})\dot{\gamma}, F_i\rangle dt \geq k. \tag{4}$$

Let $\tilde{\gamma}$ be a lift of γ to the universal cover, $\tilde{M}$ of M. Let $\tilde{F}_i$ be the lift of F_i. Set $\tilde{W}_i = \sin\frac{\pi t}{l}\tilde{F}_i$. Arguing as in the proof of the Bonnet-Myers Theorem, we find that the sum of the indices

$$\begin{aligned} 0 \leq \sum_{i=1}^{k-1} I(\tilde{W}_i, \tilde{W}_i) &= -\sum_{i=1}^{k-1}\int_0^l \langle \tilde{W}_i, \tilde{W}_i'' + R(\tilde{W}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}\rangle dt \\ &= \sum_{i=1}^{k-1}\int_0^l \sin^2\left(\frac{\pi t}{l}\right)\left(\frac{\pi^2}{l^2}\langle \tilde{F}_i, \tilde{F}_i\rangle - \langle R(\tilde{F}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \tilde{F}_i\rangle\right) dt, \end{aligned} \tag{5}$$

where the inequality is due to the fact that $\tilde{\gamma}|_{[0,l]}$ is minimal. On the other hand, $\tilde{\gamma}|_{[\frac{l}{2},\frac{3l}{2}]}$ is also minimal, so arguing as above we find

$$0 \leq \sum_{i=1}^{k-1}\int_{\frac{l}{2}}^{\frac{3l}{2}} \cos^2\left(\frac{\pi t}{l}\right)\left(\frac{\pi^2}{l^2}\langle \tilde{F}_i, \tilde{F}_i\rangle - \langle R(\tilde{F}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \tilde{F}_i\rangle\right) dt. \tag{6}$$

We show below that if we make an appropriate choice of parametrization of $\tilde{\gamma}$ and an appropriate choice of $\{\tilde{F}_i\}_{i=1}^{k-1}$, then the right hand side of (6) is

$$\leq \sum_{i=1}^{k-1}\int_0^l \cos^2\left(\frac{\pi t}{l}\right)\left(\frac{\pi^2}{l^2}\langle \tilde{F}_i, \tilde{F}_i\rangle - \langle R(\tilde{F}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \tilde{F}_i\rangle\right) dt. \tag{7}$$

Taking this for granted for the moment and adding the resulting inequality to (5) gives us

$$\sum_{i=1}^{k-1}\int_0^l \frac{\pi^2}{l^2}\langle \tilde{F}_i, \tilde{F}_i\rangle - \langle R(\tilde{F}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \tilde{F}_i\rangle dt \geq 0. \tag{8}$$

Combining this with inequality 4 we get

$$\frac{\pi^2}{l}(k-1) = \sum_{i=1}^{k-1}\int_0^l \frac{\pi^2}{l^2}\langle \tilde{F}_i, \tilde{F}_i\rangle dt \geq \sum_{i=1}^{k-1}\int_0^l \langle R(\tilde{F}_i, \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, \tilde{F}_i\rangle dt \geq kl,$$

or

$$\pi^2\frac{k-1}{k} \geq l^2$$

or

$$\pi\sqrt{\frac{k-1}{k}} \geq l$$

as desired.

It remains to show that inequality (7) is valid provided we make the right choice of $\{\tilde{F}_i\}_{i=1}^{k-1}$ and choose the appropriate parametrization of $\tilde{\gamma}$.

By an abuse of notation we let $\tilde{\gamma}$ denote the geodesic extension of $\tilde{\gamma}$ to $[0, \infty)$. Let $\mathcal{F}^{k-1}\tilde{\gamma}(t)$ be the set of orthonormal $(k-1)$-frames at $\tilde{\gamma}(t)$ that are perpendicular to $\dot{\tilde{\gamma}}$ and the lift of E, and let $\mathcal{F}^{k-1}\tilde{\gamma} = \bigcup_{t\in\mathbb{R}} \mathcal{F}^{k-1}\tilde{\gamma}(t)$. Since γ is periodic and $\pi_1(M)$ is finite, $\tilde{\gamma}$ is periodic. Therefore $\mathcal{F}^{k-1}\tilde{\gamma}$ is compact.

We define a continuous function $I\colon \mathcal{F}^{k-1}\tilde{\gamma} \longrightarrow \mathbb{R}$ by

$$I(\{f_i\}_{i=1}^{k-1}) = -\sum_{i=1}^{k-1}\int_{\tau}^{\tau+\frac{l}{2}} \cos^2\left(\frac{\pi}{l}(t-\tau)\right)\langle R(P_t(f_i), \dot{\tilde{\gamma}})\dot{\tilde{\gamma}}, P_t(f_i)\rangle dt,$$

where τ is the parameter time along $\tilde{\gamma}$ of the foot point of $\{f_i\}_{i=1}^{k-1}$ and $P_t(f_i)$ denotes the extension of f_i to a parallel field along $\tilde{\gamma}$. Then inequality (7) will be valid provided we choose $\{\tilde{f}_i\}_{i=1}^{k-1}$ so that it maximizes I and (if necessary) we choose a (normal geodesic) reparametrization $\tilde{\tilde{\gamma}}$ of $\tilde{\gamma}$ with $\tilde{\tilde{\gamma}}(0) = \tilde{\gamma}(\tau)$. □

Concluding remarks

Remark 9. Notice that the proof of the main theorem really only requires that γ is a loop of minimal length in its free homotopy class. Thus it shows that in a complete, even dimensional, orientable, Riemannian manifold M with $Ric_k\ M \geq k$, *every* free homotopy class contains a representative with length $\leq \pi\sqrt{\frac{k-1}{k}}$, and in a complete, odd dimensional, nonorientable, Riemannian manifold M with $Ric_k\ M \geq k$ *every* free homotopy class that induces an orientation reversing deck transformation of $\tilde{M}$ contains a representative with length $\leq \pi\sqrt{\frac{k-1}{k}}$.

