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ON INTERTWINING BY AN OPERATOR HAVING A DENSE RANGE

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1. Throughout the paper, by an operator we mean a bounded linear transformation acting on a Hilbert space H. The algebra of all operators on H is denoted by $B(H)$.

We formulate an algebraic version of generalized Putnam-Fuglede theorem [3; Theorem 1], and we show that a paranormal contraction T is unitary, if S is a coisometry, if W is an operator having a dense range and if $TW = WS$. This is a generalization of a result due to Okubo [1].

Let $T \in B(H)$. T is *hyponormal* (resp. *cohyponormal*) if $T^*T - TT^* \geqq 0$ (resp. $TT^* - T^*T \geqq 0$). T is *dominant* if range $(T-\lambda) \subset$ range $(T-\lambda)^*$ for all $\lambda \in \sigma(T)$, the spectrum of T. This condition is equivalent to the existence of a constant M_λ for each $\lambda \in \sigma(T)$ such that

$$\|(T-\lambda)^*x\| \leqq M_\lambda \|(T-\lambda)x\|$$

for all $x \in H$. Thus every hyponormal operator is dominant. T is *paranormal* if

$$\|Tx\|^2 \leqq \|T^2x\| \|x\|$$

for all $x \in H$.

2. The following theorem is a version of [3; Theorem 1]. The proof of [3] applies to this version. We include it for completeness.

THEOREM 1. *Let T, S, and $W \in B(H)$, where W has a dense range. Assume that $TW = WS$ and $T^*W = WS^*$. Then*

(i) *T is hyponormal (resp. cohyponormal), if so is S.*

(ii) *T is isometric (resp. coisometric), if so is S. In particular, T is unitary, if so is S.*

(iii) *T is normal, if so is S.*

PROOF. Let $W^* = V^*B$ be the polar decomposition of W^*. Since W has a dense range, W^* is injective. Thus $B^2 = WW^*$ is injective, and V is coisometric. From equations $TW = WS$ and $T^*W = WS^*$, we have

$$TWW^* = WSW^* , \qquad WW^*T = WSW^* .$$

Thus, WW^* commutes with T, and so B commutes with T. Hence we have

$$BTV = TBV = TW = WS = BVS ,$$

which implies that $TV = VS$ because B is injective. Since V is coisometric, we obtain

$$T = TVV^* = VSV^* .$$

From the equations $W^*T = SW^*$ and $TB = BT$, we have

$$V^*TB = V^*BT = W^*T = SW^* = SV^*B ,$$

which implies that $V^*T = SV^*$. Hence

$$V^*VS = V^*TV = SV^*V .$$

First we assume that S is normal. Since $S^*S = SS^*$, we obtain

$$\begin{aligned} T^*T &= (VSV^*)^*(VSV^*) = VS^*V^*VSV^* = VS^*SV^*VV^* = VS^*SV^* \\ &= VSS^*V^* = VV^*VSS^*V^* = (VSV^*)(VSV^*)^* = TT^* , \end{aligned}$$

whence T is normal.

To prove (i), assume that S is hyponormal (resp. cohyponormal). Since $S^*S \geqq SS^*$ (resp. $SS^* \geqq S^*S$), the above computation implies that

$$T^*T = VS^*SV^* \geqq VSS^*V^* = TT^* ,$$
$$(\text{resp. } TT^* = VSS^*V^* \geqq VS^*SV^* = T^*T) ,$$

and the assertion of (i) follows.

To prove (ii), assume that S is isometric (resp. coisometric). Again, by the above computation,

$$T^*T = VS^*SV^* = VV^* = I , \qquad (\text{resp. } TT^* = VSS^*V^* = VV^* = I) ,$$

whence T is isometric (resp. coisometric).

The rest of the theorem is obvious.

REMARK. In Theorem 1, if W is injective and has a dense range, V is a unitary operator which implements the unitary equivalence of S and T.

The next theorem is a generalization of [1; Proposition 1].

