ON INVERSE PROBLEMS ASSOCIATED WITH SECOND-ORDER DIFFERENTIAL OPERATORS

BY

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In a by now classical theorem G. Borg [1] proved the following:

THEOREM A. *Consider the two Sturm–Liouville problems*

$$y'' + [\lambda - q(x)]\, y = 0 \tag{1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{2}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0, \tag{3}$$

where $q(x)$ *is real and integrable on* $(0, \pi]$ *and* $\sin(\gamma - \beta) \neq 0$. *Then the two spectra corresponding to the boundary conditions* (2) *and* (3) *uniquely determine* $q(x)$, *almost everywhere.*

More recently Li [3] proved the following theorem.

THEOREM B. *Consider the boundary value problem*

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{4}$$

$$y(0) = 0, \quad a\, y'(\pi) + \lambda\, y(\pi) = 0, \tag{5}$$

where $a \neq 0$ *is real and* $q(x)$ *is integrable on* $[0, \pi]$. *The spectrum of the problem* (4), (5) *uniquely determines* $q(x)$, *almost everywhere.*

At first glance it seems paradoxical that the determination of $q(x)$ depends on two spectra in Theorem A and only one spectrum in Theorem B. It is our purpose to

(1) The author gratefully acknowledges the support of the National Science Foundation under grant GP–7435.

discuss the relationship between these two theorems, to generalize Theorem B, and to investigate the expansion theorems associated with the operator in Theorem B.

Borg's proof of Theorem A is quite long and a much simpler proof has since been given by N. Levinson [2]. Li's proof of Theorem B is different from either of these and is related to techniques developed in the quantum theory of scattering. Furthermore, in a key step he refers to a result that has appeared only in the Chinese literature.

Although Borg's method is quite involved one can provide a rather simple heuristic argument based on it. For the sake of simplicity let $\alpha=\beta=0$, and $\gamma=\pi/2$ in (2) and (3). Then the asymptotic forms of the solutions of (1), (2) are $y_n \approx \sin nx$ and those of (1), (3) are $y_n \approx \sin (n+\frac{1}{2})\, x$. Suppose that

$$u'' + [\lambda - p(x)]u = 0 \tag{a}$$

has the same spectra as (1) corresponding to the boundary conditions (2) and (3) respectively. Then, using (1) and (a) one finds

$$\int_0^\pi (p-q)\, y_n u_n\, dx = 0$$

for all eigenfunctions. Using their asymptotic form one finds that

$$\int_0^\pi (p-q) \sin^2 nx\, dx = 0, \quad n=1, 2, \ldots$$

and

$$\int_0^\pi (p-q) \sin^2 (n+\tfrac{1}{2})\, x\, dx = 0, \quad n=1, 2, \ldots$$

from which if follows that

$$\int_0^\pi (p-q) \cos nx\, dx = 0, \quad n=0, 1, 2, \ldots$$

so that $p-q=0$, almost everywhere.

The eigenvalues of (4), (5) have the asymptotic form

$$\lambda_n = n - \frac{1}{\pi} \tan^{-1} a + O\left(\frac{1}{n}\right),$$

where $n=0, \pm 1, \pm 2, \ldots$. Note that the spectrum, in this case, stretches from $-\infty$ to $+\infty$, whereas the Sturm–Liouville operators are semibounded. The eigenfunctions have the asymptotic form $y_n = \sin (n-(1/\pi) \tan^{-1} a)$. Again using (7) we have

$$\int_0^\pi (p-q)\sin^2\left(n-\frac{1}{\pi}\tan^{-1}a\right)dx=0$$

from which one can show that

$$\int_0^\pi (p-q)\cos\left(2n-\frac{2}{\pi}\tan^{-1}a\right)dx=0,\ n=0,\ \pm 1,\ \pm 2,\ \ldots.$$

The above leads to

$$\int_0^\pi (p-q)\cos\left(2n\pm\frac{2}{\pi}\tan^{-1}a\right)dx=0,\ n=0,1,2,\ldots$$

from which we have again $p-q=0$ almost everywhere.

The proofs in the literature do not indicate how Theorems A and B are related and their relationship will be discussed in the sequel. A simpler proof of Theorem B will be given, using the method of [2], and also a number of generalizations will be proved. In a second part of this paper the expansion theorems associated with (4), (5) will be discussed fully.

Part I

Let $\lambda=\sigma+i\tau$, and define two solutions of (4) by the initial conditions

$$\left.\begin{aligned} y_1(0)=1, \quad y_1'(0)=0 \\ y_2(0)=0, \quad y_2'(0)=1. \end{aligned}\right\} \tag{6}$$

It is well known that $y_1(\pi,\lambda)$ and $y_2(\pi,\lambda)$ are entire functions of order 1, in terms of λ. As a matter of fact, they are entire functions of order $\frac{1}{2}$ in terms of λ^2. This follows from expansions of the type (55). Detailed proofs may be found in the treatise by Titchmarsh [4]. We denote the eigenvalues corresponding to the boundary conditions

$$y(0)=0, \quad y(\pi)=0$$

by $\{\zeta_n\}$, and those corresponding to

$$y(0)=0, \quad y'(\pi)=0$$

by $\{\zeta_n'\}$. Then

$$y_2(\pi,\lambda)=k_1\prod_{n=0}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n}\right) \tag{7}$$

$$y_2'(\pi,\lambda)=k_2\prod_{n=1}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n'}\right). \tag{8}$$

