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ON ITERATIVE SOLUTION OF NONLINEAR HEAT-CONDUCTION AND DIFFUSION PROBLEMS

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INTRODUCTION

Let G be a bounded domain in the n-dimensional Euclidean space R^n with the boundary Γ and let $S = [0, T]$ be a bounded (time-) interval. Set $Q =]0, T[\times G$ and $\Sigma =]0, T[\times \Gamma$.

We consider in the present paper initial-boundary-value problems of the form

$$(0.1) \qquad e(w)\, w' - \operatorname{div}\,(f(w) \operatorname{grad} w) = q \quad \text{in} \quad Q\,,$$
$$w(0, x) = a(x)\,, \quad x \in G\,; \quad w = b_0 \quad \text{on} \quad \Sigma\,.$$

For such problems we shall establish approximation methods under natural conditions on the material functions e and f, the right-hand side q, the initial values a and the boundary values b_0. Especially we shall show that the solution of (0.1) can be calculated by the successive solution of problems of the form

$$z_i' - \Delta z_i = r(t, x, z_{i-1}) \quad \text{in} \quad Q\,, \quad i = 1, 2, \ldots\,,$$
$$z_i(0, x) = 0\,, \quad x \in G\,; \quad z_i = 0 \quad \text{on} \quad \Sigma\,.$$

The paper consists of six sections. In the first section we introduce the notation used in the following. The formulation of the basic assumptions, the precise statement of the problem and a comparison principle for the problem (0.1) are given in the second section. In the third section we prove the main result of the paper concerning the convergence of the iteration method mentioned above. A procedure of Galerkin type for solving the problem (0.1) is considered in the fourth section. As we show

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in the fifth section, this procedure may be combined with the iteration method to obtain a projection-iteration method. In the sixth section we sketch an example.

Some results of this paper follow from more general results of the papers [2]–[4]. We shall refer to these papers at the corresponding proofs.

1. NOTATION

Let G be a bounded domain in R^n with a Lipschitzian boundary Γ. We denote by $L^2(G)$, $H^k(G)$, $H_0^1(G)$ and $H^{-1}(G)$ the usual Hilbert spaces (see e.g. [5]). The symbol $(\cdot,\cdot)$ denotes the scalar product in $L^2(G)$ and also the pairing between $H^{-1}(G)$ and $H_0^1(G)$. The symbols $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$ mean the norms in $L^2(G)$, $H_0^1(G)$ and $H^{-1}(G)$, respectively.

Let $S = [0, T]$ be a bounded (time-) interval. In what follows we shall consider functions on S which have values in the Hilbert spaces mentioned above. Let H be a Hilbert space. We denote by $C(S; H)$ the space of continuous functions $u \in (S \to H)$. The Hilbert space $L^2(S; H)$ consists of all Bochner-measurable functions $u \in (S \to H)$ for which $\int_S \|u(t)\|_H^2 \,\mathrm{d}t < \infty$. We denote by $H^k(S; H)$, $k = 1, 2$ the space of functions u belonging to $L^2(S; H)$ together with their derivatives (taken in the sense of the space $\mathscr{D}^*(S; H)$ of distributions on $]0, T[$ with values in H) up to the order k. Accordingly we write u' for the derivative of u with respect to $t \in S$.

The scalar product in $L^2(S; L^2(G))$ and the norms in $L^2(S; L^2(G))$, $L^2(S; H_0^1(G))$ and $L^2(S; H^{-1}(G))$ are denoted by

$$(u, v)_S = \int_S (u(t), v(t)) \,\mathrm{d}t\,, \quad |u|_S^2 = \int_S |u(t)|^2 \,\mathrm{d}t\,,$$

$$\|u\|_S^2 = \int_S \|u(t)\|^2 \,\mathrm{d}t\,, \quad \|r\|_{*S}^2 = \int_S \|r(t)\|_*^2 \,\mathrm{d}t\,,$$

respectively.

We set $Q =]0, T[\times G$ and $\Sigma =]0, T[\times \Gamma$. Let u be any function from $L^2(S; L^2(G))$. In what follows we shall denote by u also the function $(t, x) \mapsto (u(t))(x)$ belonging to $L^2(Q)$ and, consequently, consider "abstract" functions from $L^2(S; L^2(G))$ as "ordinary" functions simultaneously.

Let $F \in (R^1 \to R^1)$ be a given function. For any function $u \in L^2(G)$ $(L^2(S; L^2(G)))$ we denote by $F(u)$ the function $x \mapsto F(u(x))$ $((t, x) \mapsto F((u(t))(x)))$. If F is Lipschitz continuous, then we can define by $u \mapsto F(u)$ mappings of the spaces $L^2(G)$, $H^1(G)$, $L^2(S; L^2(G))$, $H^1(S; L^2(G))$ and $L^2(S; H^1(G))$ into itself. In the following lemma we collect some simple continuity properties of these mappings.

