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On Interpretability of Almost Linear Orderings

AKITO TSUBOI and KENTARO WAKAI

Abstract In this paper we define the notion of m-linearity for $m \in \omega (m > 1)$ and discuss interpretability (and noninterpretability) of m-linear orders in structures and theories.

1 Introduction We say that a structure N is $(\emptyset)$-interpretable in a structure M if N is $(\emptyset)$-definable in M^{eq}. We also say that a structure N is $(\emptyset)$-interpretable in a theory T if N is $(\emptyset)$-interpretable in some model M of T. (For the definition of eq-structures, see Shelah [5] or Hodges [2].) In [3], Nies and Hodges studied interpretability of linear orderings and showed that no infinite linear order is $\emptyset$-interpretable in the theory of $M \times M$, where $M \times M$ is defined so that an atomic formula $R((a_1, b_1), \ldots, (a_n, b_n))$ holds in the structure if and only if $R(a_1, \ldots, a_n) \wedge R(b_1, \ldots, b_n)$ holds in M. Though they proved their result as a lemma for deriving a recursion theoretic result, whether a theory interprets a certain kind of order or not will be itself an interesting problem in model theory. Let $m > 1$. In this paper, an ordered set $(M, <)$ will be called an m-linear order if there are no incomparable m elements in M. So, by definition, 2-linearity coincides with linearity. Intuitively speaking, if $m < n$ then an m-linear order can be considered closer to a linear order than an n-linear order. In Section 2, we treat interpretability with parameters and prove the following.

Result 1.1 Let T be the theory of an infinite m-linear order $(M, <)$. Then an infinite linear order is interpretable in T.

Thus if we allow parameters, the interpretability of linear order and the interpretability of m-linear orders are essentially the same. However, as is shown in Section 3, if we do not allow parameters, the situation is different.

In Section 3, we treat $\emptyset$-interpretability. We will consider reduced powers $\prod_F M$ of M and their interpretability of m-linear orders. (See [2] or Chang [1] for the definition of reduced power.) If $M = (M, <)$ is a linear order and F is an ultrafilter, then $\prod_F M \succ M$, so $\prod_F M$ is also a linear order. However, the statement that $<$ is a

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linear order is not expressible by a Horn sentence; it may not be preserved in reduced products. In fact if F is not an ultrafilter, then $<$ is no longer a linear order in $\prod_F M$. This can be seen as follows. Choose a subset A of the domain I of F with $A \notin F$ and $A^c \notin F$. For two elements a and b in M with $a < b$, let $f : I \to M$ and $g : I \to M$ be functions defined by: $f(i) = a$ $(i \in A)$, $f(i) = b$ $(i \notin A)$, $g(i) = a$ $(i \notin A)$, $g(i) = b$ $(i \in A)$. Then f and g are not comparable in $\prod_F M$. In this section, we show that if F satisfies a certain condition, then no linear order exists in $\prod_F M$ even in the sense of $\varnothing$-interpretability. Among others we show the following.

Result 1.2 Let M be any structure. Let F be a filter over I with generators $\{A_i\}_{i<\kappa} \subset F$ such that $A_j \subset A_i$ holds for any $i < j < \kappa$. Suppose that the cardinality of $\bigcap F$ is 0 or $\geq m$. Then no infinite m-linear order is $\varnothing$-interpretable in the reduced power $\prod_F M$.

Since the mth power of a structure can be considered as a reduced power via a filter generated by an m-element set, so the result by Hodges and Nies [3] stated above is a corollary to Result 1.2. On the other hand, we can show that after naming one element in a given infinite linear order $(M, <)$, an $(m+1)$-linear order is $\varnothing$-interpretable in the mth power $M^m = M \times \cdots \times M$. (So M^m is an example in which an infinite $(m+1)$-linear order is $\varnothing$-interpretable but m-linear orders are not.) We will also show that if F is the Fréchet filter over an infinite set then no infinite m-linear order is $\varnothing$-interpretable in $\prod_F M$.

