

## ON ISOMORPHISMS BETWEEN CENTERS OF INTEGRAL GROUP RINGS OF FINITE GROUPS

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ABSTRACT. For finite nilpotent groups  $G$  and  $G'$ , and a  $G$ -adapted ring  $S$  (the rational integers, for example), it is shown that any isomorphism between the centers of the group rings  $SG$  and  $SG'$  is monomial, i.e., maps class sums in  $SG$  to class sums in  $SG'$  up to multiplication with roots of unity. As a consequence,  $G$  and  $G'$  have identical character tables if and only if the centers of their integral group rings  $\mathbb{Z}G$  and  $\mathbb{Z}G'$  are isomorphic. In the course of the proof, a new proof of the class sum correspondence is given.

### 1. INTRODUCTION

It has been asked whether an automorphism of the center of the integral group ring of a finite group necessarily induces a monomial permutation on the set of the class sums (listed as Problem 14.2 in the Kourovka Notebook [8] and as Problem 41 in [17], both times attributed to S. D. Berman), but seemingly no progress was made towards a solution, except that it is annotated in [17] that A. A. Bovdi has answered it affirmatively for nilpotent groups of class at most three. In this paper, it is finally dealt with in the case of nilpotent groups.

Actually, we are treating an obvious generalization of the original question. Suppose  $G$  is a finite group and  $S$  is an integral domain of characteristic zero, and let  $\text{Cent}_S(G)$  be the center of  $SG$ , the group ring of  $G$  over  $S$ . Throughout, we shall be concerned with a  $G$ -adapted coefficient ring  $S$  (precise definitions are given in the following sections). Then, given another group  $G'$ , we ask whether an isomorphism  $\text{Cent}_S(G) \cong \text{Cent}_S(G')$  of  $S$ -algebras (if existing) is necessarily monomial, i.e., maps class sums in  $SG$  to class sums in  $SG'$  up to multiplication with roots of unity (which, in any case, can be avoided by “normalization”; see Lemma 2.2). For nilpotent  $G$ , this is answered in the affirmative (Theorem 4.2). In fact, nilpotent groups constitute a special case as the familiar Berman–Higman result assures that the central units of finite order in  $SG$  are, up to roots of unity, just the central elements in  $G$ . This suggests that an approach might exist which proceeds inductively along the upper central series of the groups. This idea is realized, resulting in an elementary, character free proof.

Dealing with isomorphisms—rather than only with automorphisms—does not cause additional difficulties and is motivated by the following question: Does an

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isomorphism  $\text{Cent}_{\mathbb{Z}}(G) \cong \text{Cent}_{\mathbb{Z}}(G')$  result in an isomorphism between the character tables of  $G$  and  $G'$ ? (The converse has been known ever since Frobenius introduced group characters.) It turns out that a monomial isomorphism of centers preserves the character degrees, thus giving rise to an isomorphism of character tables. Conversely, a degree-preserving isomorphism of centers is monomial. This is the content of the class sum correspondence, for which a proof is given in Section 3 which deviates from the known in so far as the case when  $S$  is a ring of algebraic integers is treated in a simple way, using an elementary result of Kronecker, which says that at least one of the conjugates of a nonzero algebraic integer that is not a root of unity must lie outside the unit circle (Lemmas 2.4 and 2.5). It is only in Section 3 that characters really show up, and one might get an impression about the problem in general. Section 2 contains all the generalities needed for the handling of nilpotent groups in Section 4. We would like to emphasize that no internal characterization of class sums, or even character degrees, in the center is known, so we do *not* consider  $\text{Cent}_S(G)$  as an algebra in its own right.

It seems appropriate to include a few remarks on character rings. Let  $\text{Char}_S(G)$  be the ring of  $S$ -linear combinations of the irreducible characters of  $G$ . Let  $R$  be a ring of algebraic integers. Weidman [19] and Saksonov [13] proved independently that if  $\text{Char}_R(G) \cong \text{Char}_R(G')$ , then the character tables of  $G$  and  $G'$  are the same. This can be viewed as a consequence of an internal characterization of the ordinary inner product on the character ring (see also the presentation in [1]). Actually, they showed that any isomorphism of character rings is monomial (in the supplement [20] it is shown how “normalization” is to be understood).

