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Research Article

On Intuitionistic Fuzzy Entropy of Order-α

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Using the idea of Rènyi's entropy, intuitionistic fuzzy entropy of order-α is proposed in the setting of intuitionistic fuzzy sets theory. This measure is a generalized version of fuzzy entropy of order-α proposed by Bhandari and Pal and intuitionistic fuzzy entropy defined by Vlachos and Sergiadis. Our study of the four essential and some other properties of the proposed measure clearly establishes the validity of the measure as intuitionistic fuzzy entropy. Finally, a numerical example is given to show that the proposed entropy measure for intuitionistic fuzzy set is reasonable by comparing it with other existing entropies.

1. Introduction

In 1965, Zadeh [1] proposed the notion of fuzzy set (FS) to model nonstatistical imprecise or vague phenomena. Since then, the theory of fuzzy set has become a vigorous area of research in different disciplines such as engineering, artificial intelligence, medical science, signal processing, and expert systems. Fuzziness, a feature of uncertainty, results from the lack of sharp distinction of being or not being a member of a set; that is, the boundaries of the set under consideration are not sharply defined. A measure of fuzziness used and cited in the literature is fuzzy entropy, also first mentioned in 1968 by Zadeh [2]. In 1972, De Luca and Termini [3] first provided axiomatic structure for the entropy of fuzzy sets and defined an entropy measure of a fuzzy set based on Shannon's entropy function [4]. Kaufmann [5] introduced a fuzzy entropy measure by a metric distance between the fuzzy set and that of its nearest crisp set. In addition, Yager [6] defined an entropy measure of a fuzzy set in terms of a lack of distinction between fuzzy set and its complement. In 1989, N. R. Pal and S. K. Pal [7] proposed fuzzy entropy based on exponential function to measure the fuzziness called exponential fuzzy entropy. Further, Bhandari and Pal [8] proposed fuzzy entropy of order-α corresponding to Rènyi entropy [9]. Recently, Verma and Sharma [10] have introduced a parametric generalized entropy measure for fuzzy sets called "exponential fuzzy entropy of order-α."

Atanassov [11, 12] introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the notion of fuzzy set. The distinguishing characteristic of intuitionistic fuzzy set is that it assigns to each element a membership degree, a nonmembership degree, and the hesitation degree. Firstly, Burillo and Bustince [13] defined the entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Szmidt and Kacprzyk [14] used a different approach from that of Burillo and Bustince to introduce the entropy measure for IFS based on the geometric interpretation of IFS and the distances between them. Zeng and Li [15] considered the axioms of Szmidt and Kacprzyk using the notion of IVFSs and discussed the relationship between similarity measure and entropy. Hung and Yang [16] gave their axiomatic definition of entropy for IFSs by exploiting the concept of probability. Vlachos and Sergiadis [17] proposed a measure of intuitionistic fuzzy entropy and revealed an intuitive and mathematical connection between the notions of entropy for fuzzy set and intuitionistic fuzzy set. Zhang and Jiang

[18] proposed intuitionistic (vague) fuzzy entropy by means of intersection and union of the membership degree and nonmembership degree of the intuitionistic (vague) fuzzy set.

Having come to rather rigid measures, it is natural to see other possible measures and those having parametric generalization that covers existing measures and leads to many more. The parameters involved give flexibility in applications and their values can be determined from the data itself or experiment. In this paper, a new entropy measure for IFSs is defined. This is called "*intuitionistic fuzzy entropy of order-α*."

The paper is organized as follows. In Section 2, some basic definitions related to fuzzy set theory and intuitionistic fuzzy set theory are briefly given. In Section 3, a new entropy measure called "*intuitionistic fuzzy entropy of order-α*" is proposed, and its axiomatic justification is established. Some mathematical properties of the proposed measure are also studied in this section. In Section 4, a numerical example is given to demonstrate the effectiveness of the proposed measure of intuitionistic fuzzy entropy with existing intuitionistic fuzzy entropies [13–18] by the comparison. Brief conclusions are presented in Section 5.

