



## On Inverse Eigenvalue Problems for Two Kinds of Special Banded Matrices

Wei-Ru Xu<sup>a</sup>, Guo-Liang Chen<sup>a</sup>

<sup>a</sup>*Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice,  
East China Normal University, Shanghai 200241, P. R. China*

**Abstract.** This paper presents two kinds of symmetric tridiagonal plus paw form (hereafter TPPF) matrices, which are the combination of tridiagonal matrices and bordered diagonal matrices. In particular, we exploit the interlacing properties of their eigenvalues. On this basis, the inverse eigenvalue problems for the two kinds of symmetric TPPF matrices are to construct these matrices from the minimal and the maximal eigenvalues of all their leading principal submatrices respectively. The necessary and sufficient conditions for the solvability of the problems are derived. Finally, numerical algorithms and some examples of the results developed here are given.

### 1. Introduction

In 2006, Peng [1, 2] discussed the inverse eigenvalue problems for tridiagonal matrices and paw form matrices respectively. Then Pickmann [3–5] made some refinements to Peng's papers. A remarkable idea in their papers is to construct an  $n \times n$  matrix with special structure from the given list  $(\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$ , such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the minimal and the maximal eigenvalues of all its  $j \times j$  leading principal submatrix for any  $j = 1, 2, \dots, n$ . Recently, Ref.[14] also constructed a special matrix based on the method and showed that the inverse eigenvalue problem for a symmetric quasi anti-bidiagonal matrix is equivalent to the problem for a certain symmetric tridiagonal matrix on the basis of Ref.[13]. In this paper, We mimic the analogous strategy to consider the problem of constructing two kinds of  $n$ -square real symmetric TPPF matrices

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Corresponding author: Guo-Liang Chen

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*Email addresses:* [weiruxu@foxmail.com](mailto:weiruxu@foxmail.com) (Wei-Ru Xu), [g1chen@math.ecnu.edu.cn](mailto:g1chen@math.ecnu.edu.cn) (Guo-Liang Chen)



**Problem 1.3.** Let  $(\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers, determine necessary and sufficient conditions for the existence of an  $n$  by  $n$  symmetric TPPF matrix  $A$  of the form (2), such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the minimal and the maximal eigenvalues of its  $j$ -square leading principal submatrix  $A_j$  for any  $j = 1, 2, \dots, n$ .

**Problem 1.4.** Constructing an  $n$  by  $n$  nonnegative symmetric TPPF matrix  $A$  of the form (2) when the necessary conditions in Problem 1.3 are satisfied.

These problems show that how to assign values to the entries of a matrix  $A$  of the form (1) or (2), such that it has the desired spectral properties.

It is not hard to note that the symmetric TPPF matrices are the generalized forms of Jacobi matrices. We will solve the above problems fully, the paper is organized as follows: In Section 2, we begin by introducing some basic concepts and results that will be used throughout this paper. Section 3 gives the necessary and sufficient conditions to answer Problems 1.1 and 1.2 respectively. Section 4 also presents the necessary and sufficient conditions for the solvability of Problems 1.3 and 1.4. Finally, Section 5 presents some examples to illustrate the efficiency of our results.

## 2. Preliminaries

In this section, our main goal is to introduce some important preliminaries that will be used in the following statements. Firstly, the well-known result—Cauchy’s Interlacing theorem, is quoted as follows.

**Theorem 2.1.** [15] *Let  $A$  be an  $n$ -square Hermitian matrix and let  $B$  be an  $(n - 1) \times (n - 1)$  principal submatrix of  $A$ . If the nonincreasing lists  $\{\lambda_i\}_{i=1}^n$  and  $\{\mu_i\}_{i=1}^{n-1}$  are the spectra of  $A$  and  $B$  respectively, then  $\lambda_n \leq \mu_n \leq \lambda_{n-1} \leq \mu_{n-1} \leq \dots \leq \lambda_2 \leq \mu_2 \leq \lambda_1$ .*

An immediate consequence of Cauchy’s Interlacing theorem is the following.

**Corollary 2.2.** *Let  $A$  be an  $n$ -square Hermitian matrix and let  $A_j$  be its  $j$  by  $j$  leading principal submatrix,  $j = 1, 2, \dots, n$ . Then the minimal and the maximal eigenvalues  $\lambda_j^{(1)}, \lambda_j^{(j)}$  of  $A_j$ ,  $j = 1, 2, \dots, n$ , respectively satisfy*

$$\lambda_n^{(1)} \leq \dots \leq \lambda_j^{(1)} \leq \dots \leq \lambda_1^{(1)} \leq \dots \leq \lambda_j^{(j)} \leq \dots \leq \lambda_n^{(n)}. \tag{3}$$

For simplicity in the further statements, we present the following concepts.

**Definition 2.3.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. If there exists an  $n$  by  $n$  Hermitian matrix  $A$  such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the minimal and the maximal eigenvalues of its  $j$  by  $j$  leading principal submatrix  $A_j$ ,  $j = 1, 2, \dots, n$ , we shall say that  $A$  interlacially realizes  $\sigma$  or that  $\sigma$  is interlacing realizable.

**Lemma 2.4.** *Let  $A$  be an  $n$ -square symmetric TPPF matrix of the form (1) and let  $f_j(\lambda)$  be the characteristic polynomial of its  $j$  by  $j$  leading principal submatrix for any  $j = 1, 2, \dots, n$ . Then the sequence  $\{f_j(\lambda)\}_{j=1}^n$  satisfies the recurrence formulae*

$$f_j(\lambda) = \begin{cases} (\lambda - a_j)f_{j-1}(\lambda) - b_{j-1}^2 f_{j-2}(\lambda), & j = 1, \dots, m + 1; \\ (\lambda - a_j)f_{j-1}(\lambda) - b_{j-1}^2 \left(\prod_{i=m+1}^{j-1} (\lambda - a_i)\right) f_{m-1}(\lambda), & j = m + 2, \dots, n; \end{cases} \tag{4}$$

where  $f_0(\lambda) = 1$  and  $b_0 = 0$ .