Duality between the rings  $\text{Cent}_S(G)$  and  $\text{Char}_S(G)$  is definitive only for abelian groups. For a subfield  $k$  of the complex numbers, the structure of  $\text{Cent}_k(G)$  and  $\text{Char}_k(G)$  has been described in [18], where it is shown that if  $p$  is an odd prime and  $G$  is a  $p$ -group, then  $\text{Cent}_k(G) \cong \text{Char}_k(G)$ , with the assumption  $p \neq 2$  being necessary (cf. also [2]). In [15],  $p$ -adic class algebras have been compared with  $p$ -adic character rings. If  $S$  is  $G$ -adapted, then  $\text{Cent}_S(G) \cong \text{Char}_S(G')$  if and only if  $G$  and  $G'$  are isomorphic abelian groups.

## 2. GENERALITIES

Let  $G$  and  $G'$  be finite groups, and let  $R$  be a ring of algebraic integers in the field of complex numbers  $\mathbb{C}$ . A *class sum* in  $RG$  is, for a group element  $g$  of  $G$ , the sum of its  $G$ -conjugates in  $RG$ . Note that the class sums in  $RG$  form an  $R$ -basis of  $\text{Cent}_R(G)$ .

Suppose there exists an  $R$ -algebra isomorphism  $\varphi: \text{Cent}_R(G) \rightarrow \text{Cent}_R(G')$ . In this section we shall derive some basic facts about it. Of course,  $\varphi$  extends to a  $\mathbb{C}$ -algebra isomorphism  $\varphi: \text{Cent}_{\mathbb{C}}(G) \rightarrow \text{Cent}_{\mathbb{C}}(G')$ , and as such,  $\varphi$  maps primitive idempotents (corresponding to irreducible characters) to primitive idempotents and is completely determined by this operation.

We let  $g_1, \dots, g_h$  with  $g_1 = 1$  be representatives of the classes of  $G$  and write  $\mathcal{C}_i$  for the class sum of  $g_i$  in  $RG$ . We let  $\varepsilon: RG \rightarrow R$  be the augmentation homomorphism and  $\varepsilon_1: RG \rightarrow R$  be the usual trace map. These maps are defined, using the group basis  $G$ , by  $\varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$  and  $\varepsilon_1(\sum_{g \in G} a_g g) = a_1$  (all coefficients  $a_i$  in  $R$ ). Throughout, we dispose of similar notation for  $G'$  using primes, i.e.,  $g'_1, \dots, g'_{h'}$  are representatives of the classes of  $G'$  (with  $g'_1 = 1$ ), and so forth.

We also write  $\mathcal{C}_g$  for the class sum of a specific element  $g$  of  $G$  (accordingly,  $\mathcal{C}_{g'}$  for  $g'$  in  $G'$  is defined) and let  $|\mathcal{C}_g|$  be its length.

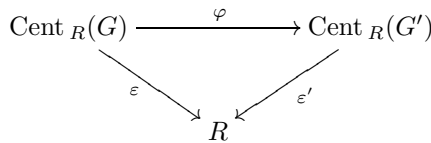
**Lemma 2.1.** *The groups  $G$  and  $G'$  have the same order and the same number of conjugacy classes. The isomorphism  $\varphi$  maps the set of linear characters of  $G$  onto the set of linear characters of  $G'$ .*

*Proof.* Since  $\text{Cent}_R(G)$  and  $\text{Cent}_R(G')$  have the class sums as  $R$ -bases, the groups  $G$  and  $G'$  have the same number of conjugacy classes. Without loss of generality we can assume that  $|G| \geq |G'|$ . Let  $I$  be the ring of all algebraic integers in  $\mathbb{C}$ . Let  $\lambda$  be a linear character of  $G$  and  $\chi'$  its image under  $\varphi$ . If  $e_\lambda$  denotes the block idempotent in  $\mathbb{C}G$  corresponding to  $\lambda$ , then

$$\left(\frac{|G'|}{\chi'(1)}e_\lambda\right)\varphi = \sum_{i=1}^h \chi'(g_i^{-1})\mathcal{C}'_i \in \text{Cent}_I(G'),$$

so  $\frac{|G'|}{\chi'(1)}e_\lambda \in \text{Cent}_I(G)$  and  $\frac{|G'|}{\chi'(1)}/|G| = \varepsilon_1\left(\frac{|G'|}{\chi'(1)}e_\lambda\right) \in I$  (here we used that  $\lambda(1) = 1$ ). Thus  $\frac{|G'|}{\chi'(1)}/|G|$  is a natural integer, and by our assumption on the orders of  $G$  and  $G'$ , this is only possible if  $\chi'(1) = 1$  and  $|G| = |G'|$ .  $\square$

