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MATHEMATICS

On invariant submanifolds of trans-Sasakian manifolds

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Abstract. The object of the present paper is to find necessary and sufficient conditions for invariant submanifolds of trans-Sasakian manifolds to be totally geodesic. As a remark, particular cases of submanifolds of α-Sasakian and β-Kenmotsu manifolds are considered and the difference between the conditions for submanifolds of α-Sasakian and β-Kenmotsu manifolds to be totally geodesic is shown.

Key words: trans-Sasakian manifolds, second fundamental form, recurrent, invariant submanifold, totally geodesic.

1. INTRODUCTION

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzalez, and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds [6]. Again, in the Gray–Hervella classification of almost Hermite manifolds [12], there appears a class W_4 of Hermitian manifolds that are closely related to locally conformal Kähler manifolds. An almost contact metric structure on a manifold $\tilde{M}$ is called a trans-Sasakian structure [19] if the product manifold $\tilde{M} \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ [16,17] coincides with the class of trans-Sasakian structures of type (α, β). In fact, in [17], the local nature of two subclasses, namely C_5 and C_6 structures, of trans-Sasakian structures are characterized completely. We note that trans-Sasakian structures of type (0,0), $(0, \beta)$, and $(\alpha, 0)$ are cosymplectic [2], β-Kenmotsu [13], and α-Sasakian [13], respectively.

In [10], we find the importance of Sasakian manifolds in supergravity and magnetic theory. Tanno [22] showed that Kenmotsu manifolds are warped product space. It is well known that the notion of warped product plays some important role in differential geometry as well as in physics. For instance, the best relativistic model of Schwarzschild space-time that describes the outer space around a massive star or a black hole is given as warped product [1,5,18]. Again, we know that a trans-Sasakian manifold of dimension ≥ 5 is either a co-symplectic or α-Sasakian or β-Kenmotsu manifold [16]. Hence, we claim that the study of trans-Sasakian manifolds is important from the physical point of view.

Nowadays, the study of submanifold theory is growing rapidly. Invariant submanifolds play a crucial role in many applied branches of mathematics. For instance, the method of invariant submanifolds is used in the study of non-linear autonomous systems [11].

There is a well-known result of Kon that an invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of the immersion is covariantly constant [15]. In general, an invariant submanifold of a Sasakian manifold is not totally geodesic. For example, the circle bundle

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(S,Q^n) over an n-dimensional complex projective space $\mathbb{C}P^{n+1}$ is an invariant submanifold of a $(2n+3)$-dimensional Sasakian space-form $S^{2n+3}(c)$ with $c > -3$, which is not totally geodesic [23]. On the other hand, any submanifold of a Kenmotsu manifold is totally geodesic if and only if the second fundamental form of the immersion is covariantly constant and the submanifold is tangent to the structure vector field ξ [14]. Recently, Sular and Özgur [21] studied submanifolds of Kenmotsu manifolds and proved some equivalent conditions regarding the submanifolds to be totally geodesic. Since the trans-Sasakian manifold generalizes both Sasakian and Kenmotsu manifolds, we are naturally motivated to find the conditions under which a submanifold of a trans-Sasakian manifold is totally geodesic. To this end, after the preliminaries in Section 2, we study invariant submanifolds of trans-Sasakian manifolds in Section 3 and prove some necessary and sufficient conditions for such submanifolds to be totally geodesic. Also, in this section an interesting corollary and an important remark concerning submanifolds of α-Sasakian and β-Kenmotsu manifolds are added. In this connection we mention that recently trans-Sasakian manifolds and their submanifolds were studied in [8,9].