**Lemma 2.5.** Let  $A$  be an  $n$ -square symmetric TPPF matrix of the form (2) and let  $g_j(\lambda)$  be the characteristic polynomial of its  $j$  by  $j$  leading principal submatrix for any  $j = 1, 2, \dots, n$ . Then the sequence  $\{g_j(\lambda)\}_{j=1}^n$  satisfies the recurrence formulae

$$g_j(\lambda) = \begin{cases} \lambda - a_1, & j = 1; \\ (\lambda - a_2)g_1(\lambda) - b_1^2, & j = 2; \\ (\lambda - a_j)g_{j-1}(\lambda) - b_{j-1}^2 \left(\prod_{i=2}^{j-1} (\lambda - a_i)\right), & j = 3, \dots, p; \\ (\lambda - a_j)g_{j-1}(\lambda) - b_{j-1}^2 g_{j-2}(\lambda), & j = p + 1, \dots, n. \end{cases} \quad (5)$$

**Lemma 2.6.** [16] A sequence of real polynomials  $\varphi_n(x), \varphi_{n-1}(x), \dots, \varphi_0(x)$  is referred to as a Sturm sequence in a interval  $(a, b)$ , if the following properties are satisfied:

1.  $\varphi_0(x)$  does not have real root in the interval  $(a, b)$ ;
2. Any two adjacent polynomials in the sequence can not have any same root in the interval  $(a, b)$ ;
3. Let  $x_0 \in (a, b)$  be a root of a certain intermediate polynomial  $\varphi_i(x)$ ,  $i = 1, 2, \dots, n - 1$ , in the sequence, then  $\varphi_{i-1}(x_0)$  and  $\varphi_{i+1}(x_0)$  have opposite signs.

### 3. Solvability Conditions of Problems 1.1 and 1.2

Based on the discussion about Hermitian matrices in [15, pp.115]. We can derive the following result.

**Lemma 3.1.** Let  $A$  be an  $n$ -square symmetric TPPF matrix of the form (1). Let  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  be the minimal and the maximal eigenvalues of its  $j$ -square leading principal submatrix  $A_j$ ,  $j = 1, 2, \dots, n$ , respectively. Then we have

$$\lambda_n^{(1)} < \dots < \lambda_j^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_j^{(j)} < \dots < \lambda_n^{(n)}, \quad (6)$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = 1, 2, \dots, j; \quad j = 2, \dots, n. \quad (7)$$

*Proof.* From Lemmas 2.4 and 2.6, we know that  $\{f_j(\lambda)\}_{j=0}^{m+1}$  is a Sturm sequence. According to Corollary 2.2, we without difficulty obtain that

$$\lambda_{m+1}^{(1)} < \dots < \lambda_j^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_j^{(j)} < \dots < \lambda_{m+1}^{(m+1)}. \quad (8)$$

In addition, from [14, Theorem 2] we also obtain that

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = 1, 2, \dots, j; \quad j = 2, \dots, m + 1. \quad (9)$$

Now it remains us to prove that if the following inequalities

$$\lambda_j^{(1)} < \lambda_{j-1}^{(1)} < \dots < \lambda_{m+2}^{(1)} < \lambda_{m+1}^{(1)} < \lambda_{m+1}^{(m+1)} < \lambda_{m+2}^{(m+2)} < \dots < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}, \quad j = m + 2, \dots, n, \quad (10)$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = m + 2, \dots, j; \quad j = m + 2, \dots, n. \quad (11)$$

hold, then the proof is completed.

From Corollary 2.2 and the inequalities (8) and (9), we also know

$$\lambda_j^{(1)} \leq \lambda_{j-1}^{(1)} \leq \dots \leq \lambda_{m+2}^{(1)} \leq \lambda_{m+1}^{(1)} < \lambda_{m+1}^{(m+1)} \leq \lambda_{m+2}^{(m+2)} \leq \dots \leq \lambda_{j-1}^{(j-1)} \leq \lambda_j^{(j)}, \quad j = m + 2, \dots, n, \quad (12)$$

and

$$\lambda_j^{(1)} \leq a_i \leq \lambda_j^{(j)}, \quad i = m + 2, \dots, j; \quad j = m + 2, \dots, n. \tag{13}$$

By inductive hypothesis and contradiction, the discussion shows as follows.

(1) If  $j = m + 2$ , by Lemma 2.4 we have

$$f_{m+2}(\lambda) = (\lambda - a_{m+2})f_{m+1}(\lambda) - b_{m+1}^2(\lambda - a_{m+1})f_{m-1}(\lambda). \tag{14}$$

Based on (12), we assume  $\lambda_{m+2}^{(1)} = \lambda_{m+1}^{(1)}$ , then by (14) we know

$$f_{m+2}(\lambda_{m+1}^{(1)}) = -b_{m+1}^2(\lambda_{m+1}^{(1)} - a_{m+1})f_{m-1}(\lambda_{m+1}^{(1)}) = 0.$$

From (9) we get  $\lambda_{m+1}^{(1)} < a_{m+1}$ , then  $f_{m-1}(\lambda_{m+1}^{(1)}) = 0$ . But  $\lambda_{m+1}^{(1)} < \lambda_{m-1}^{(1)}$  from (8), it means that  $\lambda_{m+1}^{(1)}$  is not a root of  $f_{m-1}(\lambda)$ . Then we obtain a contradiction, i.e.  $\lambda_{m+2}^{(1)} < \lambda_{m+1}^{(1)}$ . Similarly,  $\lambda_{m+1}^{(m+1)} < \lambda_{m+2}^{(m+2)}$ . Finally, the inequality (12) is strict when  $j = m + 2$ , i.e. the inequality (10) holds.

Assume  $\lambda_{m+2}^{(1)} = a_{m+2}$  or  $\lambda_{m+2}^{(m+2)} = a_{m+2}$  based on (13) again, by (14) we know  $f_{m+2}(a_{m+2}) = -b_{m+1}^2(a_{m+2} - a_{m+1})f_{m-1}(a_{m+2}) = 0$ . But from (8) and (10) we know that  $\lambda_{m+2}^{(1)} < \lambda_{m-1}^{(1)} < \lambda_{m-1}^{(m-1)} < \lambda_{m+2}^{(m+2)}$ . That is to say  $a_{m+2}$  is not a root of  $f_{m-1}(\lambda)$ , i.e.  $a_{m+2} = a_{m+1}$ . It contradicts  $\lambda_{m+2}^{(1)} < \lambda_{m+1}^{(1)} < a_{m+1} < \lambda_{m+1}^{(m+1)} < \lambda_{m+2}^{(m+2)}$  from (9) and (10). So we finally obtain  $\lambda_{m+2}^{(1)} < a_{m+2} < \lambda_{m+2}^{(m+2)}$ , i.e. the inequality (11) holds.