We shall say that  $\varphi$  is *monomial* if there is a permutation  $\pi$  on  $\{1, 2, \dots, h\}$  and roots of unity  $\xi_1, \dots, \xi_h$  such that  $\mathcal{C}_i\varphi = \xi_{i\pi}^{-1}\mathcal{C}'_{i\pi}$  for  $1 \leq i \leq h$ . Note that then the assignment  $\lambda'(g'_i) = \xi_i$  defines a linear character  $\lambda'$  of  $G'$  since the idempotent  $\frac{1}{|G|}\sum_{i=1}^h \mathcal{C}_i$  corresponding to the principal character is mapped under  $\varphi$  to  $\frac{1}{|G'|}\sum_{i=1}^h \xi_i^{-1}\mathcal{C}'_i$ . We shall say that  $\varphi$  is *normalized* if the following diagram is commutative:



We write  $\mathbb{1}$  and  $\mathbb{1}'$  for the principal characters of  $G$  and  $G'$ , respectively. Note that  $\varphi$  is normalized if and only if  $\varphi$  sends  $\mathbb{1}$  to  $\mathbb{1}'$ . If  $\varphi$  is monomial and normalized, it maps class sums to class sums.

We can turn our attention to normalized isomorphisms:

**Lemma 2.2.** *Let  $\lambda'$  be the linear character of  $G'$  to which  $\mathbb{1}$  is mapped under  $\varphi$  (cf. Lemma 2.1). Then a monomial  $R$ -algebra automorphism  $\alpha$  of  $\text{Cent}_R(G')$  is defined by the assignment  $\mathcal{C}'_i\alpha = \lambda'(g'_i)\mathcal{C}'_i$ , and  $\varphi\alpha$  is a normalized isomorphism.*

*Proof.* Note that  $\varphi$  maps  $\sum_i \mathcal{C}_i$  to  $\sum_i \lambda'(g_i^{-1})\mathcal{C}'_i$ , so  $\alpha$  is clearly an  $R$ -linear map such that  $(\sum_i \mathcal{C}_i)\varphi\alpha = \sum_i \mathcal{C}'_i$ , i.e.,  $\varphi\alpha$  sends  $\mathbb{1}$  to  $\mathbb{1}'$ . It remains to show that  $\alpha$  is multiplicative. Take class sums  $\mathcal{C}'_i$  and  $\mathcal{C}'_j$  and write  $\mathcal{C}'_i\mathcal{C}'_j = \sum_k c'_{ijk}\mathcal{C}'_k$  with integers  $c'_{ijk}$ . Then

$$(\mathcal{C}'_i\mathcal{C}'_j)\alpha = \sum_k c'_{ijk}\lambda'(g'_k)\mathcal{C}'_k, \quad (\mathcal{C}'_i)\alpha(\mathcal{C}'_j)\alpha = \sum_k c'_{ijk}\lambda'(g'_i)\lambda'(g'_j)\mathcal{C}'_k,$$

so we have to check that  $\lambda'(g'_k) = \lambda'(g'_i)\lambda'(g'_j)$  whenever  $c'_{ijk} \neq 0$ . But this is obvious by the definition of the  $c'_{ijk}$  since  $\lambda'$  is a linear character, i.e., has the commutator subgroup of  $G$  in its kernel.  $\square$

We write  $\widehat{N}$  for the sum in  $RG$  of the elements of a subgroup  $N$  of  $G$ .

**Lemma 2.3.** *Let  $\varphi$  be normalized. Assume that there are normal subgroups  $N$  and  $N'$  of  $G$  and  $G'$ , respectively, such that  $\varphi$  maps  $\text{Cent}_R(G) \cap RN$  onto  $\text{Cent}_R(G') \cap RN'$ . Then  $\widehat{N}\varphi = \widehat{N}'$  and there exists a commutative diagram of algebra homomorphisms*

$$\begin{CD} \text{Cent}_R(G) @>\varphi>> \text{Cent}_R(G') \\ @V\pi VV @VV\pi'V \\ \text{Cent}_R(G/N) @>\bar{\varphi}>> \text{Cent}_R(G'/N) \end{CD}$$

where the vertical maps are the natural ones and the bottom map  $\bar{\varphi}$  is again a normalized isomorphism.