2. PRELIMINARIES

Let $\tilde{M}$ be an almost contact metric manifold of dimension $2\tilde{n}+1$, that is, a $(2\tilde{n}+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ,ξ,η,g). By definition ϕ,ξ,η are tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$, respectively, and g is a Riemannian metric such that [2]

$$\phi^2 X = -X + \eta(X)\xi, \;\; \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all differentiable vector fields X,Y on $\tilde{M}$. Then also

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; \eta(X) = g(X,\xi). \tag{2.3}$$

Let Φ be the fundamental 2-form defined by $\Phi(X,Y) = g(X,\phi Y)$ for all differentiable vector fields X,Y on $\tilde{M}$. An almost contact metric structure (ϕ,ξ,η,g) on $\tilde{M}$ is called trans-Sasakian structure [19] if $(\tilde{M}\otimes\mathbb{R},J,G)$ belongs to the class W_4 [12], where J is the almost complex structure on $\tilde{M}\otimes\mathbb{R}$ defined by

$$J(X, f d/df) = (\phi X - f\xi, \eta(X) d/dt),$$

for all vector fields X on $\tilde{M}$ and smooth functions f on $\tilde{M}\otimes\mathbb{R}$ and G is the product metric on $\tilde{M}\otimes\mathbb{R}$. This may be expressed by the condition [3]

$$(\tilde{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

for smooth functions α and β on $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. We say that the trans-Sasakian structure is of type (α,β). From (2.4) it follows that

$$\tilde{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\tilde{\nabla}_X\eta)Y = -\alpha g(\phi X,Y) + \beta g(\phi X,\phi Y). \tag{2.6}$$

Let $f:(M,g)\to(\tilde{M},g)$ be an isometric immersion from an n-dimensional Riemannian manifold M to a $(2\tilde{n}+1)$-dimensional trans-Sasakian manifold $\tilde{M}$. Then we have [4]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.7}$$

$$\tilde{\nabla}_X N = -A_N X + \tilde{\nabla}^{\perp}_X N, \tag{2.8}$$

for all vector fields X,Y tangent to M and normal vector field N on M, where ∇ is the Riemannian connection on M defined by the induced metric g and $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$ of M; h is the second fundamental form of the immersion.

A submanifold M of $\tilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for every vector X tangent to M at every point on M. The submanifold is called totally geodesic if its second fundamental form vanishes identically on it [23].

The first and the second covariant derivative of the second fundamental form h are given by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \tag{2.9}$$

and

$$(\bar{\nabla}_X \bar{\nabla}_Y h)(Z,W) = \nabla^{\perp}_X((\bar{\nabla}_Y h)(Z,W)) - (\bar{\nabla}_Y h)(\nabla_X Z,W) - (\bar{\nabla}_Y h)(Z,\nabla_X W) - (\bar{\nabla}_{\nabla_X Y} h)(Z,W), \tag{2.10}$$

respectively, where $\bar{\nabla}$ is called the van der Waerden Bortolotti connection of M [4]. It is mentioned that if $(\bar{\nabla}_Y \bar{\nabla}_X h)(Y,Z) = 0$, then the submanifold is said to have a parallel third fundamental form. If ξ is tangent to M by (2.7), then

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X,\xi),$$

which implies by (2.5)

$$-\alpha\phi X + \beta(X - \eta(X)\xi) = \nabla_X \xi + h(X,\xi).$$

By equating the tangential and normal parts, from the above we get the following cases:

Case I.

$$h(X,\xi) = 0, \ \nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.11}$$

for each vector X tangent to M, when the submanifold is invariant.

Case II.

$$h(X,\xi) = 0, \ \nabla_X \xi = -\alpha\phi X \tag{2.12}$$

for each vector X tangent to M, when the submanifold is invariant and $\beta = 0$.

Case III.

$$h(X,\xi) = 0, \ \nabla_X \xi = \beta(X - \eta(X)\xi) \tag{2.13}$$

for each vector X tangent to M, when $\alpha = 0$. Here the condition of invariance is not required. We need only ξ to be tangent to M.