2 *Interpretability*

In this paper, we assume that $m \in \omega$ and $m > 1$.

Definition 2.1 An ordered set will be called *m-linear* if each subset of cardinality m has two comparable elements.

Example 2.2 Let M be a linear order and N an n-element poset. If we impose lexicographic orders on $M \times N$ and $N \times M$, then they both become $(n+1)$-linear.

In Ikeda [4], the notion of almost $\aleph_0$-categoricity was introduced and it was shown that if an almost $\aleph_0$-categorical theory has exactly three countable models then a dense linear ordering is interpretable in the theory. The first author of the present paper thought that by starting from an m-linear order one can construct a theory with a finite number of countable models and without linear ordering. Such a theory gives a counterexample to the conjecture stated in [4]. However, it was not the case. In fact we can show the following.

Theorem 2.3 *Let T be the theory of an infinite m-linear order $(M, <)$. Then an infinite linear order is interpretable in T.*

Proof: We write $a \perp b$ for $a \not\leq b$ and $a \not\geq b$. We show the following statement by induction on $m < \omega$.

(∗) If M is an $\aleph_0$-saturated model of T in which an infinite m-linear order is interpretable, then an infinite linear order is interpretable in M.

If $m = 2$, then the notion of m-linear order coincides with that of linear order. So let $m > 2$ and suppose that we have shown (∗) for $m - 1$. We need to show (∗) for m. Let $n(M) = \sup\{|S_a^M| : a \in M\}$, where $S_a^M = \{b \in M : b \perp a\}$. If $n(M) \geq \aleph_0$, then by compactness, there is an element $a \in M$ with $|S_a^M| \geq \aleph_0$. $(S_a^M, <)$ is clearly $(m-1)$-linear order. By the induction hypothesis, an infinite linear order is interpretable in S_a^M. (This is the only part where we need parameters. Hence if $n(M) < \aleph_0$, then an infinite linear order is interpretable in M without parameters.) Hence we may assume that $n(M) < \aleph_0$. Now we prove $(*)$ for m by induction on $n(M)$. If $n(M) = 0$, then M is in fact a linear order and we are done. So suppose that we have shown $(*)$ for M with $n(M) < n$. Let M be an $\aleph_0$-saturated m-linear order. Suppose $n(M) = n$. We put $M_a = \{b \in S_a^M : b \text{ is a minimal element of } S_a^M\}$. Now define a binary relation $\leq^*$ on M by $a \leq^* b$ if and only if $a \leq b$ or $a \in M_b$.

Claim 2.4 *$\leq^*$ is transitive.*

Suppose $a \leq^* b \leq^* c$. We have four cases to be considered. In each of the cases, we will show $a \leq^* c$.

Case 1: $a \leq b \leq c$. In this case, we have $a \leq c$ by the transitivity of $\leq$.

Case 2: $a \leq b, b \in M_c$. First suppose $a \not\perp c$. If $c < a$, then we would have $c < b$, contradicting $b \in M_c$. So we have $a \leq c$. Next suppose $a \perp c$. Then by the minimality of b in M_c, we have $a = b$. Hence $a \in M_c$.

Case 3: $a \in M_b, b \leq c$. First suppose $a \not\perp c$. If $c < a$, then we would have $b < a$, contradicting $a \in M_b$. So we have $a \leq c$. Next suppose $a \perp c$. For showing $a \in M_c$, let $d < a$. We need to show $d \not\perp c$. By the minimality of a in S_b^M, we have $d \not\perp b$. If $b \leq d$, then we would have $b < a$, contradicting $a \in M_b$. So we have $d < b$. Hence $d < c$. Thus a is minimal in S_c^M, that is, $a \in M_c$.