*Proof.* Let  $e_N$  be the block idempotent of  $\mathbb{C}N$  corresponding to the principal character. As a central idempotent of  $\mathbb{C}G$ , it is mapped by  $\varphi$  to a central idempotent  $f$  of  $\mathbb{C}G'$ . By assumption,  $f$  lies in  $\mathbb{C}N'$ ; it is primitive in  $\mathbb{C}N'$  as  $e_N$  is primitive in  $\mathbb{C}N$ . Since  $\varphi$  is normalized, it follows that  $f$  corresponds to the principal character of  $N'$ . So  $\widehat{N}\varphi = \frac{|N|}{|N'|}\widehat{N}'$ , showing that  $\frac{|N|}{|N'|} \in R \cap \mathbb{Q} = \mathbb{Z}$ . Similarly,  $\frac{|N'|}{|N|} \in \mathbb{Z}$ , so  $|N| = |N'|$  and  $\widehat{N}\varphi = \widehat{N}'$ .

Set  $\bar{G} = G/N$ , and let  $\pi: RG \rightarrow R\bar{G}$  be the  $R$ -algebra homomorphism extending the natural homomorphism  $G \rightarrow \bar{G}$ . Note that the kernel of  $\pi$  is the annihilator of  $\widehat{N}$ . Also considering the analogously defined map  $\pi'$ , it follows from  $\widehat{N}\varphi = \widehat{N}'$  that  $\varphi$  induces an isomorphism  $\bar{\varphi}: \text{Cent}_R(G)\pi \rightarrow \text{Cent}_R(G')\pi'$ . We proceed to show that  $\bar{\varphi}$  can be extended to an isomorphism  $\text{Cent}_R(\bar{G}) \rightarrow \text{Cent}_R(\bar{G}')$ . Let  $g \in G$ , and let  $\mathcal{C}_{\bar{g}}$  be the class sum of  $\bar{g}$  in  $R\bar{G}$ . Since  $\pi$  maps conjugacy classes of  $G$  onto conjugacy classes of  $\bar{G}$ , we have  $\mathcal{C}_{\bar{g}}\pi = m\mathcal{C}_{\bar{g}}$  for some  $m \in \mathbb{N}$ . So  $\text{Cent}_R(G)\pi$  contains a  $\mathbb{C}$ -basis of  $\text{Cent}_{\mathbb{C}}(\bar{G})$ , and  $\bar{\varphi}$  uniquely extends to an isomorphism  $\bar{\varphi}: \text{Cent}_{\mathbb{C}}(\bar{G}) \rightarrow \text{Cent}_{\mathbb{C}}(\bar{G}')$ . With  $\varphi$ , also  $\bar{\varphi}$  is normalized. It remains to show that  $\mathcal{C}_{\bar{g}}\bar{\varphi} \in R\bar{G}'$ . Since  $\mathcal{C}_{\bar{g}}\bar{\varphi} = \frac{1}{m}(\mathcal{C}_g\varphi\pi')$ , this is equivalent to  $\mathcal{C}_g\varphi\pi' \in mR\bar{G}'$  or  $(\mathcal{C}_g\widehat{N})\varphi\pi' \in m|N|R\bar{G}'$ . Since  $\mathcal{C}_g\widehat{N} \in m\mathbb{Z}G$  we have  $(\mathcal{C}_g\widehat{N})\varphi = (\mathcal{C}_g\varphi)\widehat{N}' \in mRG \cap (RG')\widehat{N}'$ , so  $(\mathcal{C}_g\varphi)\widehat{N}' = mx\widehat{N}'$  for some  $x \in RG'$ , and it follows that  $(\mathcal{C}_g\widehat{N})\varphi\pi' = ((\mathcal{C}_g\varphi)\widehat{N}')\pi' = (mx\widehat{N}')\pi' = m|N'|(x\pi')$  as desired.  $\square$

We define an anti-automorphism  $\circ$  on  $\mathbb{C}G$  in the usual way by  $(\sum_{g \in G} a_g g)^\circ = \sum_{g \in G} \bar{a}_g g^{-1}$  where  $\bar{a}_g$  denotes the complex conjugate of the number  $a_g$ . Note that  $\circ$  fixes each central primitive idempotent in  $\mathbb{C}G$ . This shows that  $\varphi$  commutes with these anti-automorphisms in the sense that  $(x^\circ)\varphi = (x\varphi)^\circ$  for all  $x \in \mathbb{C}G$ . Also note that  $\varepsilon_1(\mathcal{C}_i\mathcal{C}_i^\circ) = |\mathcal{C}_i|$  for any index  $i$  and  $\varepsilon_1(\mathcal{C}_i\mathcal{C}_j^\circ) = 0$  for distinct indices  $i, j$ .