2. Preliminaries

In the following, some needed basic concepts and definitions related to fuzzy sets and intuitionistic fuzzy sets are introduced.

Definition 1 (fuzzy set [1]). A fuzzy set $\widetilde{A}$ in a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is given by

$$\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_{\widetilde{A}} : X \to [0,1]$ is the membership function of $\widetilde{A}$. The number $\mu_{\widetilde{A}}(x)$ describes the degree of belongingness of $x \in X$ in $\widetilde{A}$.

De Luca and Termini [3] defined fuzzy entropy for a fuzzy set $\widetilde{A}$ corresponding to Shannon's entropy [4] as

$$H(\widetilde{A}) = -\frac{1}{n}\sum_{i=1}^{n}[\mu_{\widetilde{A}}(x_i)\log(\mu_{\widetilde{A}}(x_i)) + (1-\mu_{\widetilde{A}}(x_i))\log(1-\mu_{\widetilde{A}}(x_i))]. \tag{2}$$

Later, Bhandari and Pal [8] made a survey of information measures on fuzzy sets and proposed some new measures of fuzzy entropy. Corresponding to Rènyi entropy [9], they defined the following measure:

$$H_\alpha(\widetilde{A}) = \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log\left[\mu_{\widetilde{A}}^{\alpha}(x_i) + (1-\mu_{\widetilde{A}}(x_i))^{\alpha}\right]; \quad \alpha \neq 1, \quad \alpha > 0. \tag{3}$$

Atanassov [11, 12], as mentioned earlier, generalized Zadeh's idea of fuzzy sets, by what is called intuitionistic fuzzy sets, defined as follows.

Definition 2 (intuitionistic fuzzy set [11]). An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{4}$$

where

$$\mu_A : X \longrightarrow [0,1], \qquad \nu_A : X \longrightarrow [0,1] \tag{5}$$

with the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \quad \forall x \in X. \tag{6}$$

In this definition numbers $\mu_A(x)$ and $\nu_A(x)$, respectively, denote the *degree of membership* and *degree of nonmembership* of $x \in X$ to the set A.

For each IFS A in X, if $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, $x \in X$, then $\pi_A(x)$ represents the degree of hesitance of $x \in X$ to the set A. $\pi_A(x)$ is also called *intuitionistic index*.

Obviously, when $\pi_A(x) = 0$, that is, $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, then IFS set A becomes fuzzy set. Thus, fuzzy sets are the special cases of IFSs.

Definition 3 (set operations on intuitionistic fuzzy sets [12]). Let IFS(X) denote the family of all IFSs in the universe X, and let $A, B \in$ IFS(X) given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$
$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}. \tag{7}$$

Then usual set relations and operations are defined as follows:

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$;

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(iii) $A^C = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(iv) $A \cap B = \{\langle \mu_A(x) \wedge \mu_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(v) $A \cup B = \{\langle \mu_A(x) \vee \mu_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

Szmidt and Kacprzyk [14] extended the axioms of De Luca and Termini [3] for proposing the entropy measure in the setting of IFSs.

Definition 4 (see [14]). An entropy on IFS(X) is a real-valued functional E : IFS(X) $\to [0,1]$, satisfying the following four axioms.

(*P1*) $E(A) = 0$ if and only if A is a crisp set; that is, $\mu_A(x_i) = 0$, $\nu_A(x_i) = 1$ or $\mu_A(x_i) = 1$, $\nu_A(x_i) = 0$ for all $x_i \in X$.

(*P2*) $E(A) = 1$ if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(*P3*) $E(A) \le E(B)$ if and only if $A \subseteq B$, that is, if $\mu_A(x_i) \le \mu_B(x_i)$ and $\nu_A(x_i) \ge \nu_B(x_i)$, for $\mu_B(x_i) \le \nu_B(x_i)$, or if $\mu_A(x_i) \ge \mu_B(x_i)$ and $\nu_A(x_i) \le \nu_B(x_i)$, for $\mu_B(x_i) \ge \nu_B(x_i)$ for any $x_i \in X$.