Proposition 2.1. *An invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian.*

Proof. We see that

$$\begin{aligned}(\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y) \\ &= \nabla_X \phi Y + h(X,\phi Y) - \phi(\nabla_X Y) - \phi h(X,Y).\end{aligned} \tag{2.14}$$

From (2.4) and the above equation we get by considering the submanifold as invariant and comparing tangential and normal components

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X). \tag{2.15}$$

The above equation proves the proposition. □

Proposition 2.2. *If X is an arbitrary tangent vector of an invariant submanifold of a trans-Sasakian manifold, then $\nabla_X \xi$ is also an arbitrary tangent vector different from a vector spanned by ξ.*

Proof. From (2.11) we get

$$\nabla_X \xi = \beta X - \alpha \phi X - \beta \eta(X)\xi. \tag{2.16}$$

From the above equation we see that $\nabla_X \xi$ is a linear combination of X, ϕX, and ξ. Since an invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian, the dimension of the submanifold is odd. Hence we can consider that an orthonormal ϕ-basis [2] $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots, \frac{n-1}{2}$, n is the dimension of the submanifold, and we can write

$$X = \sum_i a_i e_i + \sum_i b_i \phi e_i + c\xi. \tag{2.17}$$

Then

$$\phi X = -\sum_i b_i e_i + \sum_i a_i \phi e_i. \tag{2.18}$$

Here a_i, b_i, and c are scalars. From (2.16), (2.17), and (2.18) we obtain

$$\nabla_X \xi = \sum_i (\beta a_i + \alpha b_i) e_i + \sum_i (\beta b_i - \alpha a_i)\phi e_i. \tag{2.19}$$

If we write $TM = D \oplus < \xi >$, we see that $\nabla_X \xi \in D$ and, if X is arbitrary, $\nabla_X \xi$ is also arbitrary, because a_i, b_i, and c are arbitrary. This completes the proof. $\square$

3. INVARIANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS WITH THE SECOND FUNDAMENTAL FORM SATISFYING SOME CONDITIONS

The covariant differential of the pth order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, is denoted by $\nabla^p T$ on a Riemannian manifold M with Levi-Civita connection ∇. According to [20], the tensor T is said to be recurrent or 2-recurrent, if the following conditions, respectively (3.1) and (3.2), hold on M:

$$(\nabla T)(X_1, X_2, \ldots, X_k; X) T(Y_1, Y_2, \ldots, Y_k) = (\nabla T)(Y_1, \ldots, Y_k; X) T(X_1, X_2, \ldots, X_k), \tag{3.1}$$

$$(\nabla^2 T)(X_1, \ldots, X_k; X, Y) T(Y_1, \ldots, Y_k; X, Y) = (\nabla^2 T)(Y_1, Y_2, \ldots, Y_k; X, Y) T(X_1, \ldots, X_k), \tag{3.2}$$

where $X, Y, X_1, Y_1, \ldots, X_k, Y_k$ are tangent to M.

From (3.1) it follows that if the tensor T is non-zero, at a point x of M there exists a unique 1-form ω or a $(0,2)$-tensor ψ, defined on a neighbourhood U of x, such that

$$\nabla T = T \otimes \omega, \quad \omega = d(\log||T||), \tag{3.3}$$

or

$$\nabla^2 T = T \otimes \psi, \tag{3.4}$$

respectively, holds on U, where $||T||$ denotes the norm of T defined by $||T||^2 = g(T,T)$. If $\nabla T = 0$, T is called parallel. The tensor T is said to be generalized 2-recurrent if

$$\begin{aligned} &((\nabla^2 T)(X_1, \ldots, X_k : X, Y) - (\nabla T \otimes \omega)(X_1, \ldots, X_k; X, Y)) \\ &\quad = ((\nabla^2 T)(Y_1, \ldots, Y_k; X, Y) - (\nabla T \otimes \omega)(Y_1, \ldots, Y_k); X, Y) T(X_1, \ldots, X_k) \end{aligned} \tag{3.5}$$

holds on M. From the above it follows that if the tensor T is non-zero, then at a point $x \in M$ there exists a unique $(0,2)$-tensor ψ, defined on a neighbourhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \omega + T \otimes \psi \tag{3.6}$$

holds on U.