**Lemma 2.4.** *Suppose that  $\varepsilon_1(\mathcal{C}_g\mathcal{C}_g^\circ) = \varepsilon'_1((\mathcal{C}_g\mathcal{C}_g^\circ)\varphi)$  for some  $g \in G$ . Write  $\mathcal{C}_g\varphi = \sum_{i=1}^h a_i\mathcal{C}'_i$  with all  $a_i$  in  $R$  and suppose further that  $a_{i_0} \neq 0$  for some index  $i_0$  with  $|\mathcal{C}'_{i_0}| \geq |\mathcal{C}_g|$ . Then  $a_{i_0}$  is a root of unity and  $\mathcal{C}_g\varphi = a_{i_0}\mathcal{C}_{i_0}$ .*

*Proof.* We have  $\varepsilon_1(\mathcal{C}_g\mathcal{C}_g^\circ) = |\mathcal{C}_g|$  and  $\varepsilon'_1((\mathcal{C}_g\mathcal{C}_g^\circ)\varphi) = \sum_{i=1}^h |a_i|^2|\mathcal{C}'_i|$ . By an elementary result due to Kronecker [7], either  $a_{i_0}$  is a root of unity or some algebraic conjugate of  $a_{i_0}$  has absolute value strictly greater than 1. From the assumptions, it follows that  $a_{i_0}$  is a root of unity and that all other coefficients  $a_i$  vanish.  $\square$

The immediate consequence is:

**Lemma 2.5.** *If  $\varepsilon_1(z) = \varepsilon'_1(z\varphi)$  for all  $z \in \text{Cent}_R(G)$ , then  $\varphi$  is monomial.*

*Proof.* By Lemma 2.2, we can assume that  $\varphi$  is normalized. For  $n \geq 0$ , let  $T_n$  be the set of class sums of elements of  $G$  of length  $n$  (so  $T_0 = \emptyset$ ), and let  $R[T_{<n}]$  be the  $R$ -span of  $T_0, \dots, T_{n-1}$ . We shall prove by induction on  $n$  that  $\varphi$  maps  $T_n$  onto  $T'_n$ . For  $n = 0$  this is an empty statement, so let  $n \geq 1$ . Suppose that  $T_n \neq \emptyset$ , and let  $g \in G$  with  $\mathcal{C}_g \in T_n$ . By the induction hypothesis,  $\varphi$  maps  $R[T_{<n}]$  bijectively onto  $R[T'_{<n}]$ . Thus  $g$  satisfies the hypotheses of Lemma 2.4, from which we conclude that  $T_n\varphi \subseteq T'_n$ . By symmetry,  $T_n\varphi = T'_n$ , and we are done.  $\square$

### 3. THE CLASS SUM CORRESPONDENCE

For a group  $G$ , an integral domain  $S$  of characteristic zero is called  $G$ -adapted if no prime divisor of the order of  $G$  is invertible in  $S$ . In this section, we keep previous notation but instead of  $R$  we take a  $G$ -adapted ring  $S$  into consideration. It is only at first sight that this is a more general assumption, and we shall derive the class sum correspondence from results of the previous section. Let  $\psi: \text{Cent}_S(G) \rightarrow \text{Cent}_S(G')$  be an  $S$ -algebra isomorphism.

First, we give the explicit formula of the matrix  $A = (a_{ij})$  which describes  $\psi$  with respect to the bases formed by the class sums:

$$\mathcal{C}_i\psi = \sum_{j=1}^h a_{ij}\mathcal{C}'_j \quad (1 \leq i \leq h).$$

Let  $\chi_1, \dots, \chi_h$  with  $\chi_1 = 1$  be the irreducible characters of  $G$ , and let  $e_l$  be the block idempotent corresponding to  $\chi_l$  (so  $e_l = \frac{\chi_l(1)}{|G|} \sum_{i=1}^h \chi_l(g_i^{-1})\mathcal{C}_i$ ). We have  $e_l\psi = e'_{l\sigma}$  for a permutation  $\sigma$  on  $\{1, 2, \dots, h\}$ . Writing both sides as linear combinations of class sums, we find that

$$(3.1) \quad (\chi_l(g_1), \dots, \chi_l(g_h))A = \frac{\chi'_{l\sigma}(1)}{\chi_l(1)}(\chi'_{l\sigma}(g'_1), \dots, \chi'_{l\sigma}(g'_h)).$$