(2) Assume the proposition holds when  $j = k$ , then we have

$$\lambda_k^{(1)} < \lambda_{k-1}^{(1)} < \dots < \lambda_{m+2}^{(1)} < \lambda_{m+1}^{(1)} < \lambda_{m+1}^{(m+1)} < \lambda_{m+2}^{(m+2)} < \dots < \lambda_{k-1}^{(k-1)} < \lambda_k^{(k)}, \tag{15}$$

and

$$\lambda_k^{(1)} < a_i < \lambda_k^{(k)}, \quad i = m + 2, \dots, k, \tag{16}$$

from (10) and (11) respectively. Now if  $j = k + 1$ , then from (12-13) and (15-16) we respectively have

$$\lambda_{k+1}^{(1)} \leq \lambda_k^{(1)} < \dots < \lambda_{m+2}^{(1)} < \lambda_{m+1}^{(1)} < \lambda_{m+1}^{(m+1)} < \lambda_{m+2}^{(m+2)} < \dots < \lambda_k^{(k)} \leq \lambda_{k+1}^{(k+1)}, \tag{17}$$

and

$$\lambda_{k+1}^{(1)} \leq a_{k+1} \leq \lambda_{k+1}^{(k+1)}. \tag{18}$$

By Lemma 2.4, we obtain

$$f_{k+1}(\lambda) = (\lambda - a_{k+1})f_k(\lambda) - b_k^2 \left( \prod_{i=m+1}^k (\lambda - a_i) \right) f_{m-1}(\lambda). \tag{19}$$

Based on (17), we assume  $\lambda_{k+1}^{(1)} = \lambda_k^{(1)}$ . Then

$$f_{k+1}(\lambda_k^{(1)}) = -b_k^2 \left( \prod_{i=m+1}^k (\lambda_k^{(1)} - a_i) \right) f_{m-1}(\lambda_k^{(1)}) = 0.$$

But from (9) and (15-16), we obtain  $\lambda_k^{(1)} < a_i, i = m + 1, \dots, k$ . Thus  $\prod_{i=m+1}^k (\lambda_k^{(1)} - a_i) \neq 0$ . So  $f_{m-1}(\lambda_k^{(1)}) = 0$ , it means that  $\lambda_k^{(1)}$  is a root of  $f_{m-1}(\lambda)$ . It contradicts  $\lambda_k^{(1)} < \lambda_{m-1}^{(1)}$  from (8) and (15). Hence  $\lambda_{k+1}^{(1)} < \lambda_k^{(1)}$ . Similarly,  $\lambda_k^{(k)} < \lambda_{k+1}^{(k+1)}$  holds. Finally, the inequality (10) is right.

Assume  $\lambda_{k+1}^{(1)} = a_{k+1}$  or  $\lambda_{k+1}^{(k+1)} = a_{k+1}$  based on (18) again, by (19) we derive

$$f_{k+1}(a_{k+1}) = -b_k^2 \left( \prod_{i=m+1}^k (a_{k+1} - a_i) \right) f_{m-1}(a_{k+1}) = 0.$$

From the above verified results, we know  $\lambda_{k+1}^{(1)} < \lambda_k^{(1)} < a_i < \lambda_k^{(k)} < \lambda_{k+1}^{(k+1)}, i = m + 1, \dots, k$ . Then  $\prod_{i=m+1}^k (a_{k+1} - a_i) \neq 0$ . Hence  $f_{m-1}(a_{k+1}) = 0$ . But  $\lambda_{k+1}^{(1)} < \lambda_{m-1}^{(1)} < \lambda_{m-1}^{(m-1)} < \lambda_{k+1}^{(k+1)}$  holds from (8) and (10), it means that  $f_{m-1}(a_{k+1}) \neq 0$ . So we obtain a contradiction. Finally, we get  $\lambda_{k+1}^{(1)} < a_{k+1} < \lambda_{k+1}^{(k+1)}$ . Hence the inequality (11) holds from (10) and (16).

(3) In conclusion, the properties (10) and (11) hold for any positive integer  $j$  when  $m + 2 \leq j \leq n$ .  $\square$

**Remark 3.2.** Based on Lemmas 2.6 and 3.1, it is not difficult to verify that: The characteristic polynomials sequence  $f_{m+1}(\lambda), f_m(\lambda), \dots, f_1(\lambda), f_0(\lambda)$  of an  $n$ -square symmetric TPPF matrix  $A$  of the form (1) is a Sturm sequence in any interval  $(a, b)$ . But the other sequence  $\{f_i(\lambda)\}_{i=m+2}^n$  may not be a Sturm sequence. While eigenvalues of  $A_i$  ( $1 \leq i \leq m + 1$ ) are simple and distinct, the properties of eigenvalues of  $A_i$  ( $m + 2 \leq i \leq n$ ) may not hold.

From Lemma 3.1 we get the following result.

**Corollary 3.3.** Let  $A$  be an  $n$ -square symmetric TPPF matrix  $A$  of the form (1). Let  $A_j$  be the  $j$  by  $j$  leading principal submatrix of  $A$  with  $f_j(\lambda), \lambda_j^{(1)}$  and  $\lambda_j^{(j)}$ , respectively, as its characteristic polynomial, minimal and maximal eigenvalues for any  $j = 1, 2, \dots, n$ . Denote by  $a_i$  ( $m + 1 \leq i \leq n - 1$ ) the diagonal entry of  $A$ . Then

1. For  $\xi < \lambda_j^{(1)}$ , we have  $(-1)^j f_j(\xi) > 0$ ;
2. For  $\xi > \lambda_j^{(j)}$ , we have  $f_j(\xi) > 0$ ;  $j = 1, 2, \dots, n$ .
3.  $(-1)^{j-m-1} \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) > 0, \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) > 0$ ;  $j = m + 2, \dots, n$ .

Firstly, the solvability of Problem 1.1 is verified as follows. Furthermore, the proof has the form of an algorithmic construction.

**Theorem 3.4.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. Then the list  $\sigma$  can be interlacily realized by an  $n \times n$  symmetric TPPF matrix  $A$  of the form (1) iff the condition (6) holds.

*Proof.* Sufficiency: Under the circumstance of the condition (6) is satisfied, we can construct an  $n$  by  $n$  symmetric matrix  $A$  of the form (1) step by step, such that it has  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  as the minimal and the maximal eigenvalues of its  $j$  by  $j$  leading principal submatrix  $A_j, j = 1, 2, \dots, n$ , respectively.

(1) When  $1 \leq j \leq m + 1$ , we can construct a symmetric tridiagonal matrix  $A_{m+1}$  from [3, Theorem 1].