In the following we consider an invariant submanifold M of the trans-Sasakian manifold $\tilde{M}$. Suppose that the second fundamental form of the submanifold M is parallel. Then

$$(\bar{\nabla}_X h)(Y,Z) = 0, \tag{3.7}$$

for X,Y,Z tangent to M. By (2.9) we obtain from the above equation

$$\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = 0.$$

Putting $Z = \xi$ in the above equation, we get by (2.11)

$$h(Y,\nabla_X \xi) = 0.$$

From Proposition 2.2 it follows that $\nabla_X \xi = U$(say) is a tangent vector of the submanifold different from a vector spanned by ξ. Therefore, $h(Y,U) = 0$. Again $h(Y,\xi) = 0$, for any tangent vector Y. Consequently, the submanifold is totally geodesic. Conversely, if the submanifold is totally geodesic, then $h(Y,W) = 0$, for arbitrary tangent vectors Y and W. Hence h is trivially parallel. The above discussion helps us to state the following:

Theorem 3.1. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is parallel.*

Let us consider that the submanifold M of $\tilde{M}$ has recurrent second fundamental form h. Then by (3.3)

$$(\bar{\nabla}_X h)(Y,Z) = \omega(X)h(Y,Z), \tag{3.8}$$

where ω is a 1-form on M. Applying (2.9) in the above equation we obtain

$$\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = \omega(X)h(Y,Z).$$

Putting $Z = \xi$, we get from the above

$$\nabla^{\perp}_X h(Y,\xi) - h(\nabla_X Y,\xi) - h(Y,\nabla_X \xi) = \omega(X)h(Y,\xi).$$

By virtue of (2.11) the above equation reduces to

$$h(Y,\nabla_X \xi) = 0.$$

Hence, as before, the submanifold is totally geodesic. The converse also holds trivially. Thus, we have the following:

Theorem 3.2. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is recurrent.*

If the submanifold has a parallel third fundamental form, we can write

$$(\bar{\nabla}_X \bar{\nabla}_Y h)(Z,W) = 0.$$

Putting $W = \xi$ and using (2.10) in the above equation we get

$$\nabla^{\perp}_X((\bar{\nabla}_Y h)(Z,\xi)) - (\bar{\nabla}_Y h)(\nabla_X Z,\xi) - (\bar{\nabla}_Y h)(Z,\nabla_X \xi) - (\bar{\nabla}_{\nabla_X Y} h)(Z,\xi) = 0.$$

By (2.9) the above equation yields

$$\begin{aligned}\nabla^{\perp}_X(\nabla^{\perp}_Y h(Z,\xi) - h(\nabla_Y Z,\xi) - h(Z,\nabla_Y\xi))&\\ -(\nabla^{\perp}_Y h(\nabla_X Z,\xi) - h(\nabla_Y\nabla_X Z,\xi) - h(\nabla_X Z,\nabla_Y\xi))&\\ -(\nabla^{\perp}_Y h(Z,\nabla_X\xi) - h(\nabla_Y Z,\nabla_X\xi) - h(Z,\nabla_Y(\nabla_X\xi)))&\\ -(\nabla^{\perp}_{\nabla_X Y} h(Z,\xi) - h(\nabla_{\nabla_X Y}Z,\xi) - h(Z,\nabla_{\nabla_X Y}\xi))&\\ =0.&\end{aligned} \tag{3.9}$$

By virtue of (2.11) the above equation yields

$$-\nabla^{\perp}_X h(Z,\nabla_Y\xi) + h(\nabla_X Z,\nabla_Y\xi) + h(\nabla_Y Z,\nabla_X\xi) + h(Z,\nabla_Y(\nabla_X\xi)) + h(Z,\nabla_{\nabla_X Y}\xi) = 0. \tag{3.10}$$

Putting $Y=\xi$ in (3.10), we get

$$h(\nabla_\xi Z,\nabla_X\xi) + h(Z,\nabla_\xi(\nabla_X\xi)) + h(Z,\nabla_{\nabla_X\xi}\xi) = 0.$$