It is also well known that for large n

$$\zeta_n \approx n^2 \tag{9}$$

$$\zeta_n' \approx (n+\tfrac{1}{2})^2. \tag{10}$$

Now we let
$$\omega(\lambda) = ak_2 \prod_{n=1}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n'}\right) + \lambda k_1 \prod_{n=0}^{\infty}\left(1-\frac{\lambda^2}{\zeta_n}\right). \tag{11}$$

The zeros of $\omega(\lambda)$ represent the eigenvalues of problem (4), (5). $\omega(\lambda)$ is an entire function of λ of order 1. No general theorem will guarantee the existence of zeros. But if we recall that, according to standard oscillation theorems, the sets $\{\zeta_n\}$ and $\{\zeta_n'\}$ interlace we observe from (11) that for large n $\omega(\lambda)$ has precisely one zero between $\sqrt{\zeta_n'}$ and $\sqrt{\zeta_{n+1}'}$. A more precise analysis carried out in part II shows that all zeros of $\omega(\lambda)$ are real, simple and asymptotically

$$\lambda_n = n - \frac{1}{\pi}\tan^{-1} a + \frac{1}{2n}\int_0^{\pi} q\,dx + o\left(\frac{1}{n}\right), \tag{12}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Note that the zeros accumulate both at $+\infty$ as well as $-\infty$.

The following asymptotic estimates are well-known [2], [3], [4].

$$\left.\begin{aligned}
y_2 &= \frac{\sin \lambda x}{\lambda} + O\left(\frac{e^{|\tau|x}}{\lambda^2}\right)\\
y_2' &= \cos \lambda x + O\left(\frac{e^{|\tau|x}}{\lambda}\right)\\
y_1(x) &= \cos \lambda x + O\left(\frac{e^{|\tau|x}}{\lambda}\right)\\
y_1'(x) &= -\lambda \sin \lambda x + O(e^{|\tau|x}).
\end{aligned}\right\} \tag{13}$$

Consider a second problem of type (4), (5).

$$u'' + [\lambda^2 - p(x)]\,u = 0 \tag{14}$$

$$u(0) = 0, \quad au'(\pi) + \lambda u(\pi) = 0, \tag{15}$$

and we suppose that the eigenvalues of (14), (15) coincide with those of (4), (5). We now define a third solution of (4) using the initial conditions

$$y_3(\pi) = -a, \quad y_3'(\pi) = \lambda. \tag{16}$$

Similarly we define u_1, u_2, u_3 as in (6) and (16). Then

$$y_2 y_3' - y_2' y_3 = \omega(\lambda) \tag{17}$$

and also

$$u_2 u_3' - u_2' u_3 = \omega(\lambda). \tag{18}$$

That the Wronskians (17) and (18) coincide is a consequence of the fact that both have the same zeros and the same asymptotic form. Being functions of order 1 and having the same zeros implies that they differ by at most an exponential factor. Both are real for real λ and have the asymptotic form

$$\omega(\lambda) \approx a \cos \lambda\pi + \sin \lambda\pi + O\left(\frac{1}{\lambda}\right)$$

so that such an exponential factor has to reduce to unity. By evaluating the Wronskian at $x=0$ and $x=\pi$ we have

$$\omega(\lambda) = -y_3(0) = a y_2'(\pi) + \lambda y_2(\pi). \tag{19}$$

Similarly we have

$$\omega(\lambda) = -u_3(0) = a u_2'(\pi) + \lambda u_2(\pi). \tag{20}$$

When $\lambda = \lambda_n$, at some eigenvalue, y_2 and y_3 are linearly dependent so that

$$y_2 = C_n y_3$$

and also

$$u_2 = D_n u_3.$$

We shall show that necessarily $C_n = D_n$. At $x=\pi$ we have

$$C_n = \frac{y_2(\pi)}{y_3(\pi)} = \frac{y_2(\pi)}{-a}, \qquad D_n = \frac{u_2(\pi)}{u_3(\pi)} = \frac{u_2(\pi)}{-a}.$$

Recall that $y_2(\pi), u_2(\pi), y_2'(\pi), u_2'(\pi)$ are even functions of λ. Then by comparing the odd parts of (19) and (20) we find that

$$u_2(\pi) = y_2(\pi).$$

It follows from the above expressions for C_n and D_n that now $C_n = D_n$. It is in this step that we use the fact that (4), (5) and (14), (15) have the same spectrum.

Now we define the two functions

$$\Phi = \frac{y_3(x)\int_0^x y_2(t)\, f(t)\, dt + y_2(x)\int_x^\pi y_3(t)\, f(t)\, dt}{\omega(\lambda)} \tag{21}$$

and
$$\Psi = \frac{u_3(x)\int_0^x y_2(t)\,f(t)\,dt + u_2(x)\int_x^\pi y_3(t)\,f(t)\,dt}{\omega(\lambda)}, \tag{22}$$

where $f(t)$ is an arbitrary square integrable function on $[0, \pi]$.