(2) If  $m + 2 \leq j \leq n$ , we show that the existence of the matrix  $A$  of the form (1) is equivalent to the solvability of the following system of equations

$$\begin{cases} f_j(\lambda_j^{(1)}) = (\lambda_j^{(1)} - a_j) f_{j-1}(\lambda_j^{(1)}) - b_{j-1}^2 \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)}) = 0, \\ f_j(\lambda_j^{(j)}) = (\lambda_j^{(j)} - a_j) f_{j-1}(\lambda_j^{(j)}) - b_{j-1}^2 \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) = 0 \end{cases} \tag{20}$$

according to Lemma 2.4.

Transform the above system of equations into the following equivalent form of matrix product

$$\begin{bmatrix} f_{j-1}(\lambda_j^{(1)}) & \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)}) \\ f_{j-1}(\lambda_j^{(j)}) & \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) \end{bmatrix} \begin{bmatrix} a_j \\ b_{j-1}^2 \end{bmatrix} = \begin{bmatrix} \lambda_j^{(1)} f_{j-1}(\lambda_j^{(1)}) \\ \lambda_j^{(j)} f_{j-1}(\lambda_j^{(j)}) \end{bmatrix}. \tag{21}$$

Provided the determinant

$$D_{2j} = f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) - f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)})$$

of the coefficient matrix in the systems of equations (20) and (21) is nonzero, the systems of the equations will have solutions. From Lemma 3.1 and Corollary 3.3 we obtain that

$$\begin{aligned} \widetilde{D}_{2j} &= (-1)^{j-1} D_{2j} = (-1)^{j-1} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) \\ &\quad + f_{j-1}(\lambda_j^{(j)}) (-1)^{j-m-1} \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) (-1)^{m-1} f_{m-1}(\lambda_j^{(1)}) \end{aligned}$$

is larger than zero. As the value of  $a_{m+1}$  has been determined, the unique solutions of  $a_j$  and  $b_{j-1}^2$  are

$$\begin{cases} a_j &= \frac{\lambda_j^{(1)} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) - \lambda_j^{(j)} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)})}{D_{2j}} \\ b_{j-1}^2 &= \frac{(-1)^{j-1} (\lambda_j^{(j)} - \lambda_j^{(1)}) f_{j-1}(\lambda_j^{(1)}) f_{j-1}(\lambda_j^{(j)})}{\widetilde{D_{2j}}} \end{cases} \quad (22)$$

,  $j = m+2, \dots, n$ . Again, from condition (6) and Corollary 3.3 we explicitly find the denominator  $(-1)^{j-1} (\lambda_j^{(j)} - \lambda_j^{(1)}) f_{j-1}(\lambda_j^{(1)}) f_{j-1}(\lambda_j^{(j)})$  of  $b_{j-1}^2$  is larger than zero. Therefore  $b_{j-1}^2 > 0$ ,  $j = m+2, \dots, n$ , except for the choice of signs for  $b_{j-1}$ .

Therefore, from (22) we know that there exists an  $n$  by  $n$  matrix  $A$  of the form (1) such that the given list  $\sigma$  can be interlacily realized.

Necessity: Suppose there exists an  $n$  by  $n$  interlacing realizable matrix  $A$  of the form (1), then from Lemma 3.1 we can obtain the condition (6). Thus the proof is completed.  $\square$

From the above results, we can construct an  $n$ -square nonnegative symmetric TPPF matrix  $A$  of the form (1), realizing the minimal and the maximal eigenvalues of its  $j$ -square leading principal submatrix  $A_j$  for any  $1 \leq j \leq n$ . The solvable necessary and sufficient conditions of Problem 1.2 are given as follows.

**Theorem 3.5.** *Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. The necessary and sufficient conditions concerning the existence of an  $n$ -square nonnegative symmetric TPPF matrix  $A$  of the form (1), such that  $\sigma$  is interlacily realized by it, are*

$$\lambda_1^{(1)} \geq 0; \quad (23)$$

$$\frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} \geq \frac{f_{j-1}(\lambda_j^{(j)}) f_{j-2}(\lambda_j^{(1)})}{f_{j-1}(\lambda_j^{(1)}) f_{j-2}(\lambda_j^{(j)}), \quad j = 2, \dots, m+1; \quad (24)$$

$$\frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} \geq \frac{f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)})}{f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}), \quad j = m+2, \dots, n; \quad (25)$$

in addition condition (6).

*Proof.* Sufficiency: Suppose the conditions (6) and (23-25) hold. Firstly, we know that there exists a matrix  $A$  of the form (1) from the sufficiency of Theorem 3.4. Hence we can derive the recurrence formulae (4) in Lemma 2.4. Now it remains us to prove the nonnegativity of  $A$ . Since  $b_{j-1}^2 > 0$  for all  $2 \leq j \leq n$ , it is equivalent to prove  $a_j \geq 0$  ( $1 \leq j \leq n$ ) and take positive sign for any  $b_{j-1}$ .

Because of  $\lambda_1^{(1)} \geq 0$ , we without difficulty obtain  $a_1 = \lambda_1^{(1)} \geq 0$ . Again, from condition (6) we obtain  $\lambda_j^{(j)} > \lambda_1^{(1)} \geq 0$  for all  $2 \leq j \leq n$ . Now we present two cases to discuss the nonnegativity of  $a_j$  when  $2 \leq j \leq n$ .

(1) When  $2 \leq j \leq m+1$ , we can obtain a nonnegative symmetric tridiagonal matrix  $A_{m+1}$  from [3, Corollary 1].

(2) When  $m+2 \leq j \leq n$ , we have

$$\frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} \geq \frac{(-1)^{j-1} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)})}{(-1)^{j-1} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)})} \quad (26)$$

through multiplying the denominator and numerator of the right hand fraction in inequality (25) by  $(-1)^{j-1}$  respectively. Again, by Corollary 3.3

$$(-1)^{j-1} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) > 0.$$

As a result, from inequality (26) we equivalently have

$$N_{2j} = (-1)^{j-1} \lambda_j^{(1)} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) + (-1)^{j-2} \lambda_j^{(j)} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)}) \geq 0.$$

Then from (22) we finally get

$$a_j = \frac{N_{2j}}{D_{2j}} \geq 0. \tag{27}$$

Therefore,  $a_j \geq 0$  for all  $1 \leq j \leq n$  and  $b_{j-1} > 0$  for all  $2 \leq j \leq n$ . That is to say the matrix  $A$  of the form (1) is nonnegative.

Necessity: Suppose there exists an  $n$ -square nonnegative interlacing realizable matrix  $A$  of the form (1), then from Lemma 3.1 we can obtain the condition (6). Because of  $a_j \geq 0$  for all  $1 \leq j \leq n$ , we can proceed as follows.