(*P4*) $E(A) = E(A^C)$.

Corresponding to (2), the De Luca and Termini [3] entropy, Vlachos and Sergiadis [17] introduced a measure of intuitionistic fuzzy entropy as

$$E_{VS}(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + \nu_A(x_i)\log\nu_A(x_i) - (1-\pi_A(x_i))\log(1-\pi_A(x_i)) - \pi_A(x_i)\right]. \quad (8)$$

Throughout this paper, we will denote the set of all intuitionistic fuzzy sets in X by IFS(X) and by FS(X) the set of all fuzzy sets defined in X.

With these background ideas and concepts, we, in the next section, introduce a new measure called "*intuitionistic fuzzy entropy of order-α*" for intuitionistic fuzzy sets, which has a parameter.

3. Intuitionistic Fuzzy Entropy of Order-α

Definition 5. The intuitionistic fuzzy entropy of order-α for intuitionistic fuzzy set A is defined as

$$E_\alpha(A) = \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log\left[(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}\pi_A(x_i)\right], \quad \alpha > 0,\ \alpha \neq 1. \quad (9)$$

Theorem 6. *The $E_\alpha(A)$ measure in (9) of the intuitionistic fuzzy entropy of order-α is an entropy measure for IFSs; that is, it satisfies all the axioms given in Definition 4 above.*

Proof. (*P1*) Let A be a crisp set with membership values being either 0 or 1 for all $x_i \in X$. Then from (9) we obtain that

$$E_\alpha(A) = 0. \quad (10)$$

Next, if $E_\alpha(A) = 0$, that is,

$$\frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log\left[(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}\pi_A(x_i)\right] = 0 \quad (11)$$

or

$$\begin{aligned}
&(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}\pi_A(x_i) = 1 \quad \forall x_i \in X \\
&(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} - 2^{1-\alpha}(\mu_A(x_i)+\nu_A(x_i)) = 1 - 2^{1-\alpha} \quad \forall x_i \in X \\
&(\mu_A(x_i)+\nu_A(x_i)) \times \left[\frac{(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))}{(\mu_A(x_i)+\nu_A(x_i))^\alpha} - 2^{1-\alpha}\right] = 1 - 2^{1-\alpha}, \quad \forall x_i \in X.
\end{aligned} \quad (12)$$

Since $\alpha > 0$, $\alpha \neq 1$, therefore (12) will hold only if $\mu_A(x_i) = 0$, $\nu_A(x_i) = 1$, or $\mu_A(x_i) = 1$, $\nu_A(x_i) = 0$, for all $x_i \in X$.

Hence, $E_\alpha(A) = 0$ if and only if $\mu_A(x_i) = 0$, $\nu_A(x_i) = 1$, or $\mu_A(x_i) = 1$, $\nu_A(x_i) = 0$, for all $x_i \in X$. This proves (*P1*).

(*P2*) First let $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$. Then from (9) we have

$$\begin{aligned}
E_\alpha(A) &= \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log\left[(\mu_A^\alpha(x_i) + \mu_A^\alpha(x_i))(\mu_A(x_i)+\mu_A(x_i))^{1-\alpha} + 2^{1-\alpha}(1-2\mu_A(x_i))\right] \\
&= \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log\left[2^{2-\alpha}\mu_A(x_i) + 2^{1-\alpha} - 2^{2-\alpha}\mu_A(x_i)\right] \\
&= \frac{1}{n(1-\alpha)} \times n(1-\alpha) = 1.
\end{aligned} \quad (13)$$