Let $\{R_n\}$ denote a sequence of squares with vertices at

$$\left(\pm\left[n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}a\right],\ \pm i\left[n+\frac{1}{2}-\frac{1}{\pi}\tan^{-1}a\right]\right).$$

These are uniformly bounded away from the zeros of $\omega(\lambda)$ by virtue of (12) for large n. We shall show that

$$\lim_{n\to\infty}\int_{R_n}(\Phi-\Psi)\,d\lambda = 0. \tag{23}$$

A typical term in the integrand of (23) is

$$\frac{(y_3-u_3)\int_0^x y_2(t)\,f(t)\,dt}{\omega(\lambda)}.$$

Using estimates of the type (13) we see that the above has the asymptotic form

$$\frac{O\left(\frac{e^{|\tau|(\pi-x)}}{\lambda^2}\right)\int_0^x \sin\lambda t\,f(t)\,dt}{a\cos\lambda\pi+\sin\lambda\pi}.$$

Using the latter we see that

$$\int_{R_n}(\Phi-\Psi)\,d\lambda = O\left(\frac{1}{n}\right)$$

from which (23) follows. Now use residue integration and the fact that all zeros of $\omega(\lambda)$ are simple and also that at λ_n

$$y_3 = C_n y_2, \quad u_3 = C_n u_2.$$

Then from (23)
$$\sum_{n=-\infty}^{\infty}\frac{C_n[y_2(x,\lambda_n)-u_2(x,\lambda_n)]\int_0^\pi y_2(t,\lambda_n)\,f(t)\,dt}{\omega'(\lambda_n)} = 0. \tag{24}$$

In part II it is shown that the eigenfunctions $y_2(t,\lambda_n)$ are independent. Note that we do not require their completeness here. We can, therefore, select $f(t)$ so that

$$\int_0^{\pi} y_2(t, \lambda_n) f(t)\, dt = 0 \quad n \neq k$$
$$= 1 \quad n = k$$

and from (24) it follows that

$$y_2(x, \lambda_n) = u_2(x, \lambda_n).$$

The latter implies that y_2 and u_2 satisfy the same differential equation. This concludes our proof of Theorem B.

Theorem B can be generalized in the following direction.

THEOREM 1. *Consider*

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{25}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad y(\pi) \cos \beta + y'(\pi) \sin \beta + f(\lambda)\, [y(\pi) \cos \gamma + y'(\pi) \sin \gamma] = 0, \tag{26}$$

where $f(\lambda)$ *is an odd real entire function of* λ *of order less than* 1 *and* $\sin(\beta - \gamma) \neq 0$. *The eigenvalues of (25) and (26) uniquely determine* $q(x)$.

Let y satisfy the initial conditions

$$y(0) = -\sin \alpha, \quad y'(0) = \cos \alpha$$

and

$$\omega(\lambda) = S(\lambda) + f(\lambda)\, T(\lambda),$$

where

$$S(\lambda) = y(\pi) \cos \beta + y'(\pi) \sin \beta$$

$$T(\lambda) = y(\pi) \cos \gamma + y'(\pi) \sin \gamma.$$

The zeros and asymptotic form of $\omega(\lambda)$ uniquely determine $\omega(\lambda)$. The even part of $\omega(\lambda)$ is $S(\lambda)$ and its odd part is $f(\lambda)\, T(\lambda)$, since clearly $S(\lambda)$ and $T(\lambda)$ are even functions of λ. Then knowing $\omega(\lambda)$ we know the zeros of $S(\lambda)$ and $T(\lambda)$. But these determine the eigenvalues of (1), (2) and (1), (3). By Theorem A these uniquely determine $q(x)$.

Theorem 1 with $f(\lambda) = a\lambda$, $a \neq 0$ can be proved directly by the same technique as Theorem B in the preceding. Except for some details there is no difference. This leads us to the following theorem.

THEOREM 2. *Theorems A and* 1 *are fully equivalent.*

In the proof of Theorem 1 it was shown that Theorem A implies Theorem 1. To prove the converse we assume Theorem 1 to be true. Suppose $S_1(\lambda), T_1(\lambda), S_2(\lambda)$,

$T_{2},(\lambda)$ correspond to two different boundary value problems of type (1), (2) and (1), (3), where $S(\lambda)$ and $T(\lambda)$ are defined as in the preceding proof. Then we form

$$\omega_1(\lambda) = S_1(\lambda) + a\lambda T_1(\lambda)$$
$$\omega_2(\lambda) = S_2(\lambda) + a\lambda T_2(\lambda).$$

If $S_1(\lambda) = S_2(\lambda)$ and $T_1(\lambda) = T_2(\lambda)$, then $\omega_1(\lambda) = \omega_2(\lambda)$. By Theorem 1 the latter fact shows that both differential equations are the same, thereby establishing Theorem A.

Part II

We now turn our attention to the problem

$$y'' + [\lambda^2 + \mu - q(x)]\, y = 0 \tag{27}$$

$$y(0) = 0, \quad ay'(\pi) + \lambda y(\pi) = 0, \tag{28}$$

where $q(x)$ is real and integrable on $[0, \pi]$ and μ and $a \neq 0$ are real parameters. To study the problem (27), (28) we shall relate it to a different problem. We introduce the function $M(x)$ defined by

$$\left.\begin{aligned} &M'' + [\mu - q(x)]\, M = 0 \\ &M(\pi) = 1, \ M'(\pi) = 0, \end{aligned}\right\} \tag{29}$$

and now restrict μ so that $M > 0$ on $[0, \pi]$. The operator corresponding to the eigenvalue problem

$$M'' + [\mu - q(x)]\, M = 0$$

and the boundary conditions

$$M(0) = 0, \quad M'(\pi) = 0 \tag{30}$$

is lower semibounded. If we denote the smallest eigenvalue of the above problem by μ_0 then for all $\mu < \mu_0$, the solution M of (29) remain positive. This is an immediate consequence of Sturm's oscillation theorem.