Lemma 1.1. *Let $F \in (R^1 \to R^1)$ be a Lipschitz continuous function. Then the mapping $u \mapsto F(u)$ is* *a Lipschitz continuous mapping of* $L^2(G)$ *into itself,*

a weakly continuous mapping of $H^1(G)$ *into itself,*

a Lipschitz continuous mapping of $L^2(S; L^2(G))$ *into itself.*

Moreover, if $u_n \mapsto u$ *strongly in* $L^2(S; L^2(G))$ *and weakly in* $L^2(S; H^1(G))$ $(H^1(S; L^2(G)))$, *then* $F(u_n) \mapsto F(u)$ *strongly in* $L^2(S; L^2(G))$ *and weakly in* $L^2(S; H^1(G))$ $(H^1(S; L^2(G)))$. *Let finally F have an upper-semicontinuous derivative, then* $u \mapsto F(u)$ *is a continuous mapping of* $H^1(G)$ $(L^2(S; H^1(G)), H^1(S; L^2(G)))$ *into itself.*

2. ASSUMPTIONS. FORMULATION OF THE PROBLEM

We consider the initial-boundary-value problem

$$e(w)\, w' - \operatorname{div}(f(w) \operatorname{grad} w) = q \quad \text{in} \quad Q\,, \tag{2.1}$$

$$w(0, x) = a(x)\,, \quad x \in G\,, \tag{2.2}$$

$$w = b_0 \quad \text{on} \quad \Sigma \tag{2.3}$$

and suppose that the following basic conditions are satisfied throughout this paper:

a) The function f is measurable on R^1 and there exist constants f_1 and f_2 such that

$$0 < f_1 \leqq f(s) \leqq f_2 < \infty \quad \text{for almost all} \quad s \in R^1\,.$$

b) The function e is measurable on R^1 and there exist constants e_1 and e_2 such that

$$0 < e_1 \leqq e(s) \leqq e_2 < \infty \quad \text{for almost all} \quad s \in R^1\,.$$

c) There exists a function $b \in L^2(S; H^1(G))$ such that $b = b_0$ on Σ.

d) The function $x \mapsto \int_0^{a(x)} e(s)\, ds$ belongs to $H^{-1}(G)$.

e) The right-hand side q belongs to $L^2(S; H^{-1}(G))$.

Remark 2.1. Many of the considerations to follow concerning (2.1) may be extended to the more general equation

$$e(x, w)\, w' - \operatorname{div}(f(t, w) \operatorname{grad} w) + g(t, x, w) = q\,.$$

In this case the functions e, f and g must satisfy the following basic conditions:

$$0 < e_1 \leqq e(x, s) \leqq e_2 < \infty\,, \quad x \in G\,, \quad s \in R^1\,,$$

$$0 < f_1 \leqq f(t, s) \leqq f_2 < \infty\,, \quad t \in S\,, \quad s \in R^1\,,$$

$$|g(t, x, s_1) - g(t, x, s_2)| \leqq g_1 |s_1 - s_2|\,, \quad t \in S\,, \quad x \in G\,, \quad s_1, s_2 \in R^1\,.$$

Remark 2.2. For the sake of simplicity, we shall restrict ourselves to Dirichlet boundary conditions. The more general results of the papers [2]–[5] may be applied also to other boundary conditions and to the determination of periodic solutions of (2.1).

We define Lipschitz continuous functions E and F by

$$E(s) = \int_0^s e(p)\,\mathrm{d}p \quad \text{and} \quad F(s) = \int_0^s f(p)\,\mathrm{d}p\,.$$

Then the equation (2.1) can be written in the form

$$(E(w))' - \Delta F(w) = q\,. \tag{2.4}$$

Definition 2.1. *We say that a function $w \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))$ is a solution of (2.1)–(2.3) if $w - b \in L^2(S; H^1_0(G))$, $w(0) = a$ and w satisfies (2.4) understood as an equation in $L^2(S; H^{-1}(G))$.*

Remark 2.3. If $w \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))$, it follows from Lemma 1.1 that $F(w) \in L^2(S; H^1(G))$, $E(w) \in H^1(S; L^2(G))$ and therefore $\Delta F(w) \in L^2(S; H^{-1}(G))$, $(E(w))' \in L^2(S; L^2(G)) \subset L^2(S; H^{-1}(G))$. Thus we see that (2.4) really makes sense in $L^2(S; H^{-1}(G))$.

The notion of a solution just defined makes it possible to state the following comparison theorem.

Theorem 2.1. *Let w be a solution of (2.1)–(2.3). Set*

$$K = \{v \mid v \in L^2(S; H^1_0(G)),\ v \geqq 0 \ \text{a.e. on}\ Q\}\,.$$

If $\int_S (q(t), v(t))\,\mathrm{d}t \leqq 0 \ \forall v \in K$, then

$$\operatorname*{ess\,sup}_G w(t) \leqq M = \max\left(\operatorname*{ess\,sup}_\Sigma b_0,\ \operatorname*{ess\,sup}_G a\right)$$

for every $t \in S$ and also $\operatorname*{ess\,sup}_Q w \leqq M$.