To put it in matrix form, let  $M$  be the monomial matrix whose  $(l, l\sigma)$  entry is  $\chi'_{l\sigma}(1)/\chi_l(1)$  (and all other entries are zero), let  $X$  be the character table of  $G$  regarding the fixed orders on classes and characters (by convention,  $X'$  is the character table of  $G'$ ). Then  $XA = MX'$ . Solving for  $A$  using the first orthogonality relation yields

$$(3.2) \quad a_{ij} = \frac{1}{|G|} \sum_{l=1}^h \frac{|\mathcal{C}_i|\chi_l(g_i^{-1})}{\chi_l(1)} \chi'_{l\sigma}(1)\chi'_{l\sigma}(g'_j).$$

We can assume that the quotient field of  $S$  is embedded in a field containing  $\mathbb{C}$ , so that  $a_{ij} \in \mathbb{Q}(\zeta)$  where  $\zeta$  is a complex primitive  $|G|$ -th root of unity. We remark that the Galois group of  $\mathbb{Q}(\zeta)$  over  $\mathbb{Q}$  acts on the entries of  $A$ , for if  $\tau$  is a Galois automorphism, with  $\zeta^\tau = \zeta^n$  for  $n \in \mathbb{N}$  (coprime to  $|G|$ ), then

$$a_{ij}^\tau = \frac{1}{|G|} \sum_{l=1}^h \frac{|\mathcal{C}_{g_i^n}|\chi_l(g_i^{-n})}{\chi_l(1)} \chi'_{l\sigma}(1)\chi'_{l\sigma}(g_j'^n).$$

The following remark, due to Saksonov, shows that all the lemmas from the previous section hold with  $R$  replaced by the more general ring  $S$ .

*Remark 3.1.* The Galois action mentioned before shows in particular that all algebraic conjugates of  $a_{ij}$  lie in  $S$ , and this implies that all  $a_{ij}$  are algebraic integers, by a lemma we can attribute to Saksonov [14, p. 190] (Lemma 3.2.2 in [6]). Thus if  $R$  is the ring generated over  $\mathbb{Z}$  by the entries of  $A$  and  $A^{-1}$ , then  $R$  is a ring of algebraic integers, and  $\psi$  restricts to an isomorphism  $\varphi: \text{Cent}_R(G) \rightarrow \text{Cent}_R(G')$ .

We are now in a position to give a quick proof of the class sum correspondence, which was treated by Berman, Glauberman, Passman and Saksonov (names that should be mentioned at least; cf. [12, (1.1)]). Glauberman and Passman [9, Theorem C] treated the case when the coefficients are algebraic integers while Saksonov’s version [14] is for  $G$ -adapted coefficient rings. Short proofs treating the case when  $\mathbb{Z}$  serves as a coefficient ring can be found in [5, (3.17)], [10, Chapter 14, Lemma 2.3], [16], and [17, (36.5)].

We say that  $\psi$  *preserves the character degrees* if  $\chi_l(1) = \chi'_{l\sigma}(1)$  for  $1 \leq l \leq h$ , or, equivalently, if  $\psi$  can be extended to an isomorphism between the complex group rings  $\mathbb{C}G$  and  $\mathbb{C}G'$ .

We state the class sum correspondence following Saksonov [14] (as reported in [6, Theorem 3.5.8]).

**Theorem 3.2** (Class sum correspondence). *Let  $\psi: \text{Cent}_S(G) \rightarrow \text{Cent}_S(G')$  be an  $S$ -algebra isomorphism. Then  $\psi$  is monomial if and only if it preserves the character degrees.*

*Proof.* By Remark 3.1, we can assume that  $S = R$  and  $\psi = \varphi$  as before. Suppose that  $\varphi$  is monomial. Then the first column of  $A$  is the transpose of  $(1, 0, \dots, 0)$ , so comparing the first entries in (3.1) shows that  $\varphi$  preserves the character degrees. Conversely, suppose that  $\varphi$  preserves the character degrees. Using the second orthogonality relation, (3.2) shows that  $a_{i1} = 0$  for  $i > 1$ , so  $\varphi$  satisfies the assumption of Lemma 2.5 and is therefore monomial. □

*Remark 3.3.* We do not know whether, in general,  $\psi$  preserves the character degrees. What is immediate from (3.1) is that the  $\chi'_{l\sigma}(1)^2/\chi_l(1)$  are integers, so that  $\chi'_{l\sigma}(1)$  and  $\chi_l(1)$  have the same prime divisors (consider also  $\psi^{-1}$ ). The question is whether heights of irreducible characters are preserved. In [11, § 0] it is pointed out that it does not seem immediately obvious that the  $p$ -defects (or heights) of irreducible characters not of height zero can be determined from the knowledge of the isomorphism type of the center of the group ring of  $G$  over a  $p$ -adic ring alone.