We now transform (27), (28) by introducing new dependent and independent variables η, ξ by means of

$$y = M\eta, \quad \xi = \int_0^x \frac{dx}{M^2}. \tag{31}$$

This results in the new boudary value problem

$$\eta'' + \lambda^2 M^4 \eta = 0 \tag{32}$$

$$\eta(0) = 0, \quad a\eta'(\varrho) + \lambda\eta(\varrho) = 0, \tag{33}$$

where $\varrho = \int_0^\pi 1/M^2 dx$. To study the problem (32), (33) we introduce the new functions

$$\left.\begin{aligned} x_1 &= -\lambda\eta \\ x_2 &= \eta'. \end{aligned}\right\} \tag{34}$$

These satisfy the differential equaitions

$$\left.\begin{aligned} x_1' &= -\lambda x_2 \\ x_2' &= \lambda M^4 x_1 \end{aligned}\right\} \tag{35}$$

and the boundary conditions

$$x_1(0) = 0, \quad a x_2(\varrho) - x_1(\varrho) = 0. \tag{36}$$

(35) can be rewritten in the form

$$L_0 X = \lambda X, \tag{37}$$

where
$$L_0 = \begin{pmatrix} 0 & \dfrac{1}{M^4}\dfrac{d}{d\xi} \\ -\dfrac{d}{d\xi} & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The effect of all these substitutions is to linearize problem (27), (28) in terms of the parameter λ. L_0 can be inverted by means of an integral operator. Then we obtain

$$\left.\begin{aligned} x_1 &= -\lambda \int_0^\xi x_2 \, d\xi \\ x_2 &= -\frac{\lambda}{a}\int_0^\varrho x_2 \, d\xi - \lambda \int_\xi^\varrho M^4 x_1 \, d\xi. \end{aligned}\right\} \tag{38}$$

We shall denote the integral operator defined in (38) by $\mathcal{G}_0$ so that (38) can be rewritten as

$$X = \lambda \mathcal{G}_0 X.$$

We now introduce the Hilbert space H consisting of all vectors

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ for which } \int_0^\varrho [|x_1|^2 + |x_2|^2] \, d\xi < \infty.$$

As a suitable inner product we introduce

$$(X, Y) = \int_0^\varrho (M^4 x_1 \bar{y}_1 + x_2 \bar{y}_2) \, d\xi \tag{39}$$

and the norm
$$\|X\| = \sqrt{\int_0^{\varrho} [M^4 |x_1|^2 + |x_2|^2]\, d\xi}. \tag{40}$$

In view of the fact that M is a continuous and positive function on $[0, \pi]$, $\|X\|$ as defined in (40) is equivalent to

$$\sqrt{\int_0^{\varrho} [|x_1|^2 + |x_2|]^2\, d\xi}.$$

A simple exercise shows that
$$(L_0 X, Y) = (X, L_0 Y) \tag{41}$$

if X and Y are absolutely continuous and it follows also that

$$(\mathcal{G}_0 X, Y) = (X, \mathcal{G}_0 Y), \tag{42}$$

for general X, Y in H. From (38) and (42) it follows that $\mathcal{G}_0$ is a compact, selfadjoint operator defined on H. From (38) it is also evident that the nullspace of $\mathcal{G}_0$ is empty. In other words, the only solution of $\mathcal{G}_0 X = 0$ is $X = 0$.

Using the standard theory of compact, selfadjoint operators we can conclude that $\mathcal{G}_0$ has real eigenvalues, its eigenfunctions form a complete orthonormal set in H. Since H is infinite dimensional $\mathcal{G}_0$ must have an infinity of eigenvalues. One can also show that all eigenvalues are simple. If we had two eigenfunctions corresponding to (36) and (37) we could form a linear combination satisfying (37) and the boundary conditions

$$x_1(0) = 0, \quad x_2(0) = 0.$$

Using (38) we can then show that x_1 satisfies

$$x_1'' + \lambda^2 M^4 x_1 = 0$$
$$x_1(0) = x_1'(0) = 0.$$

It follows that $x_1 = 0$ and also $x_2 = 0$. Hence the two eigenfunctions are identical.

Let F be any function in H and let X_n denote the normalized eigenfunctions and λ_n the eigenvalues of L_0. Then

$$F = \sum_{n=1}^{\infty} \alpha_n X_n \tag{43}$$

$$\alpha_n = (F, X_n) = \int_0^{\varrho} [M^4 f_1 \overline{x_1^{(n)}} + f_2 \overline{x_2^{(n)}}]\, d\xi. \tag{44}$$

If, in particular, we select $f_2 = 0$ we obtain from (43)

$$f_1 = \sum_{n=1}^{\infty} \alpha_n x_1^{(n)}$$

$$0 = \sum_{n=1}^{\infty} \alpha_n x_2^{(n)}.$$

If in the first of these we return to our original variables as defined in (27), (28), we obtain

$$f_1 = \sum_{n=1}^{\infty} \alpha_n \frac{(-\lambda_n)}{M} y_n, \qquad \alpha_n = \int_0^{\pi} f_1(-\lambda_n)\, M y_n \, dx. \tag{45}$$

We shall summarize these results in the following theorems.

THEOREM 3. *The operator L_0, defined by (36), (37), or equivalently $\mathcal{G}_0$, defined by (38), acting on the space H has an infinity of simple, real eigenvalues. The eigenfunctions form a complete orthonormal set with respect to the inner product (39).*

THEOREM 4. *The operator defined by (27), (28) has an infinity of simple real eigenvalues. The associated eigenfunctions are complete and functions that are square integrable on $[0, \pi]$ can be expanded in series of the type (45). It is assumed that $\mu < \mu_0$, which is defined by (29), (30).*

Note that the eigenfunctions associated with (27), (28) are not orthonormal.