If $\int_S (q(t), v(t))\,\mathrm{d}t \geqq 0 \ \forall v \in K$, then

$$\operatorname*{ess\,inf}_G w(t) \geqq m = \min\left(\operatorname*{ess\,inf}_\Sigma b_0,\ \operatorname*{ess\,inf}_G a\right)$$

for every $t \in S$ and also $\operatorname*{ess\,inf}_Q w \geqq m$.

Proof. We confine ourselves to a proof of the first statement. The proof of the second statement is analogous. Let w be a solution of (2.1)–(2.3). We set $w_1 = E(w)$, $d = E(M)$ and define $z = z(t, x)$ by

$$z(t, x) = \tfrac{1}{2}(|w_1(t, x) - d| + w_1(t, x) - d) = \max(w_1(t, x), d) - d\,.$$

Obviously, we have

$$\operatorname*{ess\,inf}_{Q} z \geqq 0 \quad \text{and} \quad z(0, x) = 0 \quad \text{for almost all} \quad x \in G\,.$$

Using the Lipschitz continuity of the functions E and $x \mapsto |x|$, we obtain by Lemma 1.1 $z \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))$. Since

$$w_1 = E(w) = E(b_0) \quad \text{on} \quad \Sigma\,,$$

it follows that $z \in L^2(S; H_0^1(G))$. Thus we see that $z \in K$. Now, applying the fact that either $z(t, x) = 0$ or $z(t, x) = w_1(t, x) - d$ for all $(t, x) \in Q$, we conclude from (2.4) that for any $t \in S$

$$\begin{aligned}
(2.5) \qquad 0 &= \int_0^t ((E(w))' - \Delta F(w) - q, z)\,\mathrm{d}s \\
&\geqq \int_0^t ((E(w))' - \Delta F(w), z)\,\mathrm{d}s \\
&= \int_0^t \int_G (w_1' z + f(w) \operatorname{grad} w \operatorname{grad} z)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_G \left(z' z + \frac{f(w)}{e(w)} \operatorname{grad} w_1 \operatorname{grad} z\right) \mathrm{d}x\,\mathrm{d}s \\
&= \int_G \frac{1}{2} \int_0^t (z^2)'\,\mathrm{d}s\,\mathrm{d}x + \int_0^t \int_G \frac{f(w)}{e(w)} |\operatorname{grad} z|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\geqq \frac{1}{2} \int_G |z(t, x)|^2\,\mathrm{d}x + \frac{f_1}{e_2} \int_0^t \|z\|^2\,\mathrm{d}s\,.
\end{aligned}$$

This implies $z(t, x) = 0$, i.e. $w_1 \leqq d$, for almost all $x \in G$ and consequently, $\operatorname*{ess\,sup}_{G} w(t) \leqq M$ for every $t \in S$. For $t = T$ we obtain from (2.5) and Friedrichs' inequality

$$\int_Q z^2\,\mathrm{d}Q = 0\,.$$

This means $z = 0$, i.e. $w_1 \leqq d$ almost everywhere and thus $\operatorname*{ess\,sup}_{Q} w \leqq M$. Theorem 2.1 is proved.

Remark 2.4. Let $q = 0$. Then it follows from Theorem 2.1 that $m \leqq w \leqq M$ almost everywhere on Q. Hence the functions f and e may be extended outside the interval $[m, M]$ arbitrarily, for instance by constants. By this, evidently, the problem (2.1)–(2.3) remains unchanged.

Besides the initial-boundary-value problem (2.1)–(2.3) we now consider the initial-value problem

$$(2.6) \qquad Au' - \Delta u = r\,, \quad u(0) = 0\,, \quad u \in L^2(S; H_0^1(G)) \cap H^1(S; L^2(G))\,,$$

where $r = r(t) = \int_0^t (q(s) + \Delta F(b(s)))\,\mathrm{d}s + E(a)$ and the operator A is defined by

$$Av = E(F^{-1}(v + F(b))) \quad \forall v \in L^2(S; L^2(G))\,.$$

The problems (2.1)–(2.3) and (2.6) are equivalent in the following sense.

Theorem 2.2. *If the function w is a solution of the problem* (2.1)–(2.3), *then the function*

$$u(t) = \int_0^t (F(w(s)) - F(b(s)))\,\mathrm{d}s$$

is a solution of the problem (2.6).

Conversely, let u be a solution of (2.6) *and let, in addition, $u' + F(b) \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))$. Then the function*

$$w = F^{-1}(u' + F(b))$$

is a solution of (2.1)–(2.3).

Proof. Let w be a solution of (2.1)–(2.3). Then, obviously, we have

$$u(t) = \int_0^t (F(w) - F(b))\,\mathrm{d}s \in H^1(S; H_0^1(G)) \quad \text{and} \quad u(0) = 0\,.$$