We remark that the bilinear form  $(x, y) = \varepsilon'_1((x\psi)(y\psi)^{\sigma'})$  on  $\text{Cent}_S(G)$  is given with respect to the basis formed by the class sums by the matrix  $AD'A^*$ , where  $D' = \text{diag}(|\mathcal{C}'_1|, \dots, |\mathcal{C}'_h|)$  and  $A^*$  is the hermitian transpose of  $A$ . The  $(i, j)$  entry is given by

$$(AD'A^*)_{ij} = \frac{1}{|G|} \sum_{l=1}^h \chi'_{l\sigma}(1)^2 \frac{|\mathcal{C}'_i|\chi_l(g_i^{-1})}{\chi_l(1)} \frac{|\mathcal{C}'_j|\chi_l(g_j)}{\chi_l(1)}.$$

*Remark 3.4.* The groups  $G$  and  $G'$  have identical character tables if the isomorphism  $\psi: \text{Cent}_S(G) \rightarrow \text{Cent}_S(G')$  is monomial and degree preserving. This is easily seen as follows. Choose an ordering of the irreducible characters such that  $\psi$  maps the block idempotent  $e_i$  belonging to the character  $\chi_i$  to the block idempotent  $e'_i$  belonging to the character  $\chi'_i$ . By the definition of a monomial isomorphism, there is a linear character  $\lambda'$  of  $G'$  such that, after suitable ordering of the classes,

$\mathcal{C}_j\psi = \lambda'(g'_j)\mathcal{C}'_j$ , and by Lemma 2.2,  $|\mathcal{C}_j| = |\mathcal{C}'_j|$ . Since  $\psi$  is degree preserving, we can set  $q_{ij} = |\mathcal{C}_j|/\chi_i(1) = |\mathcal{C}'_j|/\chi'_i(1)$ , and obtain

$$q_{ij}\chi_i(g_j)e'_i = (q_{ij}\chi_i(g_j)e_i)\psi = (\mathcal{C}_j e_i)\psi = \lambda'(g'_j)\mathcal{C}'_j e'_i = \lambda'(g'_j)q_{ij}\chi'_i(g'_j)e'_i.$$

So  $\chi_i(g_j) = (\lambda' \otimes \chi'_i)(g'_j)$ , and the  $\lambda' \otimes \chi'_i$  are, of course, again the irreducible characters of  $G'$ .

4. NILPOTENT GROUPS

We keep the notion introduced in the previous sections. The following lemma is designed for an application to nilpotent groups.

**Lemma 4.1.** *Keep the notation and hypotheses from Lemma 2.3. Let  $M$  be the normal subgroup of  $G$  containing  $N$  so that  $M/N$  is the center of  $G/N$ , and define the normal subgroup  $M'$  of  $G'$  analogously. Suppose that  $\varphi$  maps the class sums of elements of  $N$  onto the class sums of elements of  $N'$ . Then  $\varphi$  maps the class sums of elements of  $M$  onto the class sums of elements of  $M'$ .*

*Proof.* Set  $\bar{G} = G/N$  and  $\bar{G}' = G'/N'$ . By the Berman–Higman result (from [3] and [4, p. 27]) and Lemma 2.3, we know that  $\bar{\varphi}$  maps the class sums of elements of  $\bar{M}$  (i.e., the central elements of  $\bar{G}$ ) onto the class sums of elements of  $\bar{M}'$  (i.e., the central elements of  $\bar{G}'$ ).

For  $n \geq 0$ , let  $T_n$  be the set of class sums of elements of  $M$  of length  $n$  (so  $T_0 = \emptyset$ ), and define  $T'_n$  analogously. We shall prove by induction on  $n$  that  $\varphi$  maps  $T_n$  onto  $T'_n$ . For  $n = 0$  this is an empty statement, so let  $n \geq 1$ . Suppose that  $T_n \neq \emptyset$ , and let  $m \in M$  with  $\mathcal{C}_m \in T_n$ . By the Berman–Higman result, there is  $m' \in M'$  with  $\mathcal{C}_{\bar{m}}\bar{\varphi} = \mathcal{C}_{\bar{m}'}$ , and we can write

$$(4.1) \quad \mathcal{C}_m\varphi = \underbrace{\left( \sum_{i=1}^k r_i \mathcal{C}_{m'_i} \right)}_{\bar{m}'_i = \bar{m}' \text{ for all } i} + \underbrace{\left( \sum_{j=1}^l s_j \mathcal{C}_{h'_j} \right)}_{\text{in the kernel of } \pi'}$$