Next, let $E_\alpha(A) = 1$; that is,

$$\sum_{i=1}^{n}\log\left[(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}\pi_A(x_i)\right] = n(1-\alpha) \quad (14)$$

or

$$\log\left[(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}\pi_A(x_i)\right] = (1-\alpha) \quad \forall x_i \in X \quad (15)$$

or

$$\left[(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i))(\mu_A(x_i)+\nu_A(x_i))^{1-\alpha} + 2^{1-\alpha}(1-\mu_A(x_i)-\nu_A(x_i))\right] = 2^{1-\alpha} \quad \forall x_i \in X \quad (16)$$

or

$$(\mu_A(x_i)+\nu_A(x_i)) \times \left[\frac{\mu_A^\alpha(x_i)+\nu_A^\alpha(x_i)}{2}-\left(\frac{\mu_A(x_i)+\nu_A(x_i)}{2}\right)^\alpha\right]=0. \quad \forall x_i \in X. \tag{17}$$

Equation (17) will hold if

$$\mu_A(x_i)+\nu_A(x_i)=0 \Longrightarrow \mu_A(x_i) = \nu_A(x_i)=0 \quad \forall x_i \in X \tag{18}$$

or

$$\left[\frac{\mu_A^\alpha(x_i)+\nu_A^\alpha(x_i)}{2}-\left(\frac{\mu_A(x_i)+\nu_A(x_i)}{2}\right)^\alpha\right]=0. \tag{19}$$

Now consider the following function:

$$f(z)=z^\alpha \quad \text{where } z\in[0,1]. \tag{20}$$

Differentiating (20) with respect to z, we get

$$f'(z)=\alpha z^{\alpha-1}, \quad f''(x)=\alpha(\alpha-1)z^{\alpha-2}. \tag{21}$$

Since $f''(z)>0$ when $\alpha>1$ and $f''(z)<0$ when $\alpha<1$, $f(z)$ is convex (∪) or concave (∩) function according to $\alpha>1$ or $\alpha<1$.

Therefore, for any two points z_1 and z_2 in $[0,1]$, the following inequalities hold:

$$\frac{f(z_1)+f(z_2)}{2}-f\left(\frac{z_1+z_2}{2}\right)\geq 0 \quad \text{for } \alpha>1, \qquad \frac{f(z_1)+f(z_2)}{2}-f\left(\frac{z_1+z_2}{2}\right)\leq 0 \quad \text{for } \alpha<1 \tag{22}$$

with the equality holding in (22) only for $z_1=z_2$. Therefore, from (18), (19), and (22), we conclude that (17) holds only if $\mu_A(x_i)=\nu_A(x_i)$ for all $x_i\in X$.

(*P3*) In order to show that (9) satisfies (*P3*), it suffices to prove that the function

$$g(x,y)=\frac{1}{(1-\alpha)}\log\left[(x^\alpha+y^\alpha)(x+y)^{1-\alpha} + 2^{1-\alpha}(1-x-y)\right], \tag{23}$$

where $x,y\in[0,1]$, is an increasing function with respect to x and decreasing with respect to y. Taking the partial derivatives of g with respect to x and y, respectively, yields

$$\frac{\partial g(x,y)}{\partial x} = \frac{(1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}x^{\alpha-1}-2^{1-\alpha}}{(1-\alpha)(x+y)^{1-\alpha}(x^\alpha+y^\alpha)+2^{1-\alpha}(1-x-y)}, \tag{24}$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{(1-\alpha)(x+y)^{-\alpha}(x^\alpha+y^\alpha)+\alpha(x+y)^{1-\alpha}y^{\alpha-1}-2^{1-\alpha}}{(1-\alpha)(x+y)^{1-\alpha}(x^\alpha+y^\alpha)+2^{1-\alpha}(1-x-y)}. \tag{25}$$

For critical point of g, we set $\partial g(x,y)/\partial x = 0$ and $\partial g(x,y)/\partial y=0$. This gives

$$x=y. \tag{26}$$

Also, from (24) and (26), we have

$$\frac{\partial g(x,y)}{\partial x}\geq 0 \quad \text{when } x\leq y,\ 0<\alpha<1 \text{ as also for } \alpha>1, \qquad \frac{\partial g(x,y)}{\partial x}\leq 0 \quad \text{when } x\geq y,\ 0<\alpha<1 \text{ as also for } \alpha>1, \tag{27}$$

for any $x,y\in[0,1]$. Thus $g(x,y)$ is increasing with respect to x for $x\leq y$ and decreasing when $x\geq y$.