Further, it follows from (2.4) that $\forall t \in S$

$$0 = \int_0^t ((E(w))' - \Delta F(w) - q)\,\mathrm{d}s = E(w(t)) - E(a) - \int_0^t (\Delta F(w) + q)\,\mathrm{d}s =$$

$$= (Au')(t) - \int_0^t \Delta(F(w) - F(b))\,\mathrm{d}s - \int_0^t (q + \Delta F(b))\,\mathrm{d}s - E(a) =$$

$$= (Au')(t) - \int_0^t \Delta u'\,\mathrm{d}s - r(t) = (Au')(t) - \Delta\int_0^t u'\,\mathrm{d}s - r(t) =$$

$$= (Au')(t) - \Delta u(t) - r(t)\,.$$

To prove the second statement, let us assume that u is a solution of (2.6) and $u' + F(b) \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))$. Then

$$w = F^{-1}(u' + F(b)) \in L^2(S; H^1(G)) \cap H^1(S; L^2(G))\,, \quad w - b \in L^2(S; H_0^1(G))\,,$$

$$Au' \in H^1(S; L^2(G)) \quad \text{and} \quad -\Delta u - r \in H^1(S; H^{-1}(G))\,.$$

Since $H^1(S; L^2(G)) \subset H^1(S; H^{-1}(G)) \subset C(S; H^{-1}(G))$, it follows by (2.6) that for every $t \in S$

$$0 = (Au')(t) - \Delta u(t) - r(t) = E(w(t)) - \Delta u(0) - \int_0^t \Delta u' \, ds - r(t) =$$

$$= E(w(t)) - E(a) - \int_0^t (\Delta F(w) + q) \, ds .$$

Hence we conclude $E(w(0)) = E(a)$, i.e. $w(0) = a$, and (2.4). Theorem 2.2 is proved.
With regard to Theorem 2.2 it is useful to introduce

Definition 2.2. *Let u be a solution of (2.6). Then we say that the function $w = F^{-1}(u' + F(b)) \in L^2(S; L^2(G))$ is a generalized solution of (2.1)–(2.3).*

Remark 2.4. The equation (2.6) is a simple example for an evolution equation nonlinear in the time derivative. Existence and uniqueness theorems for such equations have been proved by several authors, see e.g. W. Strauss [7], Y. Konishi [6] and H. Gajewski - K. Gröger [2]–[4].

3. ITERATION METHOD

From the assumptions a)–c) we easily conclude that the operator A satisfies the following basic conditions

(3.1) $$(Av_1 - Av_2, v_1 - v_2)_S \geqq \frac{e_1}{f_2} |v_1 - v_2|_S^2 \quad \text{(strong monotonicity)},$$

(3.2) $$|Av_1 - Av_2|_S \leqq \frac{e_2}{f_1} |v_1 - v_2|_S \quad \text{(Lipschitz continuity)}$$

$$\text{for any} \quad v_1, v_2 \in L^2(S; L^2(G)) .$$

Moreover, A is a potential operator. (Obviously, A has the potential

$$\varphi(v) = \int_Q \int_0^{v + F(b)} E(F^{-1}(s)) \, ds \, dQ .)$$

Theorem 3.1. *There exists a unique generalized solution w of the problem (2.1)–(2.3). Let $w_0 \in L^2(S; L^2(G))$ and $\alpha \in]0, 2f_1/e_2[$ be arbitrary. Then the sequence (w_i) defined by*

(3.3) $$w_i = F^{-1}(u_i' + F(b)), \quad i = 1, 2, \ldots,$$

$$u_i' - \alpha \Delta u_i = u_{i-1}' - \alpha(E(w_{i-1}) - r), \quad u_i(0) = 0,$$

$$u_i \in L^2(S; H_0^1(G)) \cap H^1(S; L^2(G)), \quad u_0' = F(w_0) - F(b)$$

converges to w strongly in $L^2(S; L^2(G))$. *Moreover, the following error estimates hold*:

$$|w_i - w|_S \leqq \frac{f_2}{f_1} \frac{(k(\alpha))^i}{1 - k(\alpha)} |w_1 - w_0|_S = \varepsilon_i \quad (a\ priori),$$

$$|w_i - w|_S \leqq \frac{f_2}{f_1 e_1} |r - Au_i' + \Delta u_i|_S \to 0 \quad (a\ posteriori)$$

$$\text{where} \quad k(\alpha) = \max\left(1 - \alpha\frac{e_1}{e_2},\ \alpha\frac{e_2}{f_1} - 1\right) < 1 .$$

Proof. A result of the paper [2] (Bemerkung 6) implies that the initial-value problem

$$z' - \alpha\Delta z = p + r, \quad z(0) = 0, \quad z \in L^2(S; H_0^1(G)) \cap H^1(S; L^2(G))$$

has a unique solution z for every $p \in L^2(S; L^2(G))$ and every $r \in H^1(S; H^{-1}(G))$. Consequently, we can define an operator B of

$$H^1(S; L^2(G)) \quad \text{into} \quad L^2(S; H_0^1(G)) \cap H^1(S; L^2(G)) \quad \text{by} \quad v \mapsto z = Bv$$

where z is the unique solution of

$$z' - \alpha\Delta z = v' - \alpha(Av' - r), \quad z(0) = 0, \quad z \in L^2(S; H_0^1(G)) \cap H^1(S; L^2(G)) .$$