Also notice that for the proof we really only need $Ric_k \geq k$ for all orthonormal $k+1$-frames $\{v, w_1, w_2, \ldots, w_k\}$ with $v = \dot{\gamma}$ and even this is only needed in an integral sense. If the theorem were restated along these lines, it would be optimal for all k. The relevant Ric_k's in Examples A and B are all $\geq \frac{k-1}{2}$ (if all factors have constant curvature 1). To rescale so that all of these Ric_k's are $\geq k$ we must multiply the metric by $\sqrt{\frac{k-1}{2k}}$, and hence sys_1 becomes $\pi\sqrt{\frac{k-1}{k}}$ as required.

Remark 10. The reader might recall that Weinstein has shown that any isometry of a compact, oriented, Riemannian, n-manifold with positive sectional curvature has a fixed point if either n is even and f preserves orientation or n is odd and f reverses orientation[1] ([Wein] or [doCar]). Since Synge's theorem is an easy corollary of this, it should not be surprising that with a little more work we can prove the following.

THEOREM 11. *Let M be a complete, oriented Riemannian n-manifold with $Ric_k \geq k$ and let $f\colon M \longrightarrow M$ be an isometry of M whose minimal displacement is $> \pi\sqrt{\frac{k-1}{k}}$. Then*

(i) *f reverses orientation if n is even, and*
(ii) *f preserves orientation if n is odd.*

Sketch of proof. Let p be a point where the displacement of f is minimal, and let $\gamma\colon [0, l] \longrightarrow M$ be a normal, minimal geodesic with $\gamma(0) = p$ and $\gamma(l) = f(p)$. Then the orbit of γ under the iterates of f determines a smooth geodesic extension $\tilde{\gamma}$ of γ. From here the proof is very similar to the proof of the main theorem. The extra technicalities arise from the fact that $\tilde{\gamma}$ may not be periodic, and hence the justification of inequality (7) will no longer be valid. There would be no problem if the supremum of the functional I restricted to the set of frames with foot point in the orbit of p, $\{\mathcal{F}^{k-1}_{f^i(p)}\gamma\}_{i=1}^{\infty}$, is realized. If this supremum is not realized, take a sequence of frames $\{\tilde{F}_{i,j}\}_{i,j\in\mathbb{N},1\leq j\leq k-1}$ almost realizing the supremum of I. Say the foot points of these frames is $\{f^{i_m}(p)\}_{m=1}^{\infty}$. The sequence of frames subconverges to a frame $\{\bar{F}_j\}_{j=1}^{k-1}$. Let x be the foot point of $\{\bar{F}_j\}_{j=1}^{k-1}$. Then $\{\tilde{\gamma}|_{[i_m l,(i_m+1)l]}\}_{m=1}^{\infty}$ subconverges to a geodesic $\bar{\gamma}$, and $E|_{[i_m l,(i_m+1)l]}$ subconverges to a parallel field on $\bar{\gamma}$. The proof of the main theorem now goes through with $\tilde{\gamma}$ replaced by $\bar{\gamma}$, E replaced by $\bar{E}$, and $\{F_i\}_{i=1}^{k-1}$ replaced by $\{\bar{F}_j\}_{j=1}^{k-1}$. Inequality (7) will hold by construction.

Remark 12. We sketch the proof of the pinching theorem stated on page 488.

Recall that the radius of a compact metric space X is given by $\operatorname{rad} X = \min_{x\in X}\max_{y\in X}\operatorname{dist}(x, y)$. Let $sys_1\, M$ be $\geq \pi - \varepsilon$ for some sufficiently small $\varepsilon > 0$. Then the radius of the universal cover $\tilde{M}$ of M is $\geq \pi - \varepsilon$. By Theorem C in [Cold2] and the main theorem in [Cold1] this implies that $\tilde{M}$ is Gromov-Hausdorff close to the unit sphere $S^n(1)$ and has volume almost equal to that of $S^n(1)$. By Theorem A.10 in [CheCol2], $\tilde{M}$ is diffeomorphic to S^n, and it is easy to see that the action of $\pi_1(M)$ on $\tilde{M}$ is close to the antipodal action on $S^n(1)$ in the sense of [FukYam], Definition 3.3. Therefore by Lemma 3.4 of [FukYam], M is Gromov-Hausdorff close to the constant curvature 1 metric on RP^n, and hence is diffeomorphic to RP^n by Theorem 1.12 of [CheCol2].

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[1] Actually f only has to be conformal.

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