with the class sums  $\mathcal{C}_{m'_1}, \dots, \mathcal{C}_{m'_k}, \mathcal{C}_{h'_1}, \dots, \mathcal{C}_{h'_l}$  pairwise distinct. Since  $\bar{m} \in Z(\bar{G})$  we have  $\mathcal{C}_m \mathcal{C}_m^\circ \in RN$  and so  $\varepsilon_1(\mathcal{C}_m \mathcal{C}_m^\circ) = \varepsilon'_1((\mathcal{C}_m \mathcal{C}_m^\circ)\varphi)$  by assumption, meaning that

$$(4.2) \quad |\mathcal{C}_m| = \left( \sum_{i=1}^k |r_i|^2 |\mathcal{C}_{m'_i}| \right) + \left( \sum_{j=1}^l |s_j|^2 |\mathcal{C}_{h'_j}| \right).$$

By Lemma 2.4 (application of Kronecker’s result), if  $|\mathcal{C}_{m'_i}| \geq |\mathcal{C}_m|$  for some index  $i$ , then  $\mathcal{C}_m\varphi = \mathcal{C}_{m'_i}$ . So let us assume that  $|\mathcal{C}_{m'_i}| < |\mathcal{C}_m|$  for all  $i$ . We will reach a contradiction, showing that  $T_n\varphi \subseteq T'_n$ . By symmetry, then also  $T'_n\varphi^{-1} \subseteq T_n$  and the proof will be complete. By the induction hypothesis, there are  $m_i \in M$  with  $\mathcal{C}_{m_i}\varphi = \mathcal{C}_{m'_i}$ . Then  $\bar{m}_i\bar{\varphi} = \bar{m}'_i = \bar{m}' = \bar{m}\bar{\varphi}$  shows that  $Nm = Nm_i$  for all  $i$ . Set  $\Delta = \mathcal{C}_m - \sum_{i=1}^k r_i \mathcal{C}_{m_i}$ . Then  $\Delta\varphi = \sum_{j=1}^l s_j \mathcal{C}_{h'_j}$ . Since  $\Delta\Delta^\circ \in RN$ , we have  $\varepsilon_1(\Delta\Delta^\circ) = \varepsilon'_1((\Delta\varphi)(\Delta\varphi)^\circ)$ , which gives

$$(4.3) \quad |\mathcal{C}_m| + \left( \sum_{i=1}^k |r_i|^2 |\mathcal{C}_{m_i}| \right) = \left( \sum_{j=1}^l |s_j|^2 |\mathcal{C}_{h'_j}| \right).$$

From (4.2) and (4.3) it follows that all  $r_i$  are zero, so that (4.1) gives the desired contradiction  $|\mathcal{C}_m|\bar{m}' = \mathcal{C}_m\pi\bar{\varphi} = \mathcal{C}_m\varphi\pi' = 0$ . □

**Theorem 4.2.** *Let  $G$  and  $G'$  be finite nilpotent groups, and let  $S$  be a  $G$ -adapted ring. Then all isomorphisms between  $\text{Cent}_S(G)$  and  $\text{Cent}_S(G')$  as  $S$ -algebras (if there are any) are monomial and preserve the character degrees.*

*Proof.* By Remark 3.1, we can assume that  $S = R$  and  $\psi = \varphi$  as above. By Lemma 2.2 we can further assume that  $\varphi$  is normalized, and then we only need to show that  $\varphi$  maps class sums to class sums, by Theorem 3.2.

Set  $Z_n = Z_n(G)$ , the  $n$ -th term of the upper central series of  $G$  (so  $Z_0 = 1$  and  $Z_1 = Z(G)$ ). Use similar notation for  $G'$ . We prove by induction on  $n$  that  $\varphi$  maps the class sums of elements of  $Z_n$  onto the class sums of elements of  $Z'_n$ . For  $n = 0$ , there is nothing to prove, so we can let  $n \geq 1$  when the statement follows from the induction hypothesis and Lemma 4.1, applied with  $N = Z_{n-1}$  and  $M = Z_n$  (and the corresponding normal subgroups of  $G'$ ).  $\square$

By Remark 3.4, we obtain as a corollary:

**Corollary 4.3.** *The finite nilpotent groups  $G$  and  $G'$  have identical character tables if and only if  $\text{Cent}_{\mathbb{Z}}(G) \cong \text{Cent}_{\mathbb{Z}}(G')$  as rings.*

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