Using a well-known lemma on potential operators (see [5], Lemma 4.14, Chap. III), we conclude from (3.1) and (3.2) that

$$(3.4) \qquad \begin{aligned} &|z_1' - z_2'|_S^2 + \alpha^2|\Delta(z_1 - z_2)|_S^2 \leqq \\ &\leqq |z_1' - z_2'|_S^2 - 2\alpha(\Delta(z_1 - z_2), z_1' - z_2')_S + \alpha^2|\Delta(z_1 - z_2)|_S^2 = \\ &= |z_1' - z_2' - \alpha\Delta(z_1 - z_2)|_S^2 = |v_1' - v_2' - \alpha(Av_1' - Av_2')|_S^2 \leqq \\ &\leqq (k(\alpha))^2\, |v_1' - v_2'|_S^2 \end{aligned}$$

for any $v_1, v_2 \in L^2(S; H_0^1(G)) \cap H^1(S; L^2(G))$, where $z_1 = Bv_1$ and $z_2 = Bv_2$. The set

$$X = \{v \mid v \in H^1(S; L^2(G)),\ v(0) = 0\}$$

is a Banach space with respect to the norm $\|v\|_X = |v'|_S$. The estimate (3.4) implies that the operator B considered as a mapping of X into itself is strictly contractive. Obviously, we may define the sequence (u_i) by $u_i = Bu_{i-1}$. Thus the statements of our theorem follow from Banach's fixed-point theorem, (3.4) and condition a). Theorem 3.1 is proved.

Remark 3.1. The function k has its minimum

$$k_1 = k(\alpha_1) = \frac{f_2 e_2 - f_1 e_1}{f_1 e_1 + f_2 e_2} \quad \text{at} \quad \alpha = \alpha_1 = \frac{2 f_1 f_2}{f_1 e_1 + f_2 e_2}.$$

We now state a regularity theorem for the solution of (2.6), which we need in the next section in order to establish the Galerkin method.

Theorem 3.2. *Suppose $b \in H^1(S; L^2(G))$ and $a \in L^2(G)$. Then the solution of (2.6) belongs to $H^1(S; H_0^1(G))$.*

We omit here the somewhat lengthy proof of the theorem. We note only that the proof essentially utilizes the potentiality of the operator A.

We conclude this section with an extension of Theorem 3.1.

Theorem 3.3. *Suppose $b \in H^1(S; L^2(G))$, $\Delta F(b) + q \in L^2(S; L^2(G))$ and $F(a) - F(b(0)) \in H_0^1(G)$. Then the problem (2.1)–(2.3) has a unique solution w. For arbitrary $\alpha \in]0, 2f_1/e_2[$ the sequence (w_i) defined by (3.3) converges strongly to w in $C(S; L^2(G))$, provided the starting element w_0 satisfies the conditions $w_0(0) = a$ and $w_0 \in H^1(S; L^2(G))$. Moreover, the error estimate*

$$\|w_i - w\|_{C(S; L^2(G))} \leqq c(\varepsilon_i)^{1/2}$$

holds where the constant c may be calculated explicitly. Suppose the function f is lower-semicontinuous, then the sequence (w_i) converges strongly to w also in $L^2(S; H^1(G))$. Finally, suppose f is Lipschitz continuous and $b \in L^2(S; H^2(G))$, then w belongs to $L^2(S; H^2(G))$.

Proof. Since $b \in H^1(S; L^2(G))$, we have because of Lemma 1.1 $F(b) \in H^1(S; L^2(G)) \subset C(S; L^2(G))$. Thus the operator $A \in (L^2(S; L^2(G)) \to L^2(S; L^2(G)))$ has the representation

$$(Av)(t) = A(t)\, v(t) = E(F^{-1}(v(t) + F(b(t)))) \quad \forall t \in S\,, \quad \forall v \in L^2(S; L^2(G))$$

with potential operators $A(t) \in (L^2(G) \to L^2(G))$ satisfying the estimate

$$(3.5) \qquad |A(t)\, y - A(s)\, y| \leqq \frac{e_2}{f_1} f_z |b(t) - b(s)|\; t\,,\; s \in S\,, \quad y \in L^2(G)\,.$$

Taking into account (3.1), (3.2) and (3.5) we can easily conclude our assertions from a result proved in [3] (Satz 3). (In [3] A is assumed to satisfy a slightly different condition with respect to the time dependence. However, it is easy to see that the results of [3] remain valid under the condition (3.5).)

The space $H_0^1(G)$ is separable. Let $(h_k) \subset H_0^1(G)$ be a sequence of coordinate functions complete in $H_0^1(G)$ and let

$$H_n = \operatorname{span}(h_1, \ldots, h_n).$$

H_n provided with the $L^2(G)$-scalar product is a Hilbert space.