We now turn to the problem

$$y'' + [\lambda^2 - q(x)]\, y = 0 \tag{46}$$

$$y(0) = 0, \quad a y'(\pi) + \lambda y(\pi) = 0. \tag{47}$$

If the value $\mu_0 > 0$ we can set $\mu = 0$ in (27), (28) and then the problem fits directly into the framework of Theorem 3, 4. If $\mu_0 \leq 0$ the preceding results no longer apply. Nevertheless we define M as in (29) and introduce the change of variables (31). This leads to the equation

$$\eta'' + [\lambda^2 - \mu]\, M^4 \eta = 0 \tag{48}$$

with the boundary conditions

$$\eta(0) = 0, \quad a\eta'(\varrho) + \lambda \eta(\varrho) = 0. \tag{49}$$

To linearize the problem in terms of λ we now let

$$\left.\begin{aligned} x_1 &= -\lambda \eta \\ x_2 &= \eta' + \mu \int_0^{\varrho} M^4 \eta \, d\xi. \end{aligned}\right\} \tag{50}$$

These satisfy the system

$$\left.\begin{array}{l} L_\mu X = \lambda X \\ x_1(0)=0, \quad a x_2(\varrho) - x_1(\varrho) = 0 \end{array}\right\}, \tag{51}$$

where
$$L_\mu X = \begin{pmatrix} \dfrac{x_2'}{M^4} \\ -x_1' - \mu \displaystyle\int_\xi^\varrho M^4 x_1 \, d\xi \end{pmatrix}$$

Note that
$$L_\mu = L_0 + \mu L_1,$$

where L_0 was defined in (37) and

$$L_1 X = \begin{pmatrix} 0 \\ -\displaystyle\int_\xi^\varrho M^4 x_1 \, d\xi \end{pmatrix} \tag{52}$$

is compact. The adjoint of L_1 is given by

$$L_1^* X = \begin{pmatrix} -\displaystyle\int_0^\xi x_2 \, d\xi \\ 0 \end{pmatrix}$$

so that
$$L_\mu^* = L_0 + \mu L_1^*,$$

with the same boundary conditions as in (51), since L_0 is selfadjoint.

We shall now assume that $\lambda = 0$ is not an eigenvalue of (51). In that case (50) can be rewritten as an integral equation. The case where $\lambda = 0$ leads to no fundamental complication, except that one does not work in the whole space H, but in the subspace orthogonal to the eigenfunction corresponding to $\lambda = 0$. Functions u and v are now defined as solutions of

$$x'' - \mu M^4 x = 0$$

satisfying the initial conditions

$$u(0) = 0, \quad u'(0) = 1$$

$$v(0) = 1, \quad v'(0) = 0.$$

For a suitable choice of μ $u'(\varrho) \neq 0$. Then we obtain the integral equation

$$\left.\begin{array}{l} x_1 = -\lambda u \displaystyle\int_\xi^\varrho v' x_2 \, d\xi + \frac{\lambda u v'(\varrho)}{u'(\varrho)} \int_0^\varrho u' x_2 \, d\xi - \lambda v \int_0^\xi u' x_2 \, d\xi \\ x_2 = \dfrac{-\lambda}{a u'(\varrho)} \displaystyle\int_0^\varrho u' x_2 \, d\xi - \lambda \int_\xi^\varrho M^4 x_1 \, d\xi. \end{array}\right\} \tag{53}$$

(53) replaces (51) and for $\mu = 0$ it reduces to (38). We shall denote the operator defined in (53) by

$$X = \lambda \mathcal{G}_\mu X. \tag{54}$$

Note that $\mathcal{G}_\mu$ is a compact operator on H, but in general it will not be selfadjoint.

LEMMA 1. *The eigenvalues of L_μ are real.*

Suppose $L_\mu X_i = \lambda_i X_i$, where λ_i is not real and is an eigenvalue of L_μ. Since the operator is real, if λ_i is an eigenvalue so is $\bar{\lambda}_i$. Then there exists F_i such that $L_\mu F_i = \bar{\lambda}_i F_i$ and since λ_i is an eigenvalue of L_μ, $\bar{\lambda}_i$ will be an eigenvalue of L_μ^*. Let G_i be such that $L_\mu^* G_i = \bar{\lambda}_i G_i$. It also follows from

$$\bar{\lambda}(F_i, G_i) = (L_\mu F_i, G_i) = (F_i, L_\mu^* G_i) = \lambda_i (F_i, G_i)$$

that $(F_i, G_i) = 0$. In that case, according to the Fredholm alternative the equation

$$L_\mu Z_i - \lambda_i Z_i = F_i$$

must have a solution. Using the fact that $L_\mu F_i = \bar{\lambda}_i F_i$ the latter can be reduced to the following scalar second order equation.

$$z_1'' + (\lambda_i^2 - \mu) M^4 z_1 = -(\lambda_i + \bar{\lambda}_i) M^4 f_1$$

$$z_1(0) = 0, \quad a z_1'(\varrho) + \lambda_i z_1(\varrho) = 0,$$

where z_1 and f_1 are the first components of Z_i and F_i respectively. Now consider $\bar{f}_1$ which satisfies

$$\bar{f}_1'' + (\lambda_i^2 - \mu) M^4 \bar{f}_1 = 0$$

$$\bar{f}_1(0) = 0, \quad a \bar{f}_1'(\varrho) + \lambda_i \bar{f}_1(\varrho) = 0.$$