Case 4: $a \in M_b, b \in M_c$. First suppose $a \not\perp c$. $c < a$ does not occur, because a is minimal element of S_b^M. So we have $a \leq c$. Next suppose $a \perp c$. We show $a \in M_c$ in this case. Let $d < a$. We need to show $d \not\perp c$. Since a is minimal in S_b^M, $d \not\perp b$. If $d < b$, then $d \not\perp c$ by the minimality of b in S_c^M. If $b \leq d$, then we would have $b < a$, contradicting $a \in M_b$.

Now define an equivalence relation $E(xy)$ by $x \leq^* y \wedge y \leq^* x$. Let a/E denote the equivalence class $\{b \in M : E(ba)\}$. $M/E = \{a/E : a \in M\}$ becomes an order structure by $a/E \leq^* b/E \iff_{\text{def}} a \leq^* b$. We write $a/E <^* b/E$ for $a/E \leq^* b/E$ and $a/E \neq b/E$.

Claim 2.5 $a < b \Longrightarrow a/E <^* b/E$.

Suppose $a < b$. Clearly $a/E \leq^* b/E$. Moreover, $a < b$ implies $b \not\leq a$ and $b \notin M_a$. Hence we have $b/E \not\leq^* a/E$.

Claim 2.6 *The order M/E is infinite.*

By Ramsey's Theorem and the fact that M is an infinite m-linear order, there is an infinite set $\{a_i : i < \omega\}$ with $a_i < a_j$ for all $i < j < \omega$. By Claim 2.5 above, we have $a_i/E \neq a_j/E$ for $i < j < \omega$. Thus M/E is infinite. By the induction hypothesis for proving that M/E interprets an infinite linear order, it is sufficient to show the following.

Claim 2.7 $n(M/E) < n(M)$.

Let $b/E \in M/E$ and $a_1/E, \dots, a_l/E$ be an enumeration of the set $S_{b/E}^{M/E}$. By Claim 2.5, $a_i \perp b$ for each $i = 1, \dots, l$. If $l \geq n(M)$, then there must be some i with $a_i \in M_b$. But then we have $a_i/E \leq^* b/E$, contradicting the fact that $a_i/E \in S_{b/E}^{M/E}$. □

3 Noninterpretability

Definition 3.1 Let F be a filter over a set I.

1. Let B be a subset of I. Then F_B denotes the set $\{X \cap B : X \in F\}$. (F_B is clearly a filter over B unless $F_B = \mathcal{P}(B)$.)
2. We will say that F is *m-good* if there is a bijection $\tau : I \to I$ and a subset $B \subset I$ such that
   (a) $X \in F \iff \tau(X) \in F$, for all $X \subset I$,
   (b) I is the disjoint union of $\{\tau^i(B) : i < m\}$.

Theorem 3.2 *Let M be any structure, F a filter over a set I. If F is m-good, then no m-linear order is $\varnothing$-interpretable in the reduced power $\prod_F M$.*

First let us remark the following.

Remark 3.3 Let M be an L-structure and $n \in \omega$. As usual, M^n denotes the nth power of the structure M, that is, the direct product of n copies of M. There is another notion of power of a structure. For an nm-ary L-formula φ, let R_φ be a new m-ary relation symbol. We define an $\{R_\varphi : \varphi \in L\}$-structure $M^{(n)}$ by

(1) the universe of $M^{(n)}$ is $\{(a_1, \dots, a_n) : a_1, \dots, a_n \in M\}$;
(2) $M^{(n)} \models R_\varphi(a_1, \dots, a_m) \iff M \models \varphi(a_1, \dots, a_m)$.