Definition 4.1. *A function $u_n \in H^1(S; L^2(G))$ with the representation*

$$u_n(t) = \sum_{j=1}^{n} a_n^{(j)}(t)\, h_j$$

is called the n-th Galerkin approximation of the solution u of (2.6) if u_n is a solution of

$$(4.1) \qquad (Au_n' - \Delta u_n - r, h)_S = 0 \quad \forall h \in L^2(S; H_n), \quad u_n(0) = 0.$$

Lemma 4.1. *For each $n = 1, 2, \ldots$ there exists a unique n-th Galerkin approximation u_n of the solution u of (2.6). Let u_{n0}' be an arbitrary starting element with $u_{n0}' \in L^2(S; L^2(G))$ and let $\alpha \in \left]0, 2f_1/e_2\right[$. Then the sequence $(u_{ni})_{i=1,2\ldots}$ defined by*

$$(4.2) \qquad (u_{ni}' - \alpha\Delta u_{ni}, h_k) = (u_{ni-1}' - \alpha(Au_{ni-1}' - r), h_k), \quad k = 1, 2, \ldots, n,$$

$$u_{ni}(0) = 0, \quad u_{ni} \in H^1(S; H_n)$$

converges strongly to u_n in $H^1(S; L^2(G))$. Moreover, the error estimate

$$|u_{ni}' - u_n'|_S \leqq \frac{(k(\alpha))^i}{1 - k(\alpha)} |u_{n1}' - u_{n0}'|_S, \quad k(\alpha) = \max\left(1 - \alpha\frac{e_1}{f_2}, \alpha\frac{e_2}{f_1} - 1\right)$$

holds. Finally, there exists a constant c independent of n such that

$$(4.3) \qquad \|u_n\|_{C(S; H^1_0(G))} \leqq c.$$

Proof. The apriori estimate (4.3) holds because of

$$\int_0^t \frac{e_1}{f_2} |u_n'|^2 \, ds + \tfrac{1}{2}\|u_n(t)\|^2 \leqq \int_0^t (Au_n' - AO - \Delta u_n, u_n')\, ds = \int_0^t (r - AO, u_n')\, ds \leqq$$

$$\leqq \|r(t)\|_*^2 + \frac{1}{4}\left(\|u_n(t)\|^2 + \frac{f_2}{e_1}|AO|_S^2\right) + \|r'\|_{*S}^2 + \int_0^t \left(\frac{1}{4}\|u_n(s)\|^2 + \frac{e_1}{f_2}|u_n'(s)|^2\right) ds.$$

Further, it follows from the well-known results on systems of ordinary differential equations that we can define an operator $B_n \in (H^1(S; H_n) \mapsto H^1(S; H_n))$ by $v \mapsto z = B_n v$ where z is the solution of

$$(z' - \alpha \Delta z, h_k) = (v' - \alpha(Av' - r), h_k), \quad k = 1, 2, \ldots, n,$$
$$z(0) = 0, \quad z \in H^1(S; H_n).$$

Using the lemma on potential operators which was already used to prove Theorem 3.1, we obtain for any $v_1, v_2 \in H^1(S; H_n)$ and $z_1 = B_n v_1$, $z_2 = B_n v_2$

$$|z_1' - z_2'|_S^2 + \frac{\alpha}{2} \|z_1(T) - z_2(T)\|^2 = (v_1' - v_2' - \alpha(Av_1' - Av_2'), z_1' - z_2')_S \leqq$$
$$\leqq k(\alpha) |v_1' - v_2'|_S |z_1' - z_2'|_S$$

and hence

$$|(B_n v_1)' - (B_n v_2)'|_S \leqq k(\alpha) |v_1' - v_r^2|_S.$$

Thus the operator B_n is strictly contractive on the space

$$X_n = \{v \mid v \in H^1(S; H_n),\ v(0) = 0\}$$

with respect to the norm $\|v\|_X = |v'|_S$. Since, obviously, we may write (4.2) in the form

$$u_{ni} = B_n u_{ni-1}, \quad i = 1, 2, \ldots,$$

the lemma follows from Banach's fixed point theorem.

Remark 4.1. The solution u_{ni} of (4.2) has the representation

$$u_{ni}(t) = \sum_{j=1}^{n} a_{ni}^{(j)}(t)\, h_j, \quad a_{ni}^{(j)}(0) = 0, \quad j = 1, \ldots, n.$$

Hence we can see that (4.2) is an initial-value problem for a system of linear ordinary differential equations which determines the vector $(a_{ni}^{(1)}, \ldots, a_{ni}^{(n)})$.

Theorem 4.1. *Let $b \in H^1(S; L^2(G))$, $a \in L^2(G)$ and let u_n denote the n-th Galerkin approximation of the solution u of (2.6). Then the sequence (w_n) defined by*

$$w_n = F^{-1}(u_n' + F(b)), \quad n = 1, 2, \ldots \tag{4.4}$$

converges strongly in $L^2(S; L^2(G))$ to the generalized solution w of (2.1)–(2.3).