Similarly, we see that

$$\frac{\partial g(x,y)}{\partial y}\leq 0, \quad \text{when } x\leq y,\ 0<\alpha<1 \text{ as also for } \alpha>1, \qquad \frac{\partial g(x,y)}{\partial y}\geq 0, \quad \text{when } x\geq y,\ 0<\alpha<1 \text{ as also for } \alpha>1. \tag{28}$$

Let us now consider two sets $A,B\in \mathrm{IFS}(X)$, with $A\subseteq B$. Assume that the finite universe of discourse $X=\{x_1,x_2,\ldots,x_n\}$ is partitioned into two disjoint sets X_1 and X_2 with $X_1\cup X_2=X$.

Let us further suppose that all $x_i\in X_1$ are dominated by the condition

$$\mu_A(x_i)\leq\mu_B(x_i)\leq\nu_B(x_i)\leq\nu_A(x_i), \tag{29}$$

while for all $x_i\in X_2$ satisfy

$$\mu_A(x_i)\geq\mu_B(x_i)\geq\nu_B(x_i)\geq\nu_A(x_i). \tag{30}$$

Then from the monotonicity of the function g and (9), we obtain that $E_\alpha(A)\leq E_\alpha(B)$ when $A\subseteq B$.

(*P4*) It is clear that $A^C=\{\langle x,\nu_A(x),\mu_A(x)\rangle \mid x\in X\}$ for $x_i\in X$; that is

$$\mu_{A^C}(x_i)=\nu_A(x_i), \qquad \nu_{A^C}(x_i)=\mu_A(x_i). \tag{31}$$

Then, from (9), we straightforwardly have

$$E_\alpha(A) = E_\alpha\left(A^C\right). \tag{32}$$

Hence, $E_\alpha(A)$ is a measure of intuitionistic fuzzy entropy.

This proves the theorem. □

The proposed intuitionistic fuzzy entropy of order-α satisfies the following additional properties.

Theorem 7. *Let A and B be two intuitionistic fuzzy sets defined in $X = \{x_1, x_2, \ldots, x_n\}$, where $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle \mid x_i \in X\}$, $B = \{\langle x_i, \mu_B(x_i), \nu_B(x_i)\rangle \mid x_i \in X\}$, such that they satisfy for any $x_i \in X$ either $A \subseteq B$ or $A \supseteq B$; then one has*

$$E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B). \tag{33}$$

Proof. Let us separate X into two parts X_1 and X_2, where

$$X_1 = \{x_i \in X : A \subseteq B\}, \qquad X_2 = \{x_i \in X : A \supseteq B\}. \tag{34}$$