Let $a \in (M^{(k)})^n$. Then a has the form $((a_{1,j})_{1\leq j\leq k}, \dots, (a_{n,j})_{1\leq j\leq k})$. We define the mapping $\sigma : (M^{(k)})^n \to (M^n)^{(k)}$ by $\sigma(a) = ((a_{i,1})_{1\leq i\leq n}, \dots, (a_{i,k})_{1\leq i\leq n})$. Then σ gives an isomorphism between $\{R_\varphi : \varphi \in L\}$-structures $(M^{(k)})^n$ and $(M^n)^{(k)}$. For example, if φ is a k-ary formula in L (so R_φ is a unary predicate), then we see the following hold: $(M^{(k)})^n \models R_\varphi(a) \iff M^{(k)} \models R_\varphi((a_{1,j})_{1\leq j\leq k}) \wedge \cdots \wedge R_\varphi((a_{n,j})_{1\leq j\leq k}) \iff M \models \varphi((a_{1,j})_{1\leq j\leq k}) \wedge \cdots \wedge \varphi((a_{n,j})_{1\leq j\leq k}) \iff M^n \models \varphi((a_{i,1})_{1\leq i\leq n}, \dots, (a_{i,k})_{1\leq i\leq n}) \iff (M^n)^{(k)} \models R_\varphi(\sigma(a))$.

Proof: By the assumption of F, there are $B \subset I$ and $\tau : I \to I$ witnessing the m-goodness of F. We claim that $\prod_F M \simeq (\prod_{F_B} M)^m$. We define $f : \prod_F M \to (\prod_{F_B} M)^m$ by $f((a_i)_{i\in I}/F) = ((b_{i,j})_{i\in B}/F_B)_{1\leq j\leq m}$, where $b_{i,j} = a_{\tau^{j-1}(i)}$. We will show that

$$\prod_F M \models \varphi(a^1, \dots, a^k) \iff (\prod_{F_B} M)^m \models \varphi(f(a^1), \dots, f(a^k))$$

holds for any atomic formula $\varphi(x^1, \dots, x^k)$. For simplifying the notation, we assume $k = 1$ and put $a = a^1$. Suppose that $(\prod_{F_B} M)^m \models \varphi(f(a))$ holds. If we put $X_j = \{i \in \tau^{j-1}(B) : M \models \varphi(a_i)\}$, then we have $X_j \in F_{\tau^{j-1}(B)}$ for all $1 \leq j \leq m$. Hence there is $Y_j \in F$ such that $Y_j|\tau^{j-1}(B) = X_j$ for all $1 \leq j \leq m$. So $\{i \in I : M \models \varphi(a_i)\} =$

$\bigcup_{1\leq j\leq m} X_j \supset \bigcap_{1\leq j\leq m} Y_j \in F_J$. Thus $\prod_F M \models \varphi(a)$ holds. The other direction is easily shown.

By what we have just shown, to finish our proof of the theorem, it is sufficient to show the following claim, which can be proven by refining arguments in [3].

Claim 3.4 *Let M be any structure. Then no infinite m-linear order is $\varnothing$-interpretable in the structure M^m.*

Temporarily we say that a binary relation on A is a k-linear preorder on A if (i) it is a preorder on A and (ii) it has no pairwise incomparable k elements. Then by Remark 3.3, it is sufficient to show that no m-linear preorder on M^m with infinitely many incomparable elements is $\varnothing$-definable in M^m.

By the way of contradiction, assume that there is such a preorder $\leq$ definable in M^m. For elements $a, b \in M^m$, let $a \sim b$ denote the relation $a \leq b \wedge b \leq a$. We may assume that the domain of $\leq$ is M^m itself. We may also assume that M issufficiently saturated and homogeneous.

Subclaim 3.4.1 *Let two tuples $a_1, \dots, a_{m-1}, a_m$ and $a_1, \dots, a_{m-1}, b$ have the same type in M. Then there is a permutation σ of $\{1, \dots, m\}$ such that*

$$f_\sigma((a_1, \dots, a_{m-1}, a_m)) \sim (a_1, \dots, a_{m-1}, b),$$

where f_σ is the automorphism of M^m defined by $f((x_1, \dots, x_m)) = (x_{\sigma(1)}, \dots, x_{\sigma(m)})$.