Proof. From Theorem 3.2 we know that $u \in H^1(S; H_0^1(G))$. We denote by P_n the orthogonal projector of $H_0^1(G)$ onto H_n and set $y_n = P_n u$. Obviously, we have $y_n' = (P_n u)' = P_n u'$ and, consequently,

$$\|y_n'(t)\| = \|P_n u'(t)\| \leqq \|u'(t)\| \quad \text{and} \quad \|y_n'(t) - u'(t)\| \to 0$$

for almost all $t \in S$. Hence by Lebesgue's theorem we obtain $\|y_n' - u'\|_S \to 0$. This and (4.1) implies that

$$0 = (Au_n' - Au' - \Delta(u_n - u),\ u_n' - y_n')_S =$$

$$= (Au_n' - Au' - \Delta(u_n - u),\ u_n' - u')_S + (Au_n' - Au' - \Delta(u_n - u),\ u' - y_n')_S \geqq$$

$$\geqq \frac{e_1}{f_2} |u_n' - u'|_S^2 + \tfrac{1}{2}\|u_n(T) - u(T)\|^2 - \frac{e_2}{f_1} |u_n' - u'|_S\, |u' - y_n'|_S -$$

$$- \|u_n - u\|_S\, \|u' - y_n'\|_S\,.$$

Applying the a priori estimate (4.3) and Friedrichs' inequality we find that

$$|u_n' - u'|_S \leqq c_1 \|u' - y_n'\|_S$$

where the constant c_1 is independent of n. Thus we see that

$$|w_n - w|_S \leqq \frac{1}{f_1} |u_n' - u'|_S \leqq \frac{c_1}{f_1} \|u' - y_n'\|_S \to 0\,.$$

Theorem 4.1 is proved.

We conclude this section with an extension of Theorem 4.1.

Theorem 4.2. *Suppose that $b \in H^1(S; L^2(G))$, $\Delta F(b) + q \in L^2(S; L^2(G))$, $b(0) = a$ and $h_k \in H^2(G) \cap H_0^1(G)$ for $k = 1, 2, \ldots$. Then the sequence (w_n) defined by (4.4) converges strongly in $C(S; L^2(G))$ to the solution w of (2.1)–(2.3). If, in addition, f is lower-semicontinuous, then (w_n) converges strongly to w also in $L^2(S; H^1(G))$.*

Theorem 4.2 follows from results of the paper [4] (Satz 2.3 and Bemerkung 2.3).

5. PROJECTION-ITERATION METHOD

In the following theorem we combine the iteration method (3.3) with the projection method (4.1), (4.4).

Theorem 5.1. *Suppose that $b \in H^1(S; L^2(G))$ and $a \in L^2(G)$. Let $z_0 \in L^2(S; L^2(G))$ and $\alpha \in]0, 2f_1/e_2[$ be arbitrary. Then the projection-iteration sequence (z_n) defined by*

(5.1) $$z_n = F^{-1}(v_n' + F(b))\,, \quad n = 1, 2, \ldots,$$

$$(v_n' - \alpha\Delta v_n, h_k) = (v_{n-1}' - \alpha(E(z_{n-1}) - r), h_k)\,, \quad k = 1, \ldots, n\,,$$

$$v_n(0) = 0\,, \quad v_n \in H^1(S; H_n)\,; \quad v_0 = F(z_0) - F(b)$$

converges strongly in $L^2(S; L^2(G))$ to the generalized solution w of (2.1)–(2.3).

Proof. Evidently, we may write (5.1) in the form

$$v_n = B_n v_{n-1}\,, \quad n = 1, 2, \ldots$$

where $B_n \in (X_n \to X_n)$ is the strictly contractive operator which was introduced to prove Lemma 4.1. Now the Galerkin approximation u_n is a fixed point of B_n. Moreover, it follows from the proof of Theorem 4.1 that $|u_n' - u'|_S \to 0$. Hence, using a result of [5] (Lemma 3.2, Chap. III), we obtain $|v_n' - u'|_S \to 0$. This implies

$$|z_n - w|_S \leqq \frac{1}{f_1} |v_n' - u'|_S \to 0\,.$$

Theorem 5.1 is proved.

From results of [4] (Satz 3.3 and Bemerkung 3.1) we derive the following extension of Theorem 5.1.

Theorem 5.2. *Under the assumptions of Theorem 4.2 the sequence (z_n) defined by (5.1) converges strongly in $C(S; L^2(G))$ to the solution w of (2.1)–(2.3), provided $\alpha \in]0, 2f_1/e_2[$ and the starting element z_0 satisfies the conditions $z_0 \in H^1(S; L^2(G))$ and $z_0(0) = a$. If, in addition, f is lower-semicontinuous, then (z_n) converges strongly to w also in $L^2(S; H^1(G))$.*

Remark 5.1. In order to determine the function v_n and hence the function z_n, the main task is to solve a system of linear ordinary differential equations.

Remark 5.2. The numerical realization of (5.1) is especially simple if the eigenfunctions of the $-\Delta$-operator are available. Indeed, let $h_k \in H_0^1(G) \cap H^2(G)$, $k = 1, 2, \ldots$, be the eigenfunction corresponding to the k-th eigenvalue λ_k associated with the problem $-\Delta h = \lambda h$, $h|_\Gamma = 0$. Then it holds

$$(-\Delta h_k, h_l) = \lambda_k \delta_{kl}\,.$$

Consequently, we may write (5.1) in the form

$$(5.2) \qquad z_n = F^{-1}(v_n' + F(b))\,, \quad n = 1, 2, \ldots\,, \quad v_n = \sum_{k=1}^{n} b_n^{(k)} h_k\,, \quad b_{n-1}^{(n)} = 0\,,$$