That is, for all $x_i \in X_1$,

$$\mu_A(x_i) \le \mu_B(x_i), \qquad \nu_A(x_i) \ge \nu_B(x_i) \tag{35}$$

and, for all $x_i \in X_2$,

$$\mu_A(x_i) \ge \mu_B(x_i), \qquad \nu_A(x_i) \le \nu_B(x_i). \tag{36}$$

From (9), we then have

$$\begin{aligned}
&E_\alpha(A \cup B) \\
&= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log\Big[\left(\mu_{A\cup B}^{\alpha}(x_i) + \nu_{A\cap B}^{\alpha}(x_i)\right) \times \left(\mu_{A\cup B}(x_i) + \nu_{A\cap B}(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_{A\cup B}(x_i) - \nu_{A\cap B}(x_i)\right) \Big] \\
&= \frac{1}{n(1-\alpha)} \Bigg[\Bigg\{ \sum_{x \in X_1} \log\Big[\left(\mu_B^{\alpha}(x_i) + \nu_B^{\alpha}(x_i)\right) \times \left(\mu_B(x_i) + \nu_B(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_B(x_i) - \nu_B(x_i)\right) \Big] \Bigg\} \\
&\quad + \Bigg\{ \sum_{x \in X_2} \log\Big[\left(\mu_A^{\alpha}(x_i) + \nu_A^{\alpha}(x_i)\right) \times \left(\mu_A(x_i) + \nu_A(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_A(x_i) - \nu_A(x_i)\right) \Big] \Bigg\} \Bigg],
\end{aligned}$$

$$\begin{aligned}
&E_\alpha(A \cap B) \\
&= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log\Big[\left(\mu_{A\cap B}^{\alpha}(x_i) + \nu_{A\cup B}^{\alpha}(x_i)\right) \times \left(\mu_{A\cap B}(x_i) + \nu_{A\cup B}(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_{A\cap B}(x_i) - \nu_{A\cup B}(x_i)\right) \Big] \\
&= \frac{1}{n(1-\alpha)} \Bigg[\Bigg\{ \sum_{x \in X_1} \log\Big[\left(\mu_A^{\alpha}(x_i) + \nu_A^{\alpha}(x_i)\right) \times \left(\mu_A(x_i) + \nu_A(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_A(x_i) - \nu_A(x_i)\right) \Big] \Bigg\} \\
&\quad + \Bigg\{ \sum_{x \in X_2} \log\Big[\left(\mu_B^{\alpha}(x_i) + \nu_B^{\alpha}(x_i)\right) \times \left(\mu_B(x_i) + \nu_B(x_i)\right)^{1-\alpha} + 2^{1-\alpha}\left(1 - \mu_B(x_i) - \nu_B(x_i)\right) \Big] \Bigg\} \Bigg].
\end{aligned} \tag{37}$$

Now from (37), we get

$$E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B). \tag{38}$$

This proves the theorem. □

Corollary 8. *For any $A \in IFS(X)$ and A^C the complement of intuitionistic fuzzy set A,*

$$E_\alpha(A) = E_\alpha\left(A^C\right) = E_\alpha\left(A \cup A^C\right) = E_\alpha\left(A \cap A^C\right). \tag{39}$$

Theorem 9. *$E_\alpha(A)$ attains the maximum value when the set is most intuitionistic fuzzy set and the minimum value when the set is crisp set; moreover, maximum and minimum values are independent of α.*

Proof. It has already been proved that $E_\alpha(A)$ is maximum if and only if A is most intuitionistic fuzzy set, that is, $\mu_A(x_i) = \nu_A(x_i)$, for all $x_i \in X$, and minimum when A is a crisp set. So, it is enough to prove that the maximum and minimum values are independent of α.

Let A be the most intuitionistic fuzzy set; that is, $\mu_A(x_i) = \nu_A(x_i)$, for all $x_i \in X$. Then

$$\begin{aligned}
E_\alpha(A) = \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log\Big[&\left(\nu_A^{\alpha}(x_i) + \nu_A^{\alpha}(x_i)\right) \\
&\times \left(\nu_A(x_i) + \nu_A(x_i)\right)^{1-\alpha} \\
&+ 2^{1-\alpha}\left(1 - \nu_A(x_i) - \nu_A(x_i)\right) \Big]
\end{aligned}$$

$$= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log \left[2^{1-\alpha} \right]$$
$$= 1, \quad (40)$$

which is independent of α.

On the other hand, if A is a crisp set, that is, $\mu_A(x_i) = 1$ and $\nu_A(x_i) = 0$ or $\mu_A(x_i) = 0$ and $\nu_A(x_i) = 1$, for all $x_i \in X$, then $E_\alpha(A) = 0$ for any value of α.