(1) When  $1 \leq j \leq m + 1$ , from [3, Corollary 1] the conditions (23-24) hold.

(2) When  $m + 2 \leq j \leq n$ , we have  $a_j \geq 0$ , i.e.  $a_j = \frac{N_{2j}}{D_{2j}} \geq 0$ , where

$$N_{2j} = (-1)^{j-1} \lambda_j^{(1)} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) + (-1)^{j-2} \lambda_j^{(j)} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)}).$$

By Corollary 3.3, we get  $\widetilde{D}_{2j} > 0$ . It means that  $N_{2j} \geq 0$ . So we have

$$(-1)^{j-1} \lambda_j^{(1)} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) \geq (-1)^{j-1} \lambda_j^{(j)} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)}). \tag{28}$$

From Corollary 3.3, we know that

$$(-1)^{j-1} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)}) > 0.$$

As a result, the inequality (28) is equivalent to

$$\frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} \geq \frac{(-1)^{j-1} f_{j-1}(\lambda_j^{(j)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(1)} - a_i) \right) f_{m-1}(\lambda_j^{(1)})}{(-1)^{j-1} f_{j-1}(\lambda_j^{(1)}) \left( \prod_{i=m+1}^{j-1} (\lambda_j^{(j)} - a_i) \right) f_{m-1}(\lambda_j^{(j)})}.$$

Hence, the condition (25) holds.

Thus the proof is completed.  $\square$

#### 4. Solvability Conditions of Problems 1.3 and 1.4

In this section, we firstly discuss the following properties that will be helpful to resolve Problems 1.3 and 1.4.

**Lemma 4.1.** *Let  $A$  be an  $n$ -square symmetric TPPF matrix of the form (2), such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the minimal and the maximal eigenvalues of its  $j$ -square leading principal submatrix  $A_j$ ,  $j = 1, 2, \dots, n$ . Then we have*

$$\lambda_n^{(1)} < \dots < \lambda_j^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_j^{(j)} < \dots < \lambda_n^{(n)} \tag{29}$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = 1, 2, \dots, j; \quad j = 2, \dots, n. \tag{30}$$



*Proof.* Firstly, we know that the following properties

$$\lambda_j^{(1)} < \lambda_{j-1}^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}, \quad j = 2, \dots, p, \tag{31}$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = 1, 2, \dots, j; \quad j = 2, \dots, p. \tag{32}$$

hold from [4, Lemma 4].

Then it remains us to prove if the following inequalities

$$\lambda_j^{(1)} < \dots < \lambda_{p+1}^{(1)} < \lambda_p^{(1)} < \lambda_p^{(p)} < \lambda_{p+1}^{(p+1)} < \dots < \lambda_j^{(j)}, \quad j = p + 1, \dots, n, \tag{33}$$

and

$$\lambda_j^{(1)} < a_i < \lambda_j^{(j)}, \quad i = p + 1, \dots, j; \quad j = p + 1, \dots, n \tag{34}$$

hold, then the proof is completed.

From Theorem 2.1, Corollary 2.2 and the inequalities (31-32), we know that

$$\lambda_j^{(1)} \leq \dots \leq \lambda_{p+1}^{(1)} \leq \lambda_p^{(1)} < \lambda_p^{(p)} \leq \lambda_{p+1}^{(p+1)} \leq \dots \leq \lambda_j^{(j)}, \quad j = p + 1, \dots, n, \tag{35}$$

and

$$\lambda_j^{(1)} \leq a_i \leq \lambda_j^{(j)}, \quad i = p + 1, \dots, j; \quad j = p + 1, \dots, n. \tag{36}$$

Now we prove that the above inequalities (35) and (36) are strict by inductive hypothesis and contradiction.

If  $j = p + 1$ , by Lemma 2.5 we obtain

$$g_{p+1}(\lambda) = (\lambda - a_{p+1})g_p(\lambda) - b_p^2 g_{p-1}(\lambda). \tag{37}$$

Assume  $\lambda_{p+1}^{(1)} = \lambda_p^{(1)}$ , by (37) we have  $g_{p+1}(\lambda_{p+1}^{(1)}) = -b_p^2 g_{p-1}(\lambda_p^{(1)})$ . From inequality (31) we obtain  $\lambda_p^{(1)} < \lambda_{p-1}^{(1)}$ , that is to say  $\lambda_p^{(1)}$  is not a root of  $g_{p-1}(\lambda)$ , i.e.  $g_{p-1}(\lambda_p^{(1)}) \neq 0$ . But this is a contradiction, then we have  $\lambda_{p+1}^{(1)} < \lambda_p^{(1)}$ . Similarly, we have  $\lambda_p^{(p)} < \lambda_{p+1}^{(p+1)}$ . Therefore, when  $j = p + 1$ , the inequality (35) is strict.

Assume  $\lambda_{p+1}^{(1)} = a_{p+1}$  again, by (37) we have  $g_{p+1}(\lambda_{p+1}^{(1)}) = -b_p^2 g_{p-1}(\lambda_{p+1}^{(1)})$ . From the above results we obtain  $\lambda_{p+1}^{(1)} < \lambda_p^{(1)} < \lambda_{p-1}^{(1)}$ , that is to say  $\lambda_{p+1}^{(1)}$  is not a root of  $g_{p-1}(\lambda)$ , i.e.  $g_{p-1}(\lambda_{p+1}^{(1)}) \neq 0$ . But this is a contradiction, then we have  $\lambda_{p+1}^{(1)} < a_{p+1}$ . Similarly, we have  $a_{p+1} < \lambda_{p+1}^{(p+1)}$ . Therefore, when  $j = p + 1$ , the inequality (36) is strict.

In addition, suppose the inequalities (33) and (34) hold when  $j = k$ , then we have

$$\lambda_k^{(1)} < \dots < \lambda_{p+1}^{(1)} < \lambda_p^{(1)} < \lambda_p^{(p)} < \lambda_{p+1}^{(p+1)} < \dots < \lambda_k^{(k)} \tag{38}$$

and

$$\lambda_k^{(1)} < a_i < \lambda_k^{(k)}, \quad i = p + 1, \dots, k. \tag{39}$$

Now if  $j = k + 1$ , we have

$$\lambda_{k+1}^{(1)} \leq \lambda_k^{(1)} < \dots < \lambda_{p+1}^{(1)} < \lambda_p^{(1)} < \lambda_p^{(p)} < \lambda_{p+1}^{(p+1)} < \dots < \lambda_k^{(k)} \leq \lambda_{k+1}^{(k+1)} \tag{40}$$

and

$$\lambda_{k+1}^{(1)} \leq a_{k+1} \leq \lambda_{k+1}^{(k+1)} \tag{41}$$

from (35-36) and (38-39).