$$b_n^{(k)}(t) = e^{-\alpha\lambda_k t} \int_0^t e^{\alpha\lambda_k s}((b_{n-1}^{(k)})'(s) - \alpha(E(z_{n-1}(s)) - r(s), h_k))\,\mathrm{d}s\,,$$

$$k = 1, \ldots, n\,, \quad v_0' = F(z_0) - F(b)\,.$$

In this case, in order to determine the function z_n we have mainly to calculate integrals. Finally, the following a posteriori error estimate is valid (see e.g. [4])

$$|z_n - w|_S \leqq \frac{f_2}{f_1 e_1} |r - Av_n' + \Delta v_n|_S \to 0\,.$$

We consider the motion of the interface between two diffusing substances which undergo a chemical reaction, the products of which do not take part in the diffusion process. This phenomenon leads (see e.g. Cannon and Fasano [1]) to the following problem:

To find domains $G_i(t)$, $i = 1, 2$, $t \in]0, T[$ with

$$G = G_1(t) \cup G_2(t)\,, \quad G_1(t) \cap G_2(t) = \emptyset$$

and nonnegative functions $w_i(t)$ defined on $\overline{G_i(t)}$ such that for every $t \in S$ it holds

$$
\begin{aligned}
(6.1)\qquad & e_i(w_i)\, w_i' - \operatorname{div}\,(f_i(w_i)\,\operatorname{grad}\, w_i) = 0 \quad \text{in} \quad G_i\,,\\
& w_i(0) = a_i > 0 \quad \text{in} \quad G_i(0)\,,\\
& w_i(t) = 0 \quad \text{on} \quad \Gamma_0(t) = \overline{G_1(t)} \cap \overline{G_2(t)}\,,\\
& w_i(t) = b_{i0}(t) > 0 \quad \text{on} \quad \Gamma_i(t) = \Gamma \cap \overline{G_i(t)} - \Gamma_0(t)\,,\\
& \nu\, f_1(w_1)\, \frac{\partial w_1}{\partial n_1} = f_2(w_2)\, \frac{\partial w_2}{\partial n_2} \quad \text{on} \quad \Gamma_0(t)\,.
\end{aligned}
$$

Here e_i and f_i are material functions, a_i and b_{i0} are given initial and boundary values, respectively. ν is a given positive constant. Finally, n_i denotes the direction of the exterior normal at Γ_0, relative to G_i.

In order to show that we may apply our results to (6.1) we set in (2.1)–(2.3)

$$
(6.2)\qquad e(s) = \begin{cases} e_1(s) & \text{for} \quad s \geqq 0\,,\\ e_2(-s) & \text{for} \quad s < 0\,,\end{cases} \qquad f(s) = \begin{cases} \nu f_1(s) & \text{for} \quad s \geqq 0\,,\\ f_2(-s) & \text{for} \quad s < 0\,,\end{cases}
$$

$$
a = \begin{cases} a_1 & \text{in} \quad G_1(0)\,,\\ -a_2 & \text{in} \quad G_2(0)\,,\end{cases} \qquad b_0(t) = \begin{cases} b_{10}(t) & \text{on} \quad \Gamma_1(t)\,,\\ -b_{20}(t) & \text{on} \quad \Gamma_2(t)\,.\end{cases}
$$

Then the following statement is valid (see [1]).

Let (G_i, w_i) be a (classical) solution of (6.1). Then w defined by

$$w = \begin{cases} w_1 & \text{in} \quad \bar{G}_1\,,\\ -w_2 & \text{in} \quad \bar{G}_2 \end{cases}$$

is a solution of (6.2), (2.1)–(2.3). Conversely, let w be a sufficiently smooth solution of (6.2), (2.1)–(2.3). Then the couple (G_i, w_i) defined by

$$G_1(t) = \{x \mid x \in G,\ w(t, x) > 0\}\,, \quad G_2(t) = \{x \mid x \in G,\ w(t, x) < 0\}\,,$$

$$w_1 = w \ \text{in} \ \bar{G}_1\,, \quad w_2 = -w \ \text{in} \ \bar{G}_2$$

is a solution of (6.1).

Remark 6.1. Some of our results (especially Theorem 2.1 and the existence and uniqueness assertions of Theorems 3.1 and 3.3) are related to the results which Cannon and Fasano [1] have obtained in studying problems of the form (6.1). However, we have got our results under somewhat more general assumptions and by a completely different technique.

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Souhrn

O ITERAČNÍ METODĚ ŘEŠENÍ NELINEÁRNÍ ROVNICE PRO VEDENÍ TEPLA A DIFUSNÍCH PROBLÉMŮ

H. GAJEWSKI

Předložená práce se zabývá numerickým řešením nelineární rovnice pro vedení tepla. Navrhuje se určitá iterační metoda, ve které jednotlivé iterace se dostanou řešením lineárních rovnic pro vedení tepla. Dokazuje se konvergence této iterační metody za velmi přirozených podmínek kladených na daná vstupní data původní úlohy. V další části práce se studují otázky konvergence Galerkinovy metody aplikované jednak na původní rovnici, jednak na lineární rovnice ve výše zmíněné iterační metodě.

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