By multiplying the equation for z_1 by $\bar{f}_1$ and the one for $\bar{f}_1$ by z_1, subtracting and integrating we are led to

$$(\lambda_i + \bar{\lambda}_i) \int_0^\varrho M^4 |f_1|^2 \, dt = 0.$$

We conclude, therefore, that

$$\lambda_i + \bar{\lambda}_i = 0$$

so that λ_i is pure imaginary. But in that case we have

$$x'' + [\lambda_i^2 - \mu] M^4 x = 0$$

$$x(0) = 0, \quad a x'(\varrho) + \lambda_i x(\varrho) = 0$$

and also $$ax'(\varrho) - \lambda_i x(\varrho) = 0$$

so that $$x'(\varrho) = x(\varrho) = 0.$$

The latter implies that $x \equiv 0$, which is certainly not true. It follows that all eigenvalues are real.

LEMMA 2. *The eigenspace associated with any eigenvalue is one dimensional.*

To prove this we note first that to every eigenvalue there corresponds precisely one eigenfunction. If there were two we could form a linear combination such that $x(0) = x'(0) = 0$, which implies that $x \equiv 0$. Now if the subspace were more than one dimensional and if

$$L_\mu F_i = \lambda_i F_i$$

then the equation $$L_\mu X_i - \lambda_i X_i = F_i$$

would have to have a solution. But as in the proof of Lemma 1 we can show that the above has no solution.

LEMMA 3. *The eigenfunctions of L_μ and L_μ^* form a biorthogonal set. It follows that both consist of linearly independent elements.*

Let $L_\mu X_i = \lambda_i X_i$ and $L_\mu^* Y_i = \lambda_i Y_i$. Clearly

$$\lambda_i (X_i, Y_j) = (L_\mu X_i, Y_j) = (X_i, L_\mu^* Y_j) = \lambda_j (X_i, Y_j)$$

so that $$(X_i, Y_j) = 0 \quad \text{for} \quad i \neq j.$$

If $(X_i, Y_i) = 0$ for some i, the equation

$$L_\mu Z_i - \lambda_i Z_i = X_i$$

would have a solution. But as in the proof of Lemma 1, this cannot be so that $(X_i, Y_i) \neq 0$.

We shall now turn to a consideration of the asymptotic structure of the eigenvalues of L_μ. We shall seek a solution of (27) in the form

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} y_n \\ y_0 &= \sin \lambda x \\ y_{n+1} &= \int_0^x \frac{\sin \lambda (x-t)}{\lambda} [q(t) - \mu] \, y_n(t) \, dt. \end{aligned}\right\} \tag{55}$$

By induction we note that

$$|y_n| \leq \frac{1}{|\lambda|^n n!} \int_0^x |q(t) - \mu|^n \, dt$$

so that (55) converges uniformly for real λ. We see that

$$y = \sin \lambda x + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} (q - \mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda^2}\right)$$

$$y' = \lambda \cos \lambda x + \int_0^x \cos \lambda(x-t) (q-\mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda}\right)$$

so that

$$\begin{aligned} ay'(\pi) + \lambda y(\pi) &= \lambda(a \cos \lambda\pi + \sin \lambda\pi) \\ &\quad + \int_0^\pi [a \cos \lambda(\pi - t) + \sin \lambda(\pi - t)] (q - \mu) \sin \lambda t \, dt + o\left(\frac{1}{\lambda}\right) \\ &= \lambda(a \cos \lambda\pi + \sin \lambda\pi) + \frac{a \sin \lambda\pi - \cos \lambda\pi}{2} \int_0^\pi (q - \mu) \, dt + o(1). \end{aligned}$$

From the above we note that

$$\lambda_n = n + \lambda_0 + \frac{1}{2n} \int_0^\pi (q - \mu) \, dt + o\left(\frac{1}{n}\right), \tag{56}$$

where $\tan \lambda_0 \pi = -a$ and $|\lambda_0 \pi| < \pi/2$. Note that $n = 0, \pm 1, \pm 2, \ldots$, so that the operator is not semibounded as is the case with Sturm–Liouville operators. We also observe that the μ dependence enters into the terms vanishing like $1/n$.

We now wish to show that the eigenfunctions associated with the operator L_μ, defined in (51), are complete. It will turn out to be more advantageous to work with L_μ^2, rather than L_μ. Note that formally

$$L_\mu^2 X = \begin{pmatrix} -\dfrac{x_1''}{M^4} + \mu x_1 \\ -\left(\dfrac{x_2'}{M^4}\right)' + \mu x_2 - \mu x_2(\varrho) \end{pmatrix} \tag{57}$$

and the boundary conditions associated with L_μ^2 are

$$x_1(0) = 0, \quad x_2'(0) = 0,$$

$$a x_2(\varrho) - x_1(\varrho) = 0, \quad a x_1'(\varrho) + x_2'(\varrho) = 0. \tag{58}$$

The eigenvalues of L_μ^2 are given by λ_n^2, and the eigenfunctions are the same as those of L_μ. By the preceding results all eigenvalues of L_μ^2 are real and simple. Note that we can write

$$L_\mu^2 X = (L_0^2 + \mu) X - \mu N X, \tag{59}$$

where

$$NX = \begin{pmatrix} 0 \\ x_2(\varrho) \end{pmatrix}. \tag{60}$$

Theorem 3 tells us that $L_0^2 + \mu$ is a selfadjoint operator and that its eigenfunctions form a complete orthonormal set. The operator N, given by (60) is not selfadjoint and also unbounded.