Replacing $\nabla_X\xi$ by U in the above equation, we obtain

$$h(\nabla_\xi Z,U) + h(Z,\nabla_\xi U) + h(Z,\nabla_U\xi) = 0.$$

For $Z=U$, the above equation yields

$$h(U,2\nabla_\xi U + \nabla_U\xi) = 0. \tag{3.11}$$

Clearly, $2\nabla_\xi U + \nabla_U\xi = V$(say) is tangent to M, because ∇ is a mapping from $TM\times TM \to TM$. In the proof of Proposition 2.2 we obtained that $\nabla_X\xi = U$ is a vector spanned by e_i and ϕe_i, $i=1,2,\ldots,\frac{n-1}{2}$, and U has no component in the distribution spanned by $<\xi>$. Here, $\{e_i,\phi e_i,\xi\}$ is an orthonormal ϕ-basis at any point of the tangent space of M. Therefore, we can write

$$U = \sum_i c_i e_i + \sum_i d_i\phi e_i,$$

where c_i and d_i are scalars. From the above equation it follows that $\eta(U)=0$. Again, by (2.11)

$$\begin{aligned}\nabla_U\xi &= -\alpha\phi U + \beta(U-\eta(U)\xi)\\ &= -\alpha\phi U + \beta U.\end{aligned} \tag{3.12}$$

Hence, $\nabla_U\xi \in D$, where $TM = D\oplus <\xi>$. Now

$$\begin{aligned}\nabla_\xi U &= \nabla_\xi\left(\sum_i c_i e_i + \sum_i d_i\phi e_i\right)\\ &= \sum_i c_i\nabla_\xi e_i + \sum_i d_i\nabla_\xi(\phi e_i).\end{aligned} \tag{3.13}$$

We know that $(\nabla_W g)(X,Y)=0$ for any tangent vector X,Y of M. Hence,

$$\nabla_W g(X,Y) - g(\nabla_W X,Y) - g(X,\nabla_W Y) = 0. \tag{3.14}$$

Putting $W=Y=\xi$ and $X=e_i$ in the above equation, we get

$$\nabla_\xi g(e_i,\xi) - g(\nabla_\xi e_i,\xi) - g(e_i,\nabla_\xi\xi) = 0.$$

From (2.11) it follows that $\nabla_\xi \xi = 0$. Again, $g(e_i, \xi) = 0$ because $\{e_i, \phi e_i, \xi\}$ is an orthonormal ϕ-basis. Hence from the above equation $g(\nabla_\xi e_i, \xi) = 0$, which implies that $\nabla_\xi e_i \in D$, where $TM = D \oplus <\xi>$. Similarly, putting $W = Y = \xi$ and $X = \phi e_i$ in (3.14), we can prove that $\nabla_\xi(\phi e_i) \in D$. Hence, from (3.13) it follows that $\nabla_\xi U \in D$. Therefore, from (3.11) it follows that $h(U,V) = 0$, for $U,V \in D$. Again, $h(\xi,\xi) = 0$ and $h(X,\xi) = 0$ for any tangent vector X. Hence, we conclude that $h(X,Y) = 0$ for any tangent vector X,Y of M. Thus, from the above discussion it follows that the submanifold is totally geodesic. Again, it can be trivially proved that if $h = 0$, then the submanifold has a parallel third fundamental form. So, we are in a position to state the following:

Theorem 3.3. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if the submanifold has a parallel third fundamental form.*

If the second fundamental form of the submanifold is 2-recurrent, then from (3.4)

$$(\bar{\nabla}_X \bar{\nabla}_X h)(Z,W) = \omega(X)h(Z,W).$$

For $W = \xi$, the above equation yields

$$(\bar{\nabla}_X \bar{\nabla}_Y h)(Z,\xi) = 0.$$

Therefore, by Theorem 3.3 we immediately obtain the following:

Theorem 3.4. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is 2-recurrent.*

Now, let us consider that the second fundamental form h of M is generalized 2-recurrent. Then by (3.6)

$$(\bar{\nabla}_X \bar{\nabla}_Y h)(Z,W) = \psi(X,Y)h(Z,W) + \omega(X)(\bar{\nabla}_Y h)(Z,W),$$

where ψ and ω are 2-form and 1-form, respectively. Putting $W = \xi$ in the above equation, we have with the help of (2.11)

$$(\bar{\nabla}_X \bar{\nabla}_Y h)(Z,\xi) = \omega(X)(\bar{\nabla}_Y h)(Z,\xi).$$