We may assume that a_m is different from any of $a_1, \dots, a_{m-1}$. Choose $f \in \mathrm{Aut}(M)$ which maps the tuple $a_1, \dots, a_{m-1}, b$ to the tuple $a_1, \dots, a_{m-1}, a_m$. Let $g \in \mathrm{Aut}(M^m)$ be the permutation of coordinates defined by $g((x_1, \dots, x_m)) = (x_m, x_1, \dots, x_{m-1})$. Now let $h \in \mathrm{Aut}(M^m)$ be the automorphism defined by

$$h((x_1, \dots, x_m)) = g((x_1, \dots, x_{m-2}, f^{-1}(x_{m-1}), f(x_m))).$$

The following are easily shown.

1. $h^i((a_1, \dots, a_{m-1}, b)) = (a_{m-i+1}, \dots, a_m, a_1, \dots, a_{m-i})$ if $i < m$,
2. $h^m((a_1, \dots, a_{m-1}, b)) = (a_1, \dots, a_{m-1}, b)$, and
3. $(a_1, \dots, a_{m-1}, b), h((a_1, \dots, a_{m-1}, b)), \dots, h^{m-1}((a_1, \dots, a_{m-1}, b))$ are distinct.

So by m-linearity, there are two distinct numbers $i, j \in m$ such that

$$h^i((a_1, \dots, a_{m-1}, b)) \leq h^j((a_1, \dots, a_{m-1}, b)).$$

Since h is an automorphism of the structure M^m, we may assume $i = 0$. Then we have

$$(a_1, \dots, a_{m-1}, b) \leq h^j((a_1, \dots, a_{m-1}, b)) \leq \cdots \leq h^{jk}((a_1, \dots, a_{m-1}, b))$$

for any $k \in \omega$. Let $k = m$ in the above inequality. Then, using property 2, we have

$$(a_1, \dots, a_{m-1}, b) \leq h^j((a_1, \dots, a_{m-1}, b)) \leq (a_1, \dots, a_{m-1}, b).$$

This together with property 1 concludes our proof of Subclaim 3.4.1.

Now choose an indiscernible sequence $I = (\bar{a}_i)_{i\in\omega}$ in M such that for all $i < j$, $\bar{a}_i < \bar{a}_j$ holds in M^m. Let a_{ij} denote the jth coordinate of the m-tuple $\bar{a}_i$. So $\bar{a}_i = a_{i1}, \ldots, a_{im}$.

Subclaim 3.4.2 *There is a permutation σ of $\{1, \ldots, m\}$ such that*

$$(a_{11}, \ldots, a_{mm}) \sim f_\sigma(a_{m+1,1}, \ldots, a_{m+m,m}),$$

where f_σ is the automorphism of M^m defined by $f((x_1, \ldots, x_m)) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$.

For $i = 0, \ldots, m$, we define $\bar{b}_i = b_{i1}, \ldots, b_{im}$ by

$$b_{ij} = \begin{cases} b_{ij} = a_{j,j} & \text{if } j \leq m - i \\ b_{ij} = a_{m+j,j} & \text{if } j > m - i \end{cases}.$$

By the indiscernibility of I, $\bar{b}_i$'s have the same type in M. So by Subclaim 3.4.1, we have permutations $\sigma_1, \ldots, \sigma_m$ such that for each $i \in \{1, \ldots, m\}$,

$$\bar{b}_{i-1} \sim f_{\sigma_i}(\bar{b}_i).$$

This completes our proof of Subclaim 3.4.2, since $\bar{b}_0 = a_{11}, \ldots, a_{mm}$ and $\bar{b}_m = a_{m+1,1}, \ldots, a_{m+m,m}$.