This proves the theorem. □

Particular and Limiting Cases

(1) When $\alpha \to 1$, then measure in (9) reduces to measures in (3).

(2) It may be noticed that if an intuitionistic fuzzy set is an ordinary fuzzy set, that is, for all $x_i \in X$, $\nu_A(x_i) = 1 - \mu_A(x_i)$, then the intuitionistic fuzzy entropy of order-α reduces to fuzzy entropy of order-α [13].

In the next section, we consider an example to demonstrate the performance of proposed intuitionistic fuzzy entropy of order-α by comparing with other existing measures of intuitionistic fuzzy entropy.

4. Numerical Example

Let $A = \{(x_i, \mu_A(x_i), \nu_A(x_i)) \mid x_i \in X\}$ be an IFS in $X = \{x_1, x_2, \ldots, x_n\}$. For any positive real number n, De et al. [19] defined the IFS A^n as follows:

$$A^n = \left\{ \left(x_i, (\mu_A(x_i))^n, 1 - (1 - \nu_A(x_i))^n \right) \mid x_i \in X \right\}. \quad (41)$$

Using the above operation, they also defined the concentration and dilation of A given by

$$\text{Concentration: } \mathrm{CON}(A) = A^2,$$
$$\text{Dilation: } \mathrm{DIL}(A) = A^{1/2}. \quad (42)$$

Like fuzzy sets, CON(A) and DIL(A) may be treated as "very (A)" and "more or less (A)," respectively.

Example 1. Let us consider an IFS A in $X = \{6, 7, 8, 9, 10\}$, defined by De et al. [19], as

$$A = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.5, 0.4), (9, 0.9, 0.0), (10, 1.0, 0.0)\}. \quad (43)$$

Using the operation defined in (41), we can generate the following IFSs:

$$A^{1/2}, \; A^2, \; A^3, \; A^4. \quad (44)$$

By taking into account the characterization of linguistic variables, we regarded A as "LARGE" in X. Using the above operation, we have

$$A^{1/2} = \{(6, 0.3162, 0.5528), (7, 0.5477, 0.2929), (8, 0.7071, 0.2254), (9, 0.9487, 0.0), (10, 1.0, 0.0)\},$$
$$A^2 = \{(6, 0.0100, 0.9600), (7, 0.0900, 0.7500), (8, 0.2500, 0.6400), (9, 0.8100, 0.000), (10, 1.0, 0.0)\},$$
$$A^3 = \{(6, 0.0010, 0.9920), (7, 0.0270, 0.8750), (8, 0.1250, 0.7840), (9, 0.7290, 0.000), (10, 1.0, 0.0)\},$$
$$A^4 = \{(6, 0.0001, 0.9984), (7, 0.0081, 0.9375), (8, 0.0625, 0.8704), (9, 0.6591, 0.000), (10, 1.0, 0.0)\}. \quad (45)$$

The levels represented by the above intuitionistic fuzzy sets are described as follows:

$A^{1/2}$ may be treated as "More or less LARGE,"

A^2 may be treated as "Very LARGE,"

A^3 may be treated as "Quite very LARGE,"

A^4 may be treated as "Very very LARGE."

Now we use these IFSs to compare the intuitionistic fuzzy entropy of order-α and other existing measures of intuitionistic fuzzy entropy.