By Lemma 2.5 we obtain

$$g_{k+1}(\lambda) = (\lambda - a_{k+1})g_k(\lambda) - b_k^2 g_{k-1}(\lambda). \tag{42}$$

Assume  $\lambda_{k+1}^{(1)} = \lambda_k^{(1)}$ , by (42) we have  $g_{k+1}(\lambda_{k+1}^{(1)}) = -b_k^2 g_{k-1}(\lambda_k^{(1)})$ . Furthermore, we know that  $\lambda_k^{(1)} < \lambda_{k-1}^{(1)}$  from (38). That is to say  $g_{k+1}(\lambda_{k+1}^{(1)}) \neq 0$ . But this is a contradiction. Hence we have  $\lambda_{k+1}^{(1)} < \lambda_k^{(1)}$ . Similarly,  $\lambda_k^{(k)} < \lambda_{k+1}^{(k+1)}$ . Therefore, the inequalities (35) and (40) are strict when  $j = k + 1$ .

Assume  $\lambda_{k+1}^{(1)} = a_{k+1}$  again, then we have  $g_{k+1}(\lambda_{k+1}^{(1)}) = -b_k^2 g_{k-1}(\lambda_{k+1}^{(1)})$  by (42). From the above results, we know  $\lambda_{k+1}^{(1)} < \lambda_k^{(1)} < \lambda_{k-1}^{(1)}$ . Then  $g_{k-1}(\lambda_{k+1}^{(1)}) \neq 0$ . Furthermore,  $g_{k+1}(\lambda_{k+1}^{(1)}) \neq 0$ . But this is a contradiction. Then we have  $\lambda_{k+1}^{(1)} < a_{k+1}$ . Similarly, we can get  $a_{k+1} < \lambda_{k+1}^{(k+1)}$ . Therefore, the inequalities (36) and (41) are strict when  $j = k + 1$ .

In conclusion, the inequalities (33) and (34) hold for any positive integer  $j$  when  $p + 1 \leq j \leq n$ . Therefore, the inequalities (29) and (30) hold.  $\square$

**Remark 4.2.** Set  $g_0(\lambda) = 1$ . Based on Lemmas 2.6 and 4.1, it is not difficult to verify that: The characteristic polynomials sequence  $g_n(\lambda), g_{n-1}(\lambda), \dots, g_1(\lambda), g_0(\lambda)$ , of an  $n$ -square symmetric TPPF matrix  $A$  of the form (2) may not be a Sturm sequence in any interval  $(a, b)$ .

From Lemma 4.1 we get the following results easily.

**Corollary 4.3.** Let  $A$  be an  $n$ -square symmetric TPPF matrix of the form (2), such that  $\lambda_j^{(1)}$  and  $\lambda_j^{(j)}$  are, respectively, the minimal and the maximal eigenvalues of its  $j$  by  $j$  leading principal submatrix  $A_j, j = 1, 2, \dots, n$ . Let  $g_j(\lambda)$  be the characteristic polynomial corresponding to  $A_j$ . Denote by  $a_i (2 \leq i \leq p - 1)$  the diagonal entry of  $A$ . Then

1. For  $\xi < \lambda_j^{(1)}$ , we have  $(-1)^j g_j(\xi) > 0$ ;
2. For  $\xi > \lambda_j^{(j)}$ , we have  $g_j(\xi) > 0; j = 1, 2, \dots, n$ .
3.  $(-1)^{j-2} \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i) > 0, \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) > 0; j = 3, \dots, p$ .

Then from the above results, we derive the solvable conditions of Problem 1.3 as follows. Because its proof is similar to the proof of Theorem 3.4, we omit it.

**Theorem 4.4.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. Then the list  $\sigma$  can be interlacially realized by an  $n \times n$  symmetric TPPF matrix  $A$  of the form (2) iff the condition (29) holds.

Similarly to the sufficient proof of Theorem 3.4, we obtain the entries of an  $n \times n$  symmetric TPPF matrix  $A$  of the form (2) as follows.

$$a_1 = \lambda_1^{(1)};$$

$$\begin{cases} a_2 = \frac{\lambda_2^{(1)} g_1(\lambda_2^{(1)}) - \lambda_2^{(2)} g_1(\lambda_2^{(2)})}{g_1(\lambda_2^{(1)}) - g_1(\lambda_2^{(2)})}, \\ b_1^2 = \frac{(\lambda_2^{(2)} - \lambda_2^{(1)}) g_1(\lambda_2^{(1)}) g_1(\lambda_2^{(2)})}{g_1(\lambda_2^{(1)}) - g_1(\lambda_2^{(2)})}; \end{cases} \tag{43}$$

$$\begin{cases} a_j = \frac{\lambda_j^{(1)} g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - \lambda_j^{(j)} g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}, \\ b_{j-1}^2 = \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) g_{j-1}(\lambda_j^{(1)}) g_{j-1}(\lambda_j^{(j)})}{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}; \end{cases} \tag{44}$$

for any  $j = 3, \dots, p$ , and

$$\begin{cases} a_j &= \frac{\lambda_j^{(1)} g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)}) - g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}, \\ b_{j-1}^2 &= \frac{(\lambda_j^{(j)} - \lambda_j^{(1)}) g_{j-1}(\lambda_j^{(1)}) g_{j-1}(\lambda_j^{(j)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)}) - g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}; \end{cases} \tag{45}$$

for any  $j = p + 1, \dots, n$ .

Then we can obtain the solvability conditions for Problem 1.4 as follows.

**Theorem 4.5.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. Then there exists an  $n$  by  $n$  nonnegative symmetric TPPF matrix  $A$  of the form (2), such that  $\sigma$  is interlacially realized by it, if and only if

$$\begin{aligned} \lambda_1^{(1)} &\geq 0; \quad \frac{\lambda_2^{(1)}}{\lambda_2^{(2)}} \geq \frac{g_1(\lambda_2^{(2)})}{g_1(\lambda_2^{(1)})}; \\ \frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} &\geq \frac{g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i)}, \quad j = 3, \dots, p; \\ \frac{\lambda_j^{(1)}}{\lambda_j^{(j)}} &\geq \frac{g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)})}, \quad j = p + 1, \dots, n; \end{aligned}$$

and condition (29) holds.