THEOREM 2. *Let T, V, and $W \in B(H)$, where T is a paranormal contraction, V is a coisometry and W has a dense range. Assume that $TW = WV$. Then T is a unitary operator. In particular, if W is injective and has a dense range, then V is also a unitary operator.*

PROOF. Let $x \in H$ such that $Wx \neq 0$, and define

$$y_n = WV^{*^n}x \quad (n = 0, 1, 2, \cdots) .$$

Then we have

$$Ty_{n+1} = TWV^{*^{n+1}}x = WVV^{*^{n+1}}x = WV^{*^n}x = y_n .$$

Since T is a contraction,

$$\|y_n\| = \|Ty_{n+1}\| \leqq \|y_{n+1}\| = \|WV^{*^{n+1}}x\| \leqq \|W\|\|x\|$$

and hence $\{\|y_{n+1}\|\}$ is a monotone increasing convergent sequence. By the paranormality of T, we have

$$\|y_n\|^2 = \|Ty_{n+1}\|^2 \leqq \|T^2y_{n+1}\|\|y_{n+1}\| = \|y_{n-1}\|\|y_{n+1}\|$$

and

$$1 \geqq \frac{\|y_0\|}{\|y_1\|} \geqq \frac{\|y_1\|}{\|y_2\|} \geqq \cdots \geqq \frac{\|y_{n-1}\|}{\|y_n\|} \to 1 \quad (n \to \infty) .$$

In particular, $\|y_0\| = \|y_1\|$, that is,

$$\|Wx\| = \|WV^*x\| .$$

Thus

$$\|WV^*x\| = \|Wx\| = \|WVV^*x\| = \|TWV^*x\| \leqq \|WV^*x\| ,$$

and so

$$\|WV^*x\| = \|Wx\| = \|TWV^*x\| .$$

Note that these equalities are valid for $x \in H$ such that $Wx = 0$. Hence

$$\begin{aligned} &\|T^*Wx - WV^*x\|^2 \\ &\quad = \|T^*Wx\|^2 + \|WV^*x\|^2 - (T^*Wx, WV^*x) - (WV^*x, T^*Wx) \\ &\quad \leqq 2\|Wx\|^2 - (Wx, TWV^*x) - (TWV^*x, Wx) \\ &\quad = 2\|Wx\|^2 - (Wx, WVV^*x) - (WVV^*x, Wx) \\ &\quad = 2\|Wx\|^2 - 2\|Wx\|^2 = 0 \end{aligned}$$

for all $x \in H$, and $TW^* = WV^*$. It follows from Theorem 1 that T is a coisometry. Since T is paranormal, T is unitary by [2; Lemma 3]. The rest is clear by the remark after Theorem 1.

REMARK. Our proof of Theorem 2 is a modification of the argument due to Okubo [1]. He proved Theorem 2 under the hypothesis that V is unitary.

COROLLARY 3. *Let $T \in B(H)$ be a paranormal contraction. Let $TW = WV$, where $V \in B(H)$ is a coisometry and $W \in B(H)$ is any non-*

zero operator. Then T has a nontrivial invariant subspace.

PROOF. Let $\mathfrak{M}$ be the closure of range W. If W does not have the dense range, $\mathfrak{M}$ is a nontrivial invariant subspace of T. If W has the dense range, then T is unitary by Theorem 2, and T has a nontrivial invariant subspace.

3. As an application of Theorems 1 and 2, we give an alternative proof to the following theorem.

THEOREM 4. *Let $T \in B(H)$ be a contraction. Let*

$$\mathfrak{M} = \{x \in H \mid \| T^{*^n} x \| \to 0 \ (n \to \infty)\} .$$

If T is dominant or paranormal, then $\mathfrak{M}$ is a reducing subspace for T such that $T|_{\mathfrak{M}^\perp}$ is unitary and $T|_{\mathfrak{M}}$ is completely non-unitary (i.e., $T|_{\mathfrak{M}}$ has no nontrivial reducing subspace on which $T|_{\mathfrak{M}}$ is unitary).