We now consider the equation

$$(\lambda - L_\mu^2) X = F, \tag{61}$$

with boundary conditions (58). The solution of (61) will formally be denoted by

$$X = \mathcal{G}_2 F.$$

From the preceding results we know that the operator $\mathcal{G}_2$ can be expressed as an integral operator where the kernel is a meromorphic function of λ. For all regular values of λ $\mathcal{G}_2$ is compact. Similarly we associate with the differential operator $\lambda - L_0^2 - \mu$ the integral operator $\mathcal{G}_1$. The solution of

$$(\lambda - L_0^2 - \mu) Y = F \tag{62}$$

with boundary conditions (58) is given by

$$Y = \mathcal{G}_1 F.$$

If we express F in the form

$$F = \sum_{n=-\infty}^{\infty} \alpha_n X_n,$$

where the X_n are the eigenfunctions of L_0, then

$$Y = \sum_{n=-\infty}^{\infty} \frac{\alpha_n X_n}{\lambda - \lambda_n^2 - \mu}. \tag{63}$$

We then can write

$$\frac{1}{2\pi i} \int \mathcal{G}_1 F \, d\lambda = \sum_{n=-\infty}^{\infty} \alpha_n X_n = F, \tag{64}$$

where the integral is taken over a sufficiently large contour in the λ plane. We can also use (63) to determine $\|\mathcal{G}_1\|$. Then

$$\|\mathcal{G}_1 F\| = \sqrt{\sum_{-\infty}^{\infty} \frac{|\alpha_n|^2}{|\lambda - \lambda_n^2 - \mu|^2}}$$

and

$$\frac{\|\mathcal{G}_1 F\|}{\|F\|} = \sqrt{\frac{\sum_{-\infty}^{\infty} \frac{|\alpha_n|^2}{|\lambda - \lambda_n^2 - \mu|^2}}{\sum_{-\infty}^{\infty} |\alpha_n|^2}}.$$

From the above we note that

$$\|\mathcal{G}_1\| = \sup_{F \neq 0} \frac{\|\mathcal{G}_1 F\|}{\|F\|} = \sup_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}, \tag{65}$$

(64) is a consequence of the completeness of the eigenfunctions of L_0. If we succeed in establishing

$$\frac{1}{2\pi i} \int \mathcal{G}_2 F \, d\lambda = F \tag{66}$$

we will have proved that the eigenfunctions of L_μ^2 and correspondingly those of L_μ are complete. Note the residue of $\mathcal{G}_2 F$ at $\lambda = \lambda_n$ represents the projection of F into the n'th eigenspace. The left side of (66) then yields the expansion of F in terms of the eigenfunctions of L_μ.

Using (59) we can rewrite (61) in the form

$$(\lambda - L_0^2 - \mu) X = F - \mu N X$$

and, using (62), we obtain

$$X = \mathcal{G}_1 [F - \mu N X]$$

or equivalently

$$\mathcal{G}_2 F = \mathcal{G}_1 F - \mu \mathcal{G}_1 N \mathcal{G}_2 F. \tag{67}$$

An immediate solution of (67) is given by

$$\mathcal{G}_2 F = \mathcal{G}_1 \sum_{S=0}^{\infty} [-\mu N \mathcal{G}_1]^S F \tag{68}$$

assuming that the series in (68) converges. To prove the convergence of (68) we shall estimate $\|N\mathcal{G}_1\|$.

By means of (63) we see that

$$N \mathcal{G}_1 F = \sum_n \frac{\alpha_n}{\lambda - \lambda_n^2 - \mu} \begin{pmatrix} 0 \\ x_2^{(n)}(\varrho) \end{pmatrix}$$

so that

$$\|N\mathcal{G}_1 F\| \leqslant \sqrt{\varrho \left| \sum_n \frac{\alpha_n x_2^{(n)}(\varrho)}{\lambda - \lambda_n^2 - \mu} \right|^2} \leqslant \sqrt{\varrho \sum_n [\alpha_n x_2^{(n)}(\varrho)]^2 \sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|^2}}.$$

Using (13) one can deduce for the orthonormalized eigenfunctions the asymptotic formulas

$$X_n = \begin{pmatrix} \dfrac{\sin \lambda_n x(\xi)}{M\sqrt{\pi}} \\ \dfrac{-M \cos \lambda_n x(\xi)}{\sqrt{\pi}} \end{pmatrix} + O\left(\frac{1}{\lambda_n}\right),$$

where

$$\xi = \int_0^{x(\xi)} \frac{dx}{M^2(x)}.$$

Combining the latter with (12) we find

$$x_2^{(n)}(\varrho) = \frac{-\cos \lambda_n \pi}{\sqrt{\pi}} + O\left(\frac{1}{n}\right) = \frac{(-1)^{n+1}}{\sqrt{\pi(1+a^2)}} + O\left(\frac{1}{n}\right).$$

Then

$$\|N G_1 F\| \leqslant K \|F\| \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|^2}}$$

$$\leqslant K \|F\| \, \|G_1\|^{\frac{1}{2}} \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}},$$

if we use the explicit estimate (65) for $\|G_1\|$. If follows that

$$\|N G_1\| \leqslant K \|G_1\|^{\frac{1}{2}} \sqrt{\sum_n \frac{1}{|\lambda - \lambda_n^2 - \mu|}}. \tag{69}$$