*Proof.* Similarly to the proof of Theorem 3.5, we can do it by the same method. So we omit the proof.  $\square$

Finally, if the diagonal entries in a matrix of the form (2) requires  $a_1 \leq a_2 \leq \dots \leq a_n$ , then we obtain the following.

**Theorem 4.6.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers and let it be interlacially realized by a matrix  $A$  of the form (2). Then the diagonal entries  $a_i$  ( $1 \leq i \leq n$ ) of matrix  $A$  place in ascending order if

$$\begin{aligned} \lambda_j^{(1)} + \lambda_j^{(j)} &\geq 2\lambda_1^{(1)}, \quad j = 2, 3, \\ \frac{\lambda_j^{(1)} - a_{j-2}}{\lambda_j^{(j)} - a_{j-2}} &\leq \frac{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-3} (\lambda_j^{(j)} - a_i)}{g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-3} (\lambda_j^{(1)} - a_i)}, \quad j = 4, \dots, p, \end{aligned}$$

and

$$\frac{\lambda_j^{(1)} - a_{j-1}}{\lambda_j^{(j)} - a_{j-1}} \geq \frac{g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)})}, \quad j = p + 1, \dots, n.$$

*Proof.* (1) When  $a_1 \leq a_2$ , from (43) we have

$$a_1 = \lambda_1^{(1)} \leq \frac{\lambda_2^{(1)} g_1(\lambda_2^{(1)}) - \lambda_2^{(2)} g_1(\lambda_2^{(2)})}{g_1(\lambda_2^{(1)}) - g_1(\lambda_2^{(2)})}.$$

Then we obtain that  $\lambda_2^{(1)} + \lambda_2^{(2)} \geq 2\lambda_1^{(1)}$ .

When  $a_2 \leq a_3$ , from (44) we have

$$a_2 \leq \frac{\lambda_3^{(1)} g_2(\lambda_3^{(1)})(\lambda_3^{(3)} - a_2) - \lambda_3^{(3)} g_2(\lambda_3^{(3)})(\lambda_3^{(1)} - a_2)}{g_2(\lambda_3^{(1)})(\lambda_3^{(3)} - a_2) - g_2(\lambda_3^{(3)})(\lambda_3^{(1)} - a_2)}.$$

Then we obtain that  $\lambda_3^{(1)} + \lambda_3^{(3)} \geq a_1 + a_2 = \lambda_2^{(1)} + \lambda_2^{(2)} \geq 2\lambda_1^{(1)}$ .

(2) When  $a_{j-1} \leq a_j$  for any  $j = 4, \dots, p$ , from (44) we have

$$a_{j-1} \leq \frac{\lambda_j^{(1)} g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - \lambda_j^{(j)} g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-1} (\lambda_j^{(j)} - a_i) - g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-1} (\lambda_j^{(1)} - a_i)}.$$

Then by Corollary 4.3, we obtain

$$\frac{\lambda_j^{(1)} - a_{j-2}}{\lambda_j^{(j)} - a_{j-2}} \leq \frac{g_{j-1}(\lambda_j^{(1)}) \prod_{i=2}^{j-3} (\lambda_j^{(j)} - a_i)}{g_{j-1}(\lambda_j^{(j)}) \prod_{i=2}^{j-3} (\lambda_j^{(1)} - a_i)}.$$

(3) When  $a_{j-1} \leq a_j$  for any  $j = p + 1, \dots, n$ , from (45) we have

$$a_{j-1} \leq \frac{\lambda_j^{(1)} g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)}) - g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}.$$

Then we can obtain

$$\frac{\lambda_j^{(1)} - a_{j-1}}{\lambda_j^{(j)} - a_{j-1}} \geq \frac{g_{j-1}(\lambda_j^{(j)}) g_{j-2}(\lambda_j^{(1)})}{g_{j-1}(\lambda_j^{(1)}) g_{j-2}(\lambda_j^{(j)})}$$

by Corollary 4.3.  $\square$

### 5. Example Analysis

The processes of proving Theorems 3.4-3.5 and 4.4-4.5 respectively show the algorithmic constructive procedures of the solvability of Problems 1.1-1.4. In this section, we will give some numerical examples to further analyze the efficiency of our solvable conditions for Problems 1.1-1.4. Firstly, we conclude the following results.

**Corollary 5.1.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. Then it can be interlacily realized by two kinds of symmetric TPPF matrices of the forms (1) and (2) simultaneously if and only if

$$\lambda_n^{(1)} < \dots < \lambda_j^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_j^{(j)} < \dots < \lambda_n^{(n)}.$$

*Proof.* From Theorems 3.4 and 4.4, the proof is obvious.  $\square$

**Remark 5.2.** Let a list  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  of real numbers be an interlacing realizable spectrum. In a matrix of the form (1) (or (2)), if we take the value of  $m$  (or  $p$ ) to be  $1, 2, \dots, n$ , orderly, then we can derive a wide class of realizable matrices of the two forms. Furthermore, by Perron-Frobenius theorem, the maximal positive eigenvalue of a nonnegative matrix is at least as large as the absolute value of any other eigenvalue, so we obtain  $\lambda_j^{(1)} + \lambda_j^{(j)} \geq 0, j = 1, 2, \dots, n$ , when  $\sigma$  can be interlacily realized by a nonnegative matrix of the form (1) or (2).



of the form (1) to interlacily realize  $\sigma$ .

If  $m = 5$  or  $p = 1$ , the nonnegative interlacing realizable matrix can be seen in Ref.[3].

When  $p = 6$ , the nonnegative interlacing realizable matrix of the form (2) is

$$\begin{bmatrix} 2.0000 & 3.4641 & 2.8284 & 3.7081 & 5.8854 & 7.7938 \\ 3.4641 & 3.0000 & & & & \\ 2.8284 & & 3.0000 & & & \\ 3.7081 & & & 5.2500 & & \\ 5.8854 & & & & 5.9664 & \\ 7.7938 & & & & & 5.0238 \end{bmatrix}.$$

**Remark 5.5.** Let  $A$  be a real matrix of the form (1) and let  $b_i^2 = e_i f_i, i = 1, 2, \dots, n - 1$ , and  $e_i, f_i \in \mathbb{R}$ . Then there exists a quasi symmetric TPPF matrix  $B = PAP^{-1}$ , where  $P = \text{diag}(\delta_1, \dots, \delta_m, \dots, \delta_n)$ ,

$$\delta_i = \begin{cases} 1, & i = 1; \\ \sqrt{\frac{\prod_{j=1}^{i-1} e_j}{\prod_{j=1}^{i-1} f_j}}, & i = 2, \dots, m; \\ \delta_m \sqrt{\frac{e_{i-1}}{f_{i-1}}}, & i = m + 1, \dots, n. \end{cases}$$