For the purpose of comparison, we first mention here some entropy measures for intuitionistic fuzzy sets defined by various researchers as

Burillo and Bustince's entropy [13]:

$$E_{\mathrm{BB}}(A) = \frac{1}{n} \sum_{i=1}^{n} \pi_A(x_i), \quad (46)$$

Zeng and Li's entropy [15]:

$$E_{\mathrm{ZL}}(A) = 1 - \frac{1}{n} \sum_{i=1}^{n} \left| \mu_A(x_i) - \nu_A(x_i) \right|, \quad (47)$$

Szmidt and Kacprzyk's entropy [14]:

$$E_{\mathrm{SK}}(A) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mu_A(x_i) \wedge \nu_A(x_i) + \pi_A(x_i)}{\mu_A(x_i) \vee \nu_A(x_i) + \pi_A(x_i)} \right), \quad (48)$$

Zhang and Jiang's entropy [18]:

$$E_{\mathrm{ZJ}}(A) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mu_A(x_i) \wedge \nu_A(x_i)}{\mu_A(x_i) \vee \nu_A(x_i)} \right), \quad (49)$$

TABLE 1: Values of the different entropy measures under $A^{1/2}$, A, A^2, A^3, A^4.

	$A^{1/2}$	A	A^2	A^3	A^4
E_{BB}	0.0923	0.1200	0.1320	0.1344	0.1360
E_{ZL}	0.3788	0.3600	0.3160	0.2320	0.1911
E_{SK}	0.3194	0.3073	0.3010	0.2121	0.1758
E_{VS}	0.5067	0.4931	0.3746	0.2969	0.2476
E_{ZJ}	0.2486	0.2117	0.2261	0.0949	0.0457
E_{hc}^2	0.3276	0.3400	0.2903	0.2525	0.2258
$E_r^{1/2}$	0.6609	0.6809	0.6045	0.5216	0.4567
$E_{0.3}$	0.5944	0.5885	0.5090	0.4159	0.3428
$E_{0.8}$	0.5633	0.5373	0.3973	0.2647	0.1988
E_2	0.5039	0.4721	0.3185	0.1618	0.1144
E_5	0.4272	0.4186	0.2719	0.0991	0.0646
E_{10}	0.3373	0.2996	0.2529	0.1514	0.0921
E_{15}	0.3216	0.2838	0.2468	0.1389	0.0354
E_{50}	0.2999	0.2616	0.2389	0.1217	0.0657
E_{100}	0.2941	0.2571	0.2373	0.1182	0.0627

and Hung and Yang' entropies [16]:

$$E_{hc}^2(A) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left(\mu_A^2(x_i) + \nu_A^2(x_i) + \pi_A^2(x_i)\right)\right),$$
$$E_r^{1/2}(A) = \frac{2}{n}\sum_{i=1}^{n}\log\left(\mu_A^{1/2}(x_i) + \nu_A^{1/2}(x_i) + \pi_A^{1/2}(x_i)\right). \quad (50)$$

From the viewpoint of mathematical operations, the entropy values of the above defined IFSs, $A^{1/2}$, A, A^2, A^3, A^4, have the following requirement:

$$E\left(A^{1/2}\right) > E(A) > E\left(A^2\right) > E\left(A^3\right) > E\left(A^4\right). \quad (51)$$

The values of different entropy measures under $A^{1/2}$, A, A^2, A^3, and A^4 are shown in Table 1.

Based on Table 1, we see that the entropy measures E_{ZL}, E_{SK}, E_{VS}, $E_{0.3}$, $E_{0.8}$, E_2, E_5, E_{10}, E_{15}, and E_{100} satisfy the requirement given in (51), but E_{BB}, E_{ZJ}, E_{hc}^2, and $E_r^{1/2}$ do not satisfy the requirement.

Therefore, the behavior of intuitionistic fuzzy entropy of order-α, $E_\alpha(A)$, is good for the viewpoint of structured linguistic variables.

5. Conclusions

This work introduces a new entropy measure called intuitionistic fuzzy entropy of order-α in the setting of intuitionistic fuzzy set theory. Some properties of this measure have been also studied. This measure generalizes Bhandari and Pal [8] fuzzy entropy of order-α and Vlachos and Sergiadis [17] logarithmic intuitionistic fuzzy entropy. Introduction of parameter provides new flexibility and wider application of intuitionistic fuzzy entropy to different situations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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