Let σ be one of the permutations chosen in Subclaim 3.4.2. By the indiscernibility of I, all the elements $f_{\sigma^k}(a_{mk+1,1}, \ldots, a_{m(k+1),m}) \in M^m$ ($k \in \omega$) belong to the same $\sim$-class. Choose $k > 0$ with $\sigma^k = \text{id}$. Then

$$(a_{11}, \ldots, a_{mm}) \sim (a_{mk+1,1}, \ldots, a_{m(k+1),m}).$$

Now, for each $i \in \{1, \ldots, m\}$, let g_i be a mapping such that $g_i(a_{j+i,i}) = a_{j,i}$ for all $j \in \omega$. By the indiscernibility of I, g_i can be extended to an automorphism of M. Put $g = (g_1, \ldots, g_m)$, then g is an automorphism of M^m. So we have $(a_{01}, \ldots, a_{0m}) \sim (a_{mk,1}, \ldots, a_{mk,m})$, contradicting our choice of I. □

Recall that the Fréchet filter over an infinite set I is the set $F = \{X \subset I : I - X \text{ is finite}\}$. We will say that a filter F has a *descending system of generators* if there is $\{A_i : i < \kappa\} \subset F$ such that: (i) $A_i \supset A_j$ for all $i < j < \kappa$ and (ii) for all $X \in F$ there is $i < \kappa$ with $A_i \subset X$.

Corollary 3.5

(1) If F is the Fréchet filter over I, then no m-linear order is $\emptyset$-interpretable in $\prod_F M$.
(2) If F has a descending system of generators and the cardinality of $\bigcap F$ is 0 or $\geq m$, then no m-linear order is $\emptyset$-interpretable in $\prod_F M$.

Proof: It is enough to show that F is m-good.

(1) Let $B_i (i < m)$ be a disjoint partition of I such that $|B_i| = |I|$ and $\tau : I \to I$ a bijection such that $\tau(B_i) = B_{(i+1) \bmod m}$. Then B_0 and τ witness the m-goodness.
(2) Notice that if $m > n$ then an n-linear order is obviously an m-linear order. So, for proving the corollary, we may assume that the cardinality of $\bigcap F$ is 0 or m or infinity.

Let $\{A_i : i < \kappa\}$ be a descending system of generators of F and $I_0 = \bigcap F$. We may assume that $A_0 = I$. Let $I_1 = I - I_0$. By our assumption on $|I_0|$, there is a bijection $\tau_0 : I_0 \to I_0$ and disjoint subsets $X_1, \ldots, X_m$ of I_0 with $\bigcup_{1\le i\le m} X_i = I_0$ and $\tau_0(X_i) = X_{(i \bmod m)+1}$ for $1 \le i \le m$.

We may assume that each $|(A_i - A_{i+1}) \cap I_1|$ is infinite, or a multiple of m by replacing A_i's if necessary. Hence there is a bijection $\tau_1 : I_1 \to I_1$ and disjoint subsets $Y_1, \ldots, Y_m$ of I_1 with $\bigcup_{1\le i\le m} Y_i = I_1$ and $\tau_1(Y_i \cap A_j) = Y_{(i \bmod m)+1} \cap A_j$ for $1 \le i \le m$ and $j < \kappa$. Let $\tau = \tau_0 \cup \tau_1$ and $B = X_1 \cup Y_1$. Then τ and B witness the m-goodness. □

Example 3.6 Let $M = (M, \le, a)$ be an infinite linear order with a named element a. We show that there is a $\varnothing$-definable $(m+1)$-linear order in M^m. First we may assume that in M there are infinitely many elements greater than a. Let $\varphi(x)$ be the formula expressing (i) $a \le x$ and (ii) any two elements between a and x are comparable. Then an element $(a_1, \ldots, a_m) \in M^m$ satisfying $\varphi(x)$ has the form $(a, \ldots, a, b, a, \ldots, a)$, that is, there is i_0 ($\in \{1, \ldots, m\}$) with $a_i = a$ for all $i \ne i_0$ and $a_{i_0} = b \ge a$. So $\varphi(x)$ determines an infinite set. It is clear that if $m+1$ elements are given, then we can choose two comparable elements from them.

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Institute of Mathematics
University of Tsukuba
Tsukuba-shi Ibaraki 305-8571
JAPAN
email: tsuboi@sakura.cc.tsukuba.ac.jp

Institute of Mathematics
University of Tsukuba
Tsukuba-shi Ibaraki 305-8571
JAPAN
email: wakai@math.tsukuba.ac.jp