We now consider a sequence of squares $\{R_k\}$ with vertices at

$$\left(k^2 + \mu + \int_0^\pi q\,dt\right)(\pm 1 \pm i),$$

and estimate the sum under the radical in (69) for λ on R_k. From (56) we have

$$\lambda_n^2 = (n + \lambda_0)^2 + \int_0^\pi q\,dt + o(1). \tag{70}$$

We decompose the sum into three terms

$$\sum_{n=0}^{\infty} = \sum_{n=0}^{[\frac{1}{2}k]} + \sum_{n=[\frac{1}{2}k]+1}^{[\frac{3}{2}k]} + \sum_{n=[\frac{3}{2}k]+1}^{\infty}$$

so that,

$$\sum_{n=0}^{[\frac{1}{2}k]} \frac{1}{|\lambda - \lambda_n^2 - \mu|} = O\left(\frac{1}{k}\right)$$

$$\sum_{n=[\frac{1}{2}k]+1}^{[\frac{3}{2}k]} \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant k \sup_{[\frac{1}{2}k]+1 \leqslant n \leqslant [\frac{3}{2}k]} \frac{1}{|\lambda-\lambda_n^2-\mu|} = o(1)$$

$$\sum_{n=[\frac{3}{2}k]+1}^{\infty} \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \sum_{n=[\frac{3}{2}k]+1}^{\infty} \frac{1}{(n+\lambda_0)^2+o(1)} = O\left(\frac{1}{k}\right).$$

Finally we have for all λ on R_k, with sufficiently large k

$$\|N\mathcal{G}_1\| \leqslant K_1 \|\mathcal{G}_1\|^{\frac{1}{2}}. \tag{71}$$

A more delicate analysis allows us to show that for the constant K_1 in (71) we actually have

$$K_1 = O\left(\frac{1}{k}\ln k\right)$$

but this is not necessary for our subsequent results. Now

$$\|\mathcal{G}_1\| = \sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \frac{K}{k}. \tag{72}$$

It follows, therefore, that the series in (68) converges on R_k for sufficiently large k. Note that this also establishes the existence of $\mathcal{G}_2F$, knowing only the structure of $\mathcal{G}_1$ and N.

Finally we consider the integral

$$\frac{1}{2\pi i}\int_{R_k} \mathcal{G}_2 F\, d\lambda.$$

To estimate the above we consider the general term in (68),

$$\frac{1}{2\pi i}\int \mathcal{G}_1 (N\mathcal{G}_1)^S F\, d\lambda.$$

Using (71) we see that

$$\|\mathcal{G}_1(N\mathcal{G}_1)^S\| \leqslant K_1 \|\mathcal{G}_1\|^{\frac{1}{2}S+1} = K_1 \sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|^{\frac{1}{2}S+1}},$$

and at first we restrict our attention to $\lambda = k^2+\mu+\int_0^\pi q\,dt+iy$, where $|y| \leqslant k^2+\mu+\int_0^\pi q\,dt$, so that

$$\sup_n \frac{1}{|\lambda-\lambda_n^2-\mu|} \leqslant \frac{K}{\sqrt{4k^2\lambda_0^2+y^2}}$$

for a suitable constant K. Then

$$\|\mathcal{G}_1(N\mathcal{G}_1)^S\| \leqslant \frac{K_2}{[4k^2\lambda_0^2+y^2]^{\frac{1}{4}S+\frac{1}{2}}}$$

and $$\left\| \frac{1}{2\pi i} \int_{-k^2-\mu-\int_0^\pi q\,dt}^{k^2+\mu+\int_0^\pi q\,dt} G_1 (N G_1)^S F\,dy \right\| \leqslant K_3 \|F\| \int_0^{k^2+\mu+\int_0^\pi q\,dt} \frac{dy}{[4k^2\lambda_0^2+y^2]^{\frac{1}{2}S+\frac{1}{2}}}$$

$$\leqslant \frac{K_3\|F\|}{(2k\lambda_0)^{\frac{1}{2}S}} \int_0^{\frac{1}{2}\pi} \cos^{\frac{1}{2}S-1}\theta\,d\theta \leqslant \frac{K_4}{k^{\frac{1}{2}S}} \|F\|.$$

Applying similar, and even simpler estimates, to the other three sides of R_k, we see that

$$\frac{1}{2\pi i} \int_{R_k} G_1 (N G_1)^S F\,d\lambda = O\left(\frac{1}{k^{\frac{1}{2}S}}\right).$$

Returning to (68) we have

$$\frac{1}{2\pi i} \int_{R_k} G_2 F\,d\lambda = \frac{1}{2\pi i} \int_{R_k} G_1 F\,d\lambda + O\left(\frac{1}{k^{\frac{1}{2}}}\right)$$

so that $$\lim_{k\to\infty} \frac{1}{2\pi i} \int_{R_k} G_2 F\,d\lambda = F. \tag{73}$$

We can now state the following theorem.

THEOREM 5. *The operator L_μ, defined by (51), acting on the Hilbert space H, has an infinity of real, simple eigenvalues. The eigenfunctions associated with it form a complete set. The eigenfunctions associated with its adjoint operator L_μ^* are also complete and biorthogonal to those of L_μ.*

In analogy to Theorem 4, we obtain expansion theorems associated with the eigenfunctions of (46), (47). We restate this result as follows:

THEOREM 6. *Theorem 4 applies to the problem (27), (28) without any restrictions placed on the parameter μ.*

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Received June 9, 1967