This theorem was first proved for dominant operators in [4] and for paranormal operators in [1]. Note that the statements in [4; Theorem 2] contain a slip, because $\{x \in H \mid \| T^{*^n} x \| \geqq \varepsilon_x > 0\}$ is not a linear subspace of H.

To prove Theorem 4, we need the following simple lemma.

LEMMA 5. *Let $T \in B(H)$ be a contraction. Let $\mathfrak{M} \subset H$ be an invariant subspace for T. If $T|_{\mathfrak{M}}$ is a coisometry, then $\mathfrak{M}$ reduces T.*

PROOF. Let $S = T|_{\mathfrak{M}}$, and let $x \in \mathfrak{M}$. Then, since S^* is isometric

$$\begin{aligned} \| S^*x - T^*x \|^2 &= \| S^*x \|^2 + \| T^*x \|^2 - (S^*x, T^*x) - (T^*x, S^*x) \\ &\leqq \| x \|^2 + \| x \|^2 - (TS^*x, x) - (x, TS^*x) \\ &= 2\| x \|^2 - 2\| S^*x \|^2 = 2\| x \|^2 - 2\| x \|^2 = 0 . \end{aligned}$$

Thus, $T^*x = (T|_{\mathfrak{M}})^*x \in \mathfrak{M}$ for all $x \in \mathfrak{M}$, which implies that $\mathfrak{M}$ is invariant under T^*.

PROOF OF THEOREM 4. Since $\| T \| \leqq 1$, the sequence $\{T^n T^{*^n}\}$ converges strongly to a positive contraction. Let

$$A = (\lim_{n \to \infty} T^n T^{*^n})^{1/2} .$$

Then, $\mathfrak{M} = \ker A$ and $TA^2T^* = A^2$. Since

$$\| AT^*x \|^2 = (TA^2T^*x, x) = (A^2x, x) = \| Ax \|^2$$

for all $x \in H$, there exists a partial isometry $W \in B(H)$ such that

$$AT^* = WA , \qquad W|_{\mathfrak{M}} = 0 .$$

It is easy to see that $\mathfrak{M}^\perp$ is invariant under T. Let us write the equation $AT^* = WA$ in matrix from on $H = \mathfrak{M} \oplus \mathfrak{M}^\perp$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & W_1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix},$$

whence $A_1S_3 = W_1A_1$, or $S_3^*A_1 = A_1W_1^*$. Note that $A_1 = A|_{\mathfrak{M}^\perp}$ is injective and has a dense range, and $W_1 = W|_{\mathfrak{M}^\perp}$ is an isometry.

Case 1. Assume that T is dominant. Since S_3^* is dominant and W_1^* is coisometric, S_3^* and W_1^* are unitarily equivalent normal operators by [4; Theorem 1] and the remark after Theorem 1. Thus $\mathfrak{M}^\perp$ reduces T by [3; Lemma 2]. Since W_1 is normal and isometric, W_1 is unitary and so is S_3.

Case 2. Assume that T is paranormal. Since $S_3^* = T|_{\mathfrak{M}^\perp}$ is paranormal, S_3^* is unitary by Theorem 2. Thus $\mathfrak{M}^\perp$ reduces T by Lemma 5.

It is clear that $T|_{\mathfrak{M}}$ is completely non-unitary in each case.

REMARK. In Theorem 4, A is the projection onto $\mathfrak{M}^\perp$. This was proved in [1] for a paranormal contraction.

COROLLARY 6. *Let $T \in B(H)$ be a dominant or paranormal contraction. If there exists a vector $x_0 \in H$ such that $\|T^{*n}x_0\| \geqq \varepsilon > 0$ for $n = 1, 2, 3, \cdots$, then T has a non-trivial invariant subspace.*

PROOF. Let $\mathfrak{M} = \{x \in H \mid \|T^{*n}x\| \to 0 \ (n \to \infty)\}$. By hypothesis, $\mathfrak{M} \neq H$ or $\mathfrak{M}^\perp \neq \{0\}$. By Theorem 4, $T = T_1 \oplus U$, where $U = T|_{\mathfrak{M}^\perp}$ is unitary, and thus T has a non-trivial invariant subspace.

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