Similarly, let  $A$  be a real matrix of the form (2) and let  $b_i^2 = e_i f_i, i = 1, 2, \dots, n - 1$ , and  $e_i, f_i \in \mathbb{R}$ . Then there also exists a quasi symmetric TPPF matrix  $B = PAP^{-1}$ , where  $P = \text{diag}(\delta_1, \dots, \delta_p, \dots, \delta_n)$ ,

$$\delta_i = \begin{cases} 1, & i = 1; \\ \sqrt{\frac{e_{i-1}}{f_{i-1}}}, & i = 2, \dots, p; \\ \delta_p \sqrt{\frac{\prod_{j=p}^{i-1} e_j}{\prod_{j=p}^{i-1} f_j}}, & i = p + 1, \dots, n. \end{cases}$$

These imply that we can spread the interlacing realizable matrices in our paper to a wider range of scenarios by a simple diagonal similarity, without changing their spectra.

Finally, we present some interesting perturbations on an interlacing realizable list  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(j)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$ .

**Theorem 5.6.** Let  $\sigma = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  be a list of real numbers. If it can be interlacily realized by two kinds of symmetric TPPF matrices of the forms (1) and (2) simultaneously, then

- (1)  $\sigma_1 = (\lambda_n^{(1)} - t, \dots, \lambda_j^{(1)} - t, \dots, \lambda_2^{(1)} - t, \lambda_1^{(1)}, \lambda_2^{(2)} + t, \dots, \lambda_j^{(j)} + t, \dots, \lambda_n^{(n)} + t)$  is also interlacing realizable by them for any nonnegative number  $t$ .
- (2)  $\sigma_2 = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_2^{(1)}, \lambda_1^{(1)}, \lambda_2^{(2)} + t, \dots, \lambda_j^{(j)} + t, \dots, \lambda_n^{(n)} + t)$  is also interlacing realizable by them for any nonnegative number  $t$ .
- (3)  $\sigma_3 = (\lambda_n^{(1)} - t, \dots, \lambda_j^{(1)} - t, \dots, \lambda_2^{(1)} - t, \lambda_1^{(1)}, \lambda_2^{(2)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  is also interlacing realizable by them for any nonnegative number  $t$ .
- (4)  $\sigma_4 = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_2^{(1)}, \lambda_1^{(1)} - t, \lambda_2^{(2)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  is also interlacing realizable by them for any  $0 \leq t < \lambda_1^{(1)} - \lambda_2^{(1)}$ .
- (5)  $\sigma_5 = (\lambda_n^{(1)}, \dots, \lambda_j^{(1)}, \dots, \lambda_2^{(1)}, \lambda_1^{(1)} + t, \lambda_2^{(2)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)})$  is also interlacing realizable by them for any  $0 \leq t < \lambda_2^{(2)} - \lambda_1^{(1)}$ .

- (6)  $\sigma_6 = (\lambda_n^{(1)} - l, \dots, \lambda_j^{(1)} - l, \dots, \lambda_2^{(1)} - l, \lambda_1^{(1)}, \lambda_2^{(2)} + m, \dots, \lambda_j^{(j)} + m, \dots, \lambda_n^{(n)} + m)$  is also interlacing realizable by them for any nonnegative numbers  $l$  and  $m$ .
- (7)  $\sigma_7 = (\lambda_n^{(1)} - l_n, \dots, \lambda_j^{(1)} - l_j, \dots, \lambda_2^{(1)} - l_2, \lambda_1^{(1)} + t, \lambda_2^{(2)} + m_2, \dots, \lambda_j^{(j)} + m_j, \dots, \lambda_n^{(n)} + m_n)$  is also interlacing realizable by them for any  $0 \leq t < \lambda_2^{(2)} - \lambda_1^{(1)}$  when  $0 \leq l_2 \leq l_3 \leq \dots \leq l_n$  and  $0 \leq m_2 \leq m_3 \leq \dots \leq m_n$ .
- (8)  $\sigma_8 = (\lambda_n^{(1)} - l_n, \dots, \lambda_j^{(1)} - l_j, \dots, \lambda_2^{(1)} - l_2, \lambda_1^{(1)} - t, \lambda_2^{(2)} + m_2, \dots, \lambda_j^{(j)} + m_j, \dots, \lambda_n^{(n)} + m_n)$  is also interlacing realizable by them for any  $0 \leq t < \lambda_1^{(1)} - \lambda_2^{(1)}$  when  $0 \leq l_2 \leq l_3 \leq \dots \leq l_n$  and  $0 \leq m_2 \leq m_3 \leq \dots \leq m_n$ .

*Proof.* Because from the item (7) we apparently know the items (1), (2), (3), (5) and (6), we only proof the item (7).

Because the list  $\sigma$  can be interlacially realized by two kinds of symmetric TPPF matrices of the forms (1) and (2) simultaneously, from Corollary 5.1 we have  $\lambda_n^{(1)} < \dots < \lambda_j^{(1)} < \dots < \lambda_1^{(1)} < \dots < \lambda_j^{(j)} < \dots < \lambda_n^{(n)}$ .

Then when  $0 \leq l_2 \leq l_3 \leq \dots \leq l_n$  and  $0 \leq m_2 \leq m_3 \leq \dots \leq m_n$  we respectively have  $\lambda_n^{(1)} - l_n < \dots < \lambda_j^{(1)} - l_j < \dots < \lambda_2^{(1)} - l_2$  and  $\lambda_2^{(2)} + m_2 < \dots < \lambda_j^{(j)} + m_j < \dots < \lambda_n^{(n)} + m_n$ .

Therefore the elements in the list  $\sigma_7$  satisfy  $\lambda_n^{(1)} - l_n < \dots < \lambda_j^{(1)} - l_j < \dots < \lambda_2^{(1)} - l_2 < \lambda_1^{(1)} + t < \lambda_2^{(2)} + m_2 < \dots < \lambda_j^{(j)} + m_j < \dots < \lambda_n^{(n)} + m_n$  for any  $0 \leq t < \lambda_2^{(2)} - \lambda_1^{(1)}$ . Thus the list  $\sigma_7$  is also interlacing realizable by two kinds of symmetric TPPF matrices of the form (1) and (2) simultaneously from Corollary 5.1.

Finally, the proof of the item (8) is similar to that of the item (7), so the list  $\sigma_8$  is interlacing realizable. From the item (8), we apparently know that the list  $\sigma_4$  in the item (4) is also interlacing realizable.  $\square$

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