Using (2.9) and (2.10), we get from the above equation

$$\begin{aligned}
\nabla^\perp_X(\nabla^\perp_Y h(Z,\xi) - h(\nabla_Y Z,\xi) - h(Z,\nabla_Y \xi))&\\
-(\nabla^\perp_Y h(\nabla_X Z,\xi) - h(\nabla_Y \nabla_X Z,\xi) - h(\nabla_X Z,\nabla_Y \xi))&\\
-(\nabla^\perp_Y h(Z,\nabla_X \xi) - h(\nabla_Y Z,\nabla_X \xi) - h(Z,\nabla_Y(\nabla_X \xi)))&\\
-(\nabla^\perp_{\nabla_X Y} h(Z,\xi) - h(\nabla_{\nabla_X Y} Z,\xi) - h(Z,\nabla_{\nabla_X Y}\xi))&\\
= \omega(X)(\nabla^\perp_Y h(Y,\xi) - h(\nabla_Y Z,\xi) - h(Z,\nabla_Y \xi)).& \qquad (3.15)
\end{aligned}$$

By (2.11) the above equation yields

$$\begin{aligned}
&-\nabla^\perp_X h(Z,\nabla_Y \xi) + h(\nabla_X Z,\nabla_Y \xi) + h(\nabla_Y Z,\nabla_X \xi)\\
&+ h(Z,\nabla_Y(\nabla_X \xi)) + h(Z,\nabla_{\nabla_X Y}\xi)\\
&= \omega(X)h(Z,\nabla_Y \xi). \qquad (3.16)
\end{aligned}$$

For $Y = \xi$, the above equation gives

$$h(\nabla_\xi Z,\nabla_X \xi) + h(Z,\nabla_\xi(\nabla_X \xi)) + h(Z,\nabla_{\nabla_X \xi}\xi) = 0.$$

Hence, as before, the submanifold is totally geodesic. If $h = 0$, then it can be trivially proved that the second fundamental form of the submanifold is generalized 2-recurrent. Thus we obtain the following:

Theorem 3.5. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is generalized* 2*-recurrent.*

Chinea and Prestelo [7] proved that an invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if $\tilde{R}(X,Y)h = 0$, where $\tilde{R}$ denotes the curvature transformation on the manifold $\tilde{M}$, that is, the second fundamental form h is semi-parallel. Hence we obtain the following corollary.

Corollary 3.1. *For an invariant submanifold of a trans-Sasakian manifold the following conditions are equivalent*:
(i) *the submanifold is totally geodesic*,
(ii) *the second fundamental form of the submanifold is parallel*,
(iii) *the second fundamental form of the submanifold is recurrent*,
(iv) *the second fundamental form of the submanifold is* 2*-recurrent*,
(v) *the second fundamental form of the submanifold is generalized* 2*-recurrent*,
(vi) *the third fundamental form of the submanifold is parallel*,
(vii) *the second fundamental form of the submanifold is semi-parallel.*

Remark 3.1. From Case II and Case III of Section 2, it follows that for α-Sasakian manifolds the above theorems are also valid. For β-Kenmotsu manifolds the above results are true for any submanifolds tangent to ξ, the invariant condition is not needed.

4. CONCLUSION

A trans-Sasakian manifold arose in a natural way from the classification of almost contact metric structures. The study of trans-Sasakian manifolds is important from physical point of view. Invariant submanifolds play a crucial role in many applied branches of mathematics. For an invariant submanifold of a trans-Sasakian manifold the following conditions are equivalent:
• the submanifold is totally geodesic,
• the second fundamental form of the submanifold is parallel,
• the second fundamental form of the submanifold is recurrent,
• the second fundamental form of the submanifold is 2-recurrent,
• the second fundamental form of the submanifold is generalized 2-recurrent,
• the third fundamental form of the submanifold is parallel,
• the second fundamental form of the submanifold is semi-parallel.

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Trans-Sasaki muutkonna invariantsetest alammuutkondadest

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On leitud tarvilikud ja piisavad tingimused selleks, et trans-Sasaki muutkonna invariantne alammuutkond oleks täielikult geodeetiline (teoreem 3.5). Lisaks on tõestatud, et niisuguse alammuutkonna teise ja kolmanda fundamentaalvormi teatavad omadused on ekvivalentsed (järeldus 3.1).