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ON JACOBSON RADICAL OF A $\Gamma-$SEMIRING

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Abstract. We introduce the notions of Jacobson radical of a $\Gamma-$semiring and semisimple $\Gamma-$semiring and characterize them via operator semirings.

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1. Introduction

As a continuation of our previous paper on primitive $\Gamma-$semirings, [9], we introduce here the notions of Jacobson radical of a $\Gamma-$semiring and semisimple $\Gamma-$semiring followed by their different characterizations. We obtain the relation between the Jacobson radical of a $\Gamma-$semiring S and that of the right operator semiring R of S, which we use to obtain characterizations of Jacobson radical of a $\Gamma-$semiring analogous to familiar results of the ring theory and semiring theory, [5]. Then, with the help of the notion of subdirect sum of $\Gamma-$semiring, introduced at the outset in a similar way to that in $\Gamma-$ring, [6], and using the result that "a $\Gamma-$semiring S is semisimple if and only if its right operator semiring R is semisimple", a number of characterizations of semisimple $\Gamma-$semiring is obtained.

For preliminaries of semirings, $\Gamma-$semirings, operator semirings of a $\Gamma-$semiring and $\Gamma-$rings we refer to [4], [9], [1], [6] and references therein.

Throughout this paper a $\Gamma-$semiring is assumed to be with zero, the left unity, the right unity. It is also assumed that a $\Gamma S-$semimodule is additively cancellative.

2. Subdirect sum of $\Gamma-$semirings

Let S_i be a Γ_i-semiring for $i = 1, 2$. Then an ordered pair (θ, ϕ) of mappings $(\theta : S_1 \to S_2, \phi : \Gamma_1 \to \Gamma_2)$ is called a *homomorphism* of S_1 into S_2 if (i) θ is a semigroup homomorphism from S_1 into S_2; (ii) ϕ is semigroup isomorphism from Γ_1 onto Γ_2; (iii) for every $x, y \in S_1$, every $\alpha \in \Gamma_1$, $\theta(x\alpha y) = \theta(x)\phi(\alpha)\theta(y)$; (iv) $\theta(0_{S_1}) = 0_{S_2}$. (θ, ϕ) is said to be onto if θ is also onto. Then the *kernel* of (θ, ϕ), denoted by kerθ, defined by kerθ= $\{x \in S_1 : \theta(x) = 0\}$. ker$\theta$ is a $k-$ideal of S_1. If S_1 is additively cancellative then kerθ is an $h-$ideal. Let (θ, ϕ) be a

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homomorphism of a Γ_1-semiring S_1 onto a Γ_2-semiring S_2. (θ, ϕ) is called a *semi-isomorphism* from S_1 onto S_2 if ker$\theta = \{0\}$. If $\Gamma_1 = \Gamma_2 = \Gamma$ and ϕ is the identity mapping, then we henceforth write $\phi = \tau$.

Theorem 2.1. *Let (θ, τ) be a homomorphism from $\Gamma-$semiring S_1 onto $\Gamma-$semiring S_2 with the kernel K. Then S_1/K is semiisomorphic to S_2.*

Proof. The proof is a matter of routine verification. □

For a proper ideal A of a $\Gamma-$semiring S the $\Gamma-$congruence on S, denoted by σ_A, defined as $s\sigma_A s'$ if and only if $s + a_1 + z = s' + a_2 + z$ for some $a_1, a_2 \in A$ and for some $z \in S$, is called the *Izuka* $\Gamma-$*congruence* on S defined by the ideal A. We denote the Izuka $\Gamma-$congruence class of an element r of S by $r[/]A$ and denote the set of all such $\Gamma-$congruence classes of the $\Gamma-$semiring S by $S[/]A$. If the Izuka $\Gamma-$congruence σ_A, defined by A, is proper i.e. $0[/]A \neq S$ then $S[/]A$ is a $\Gamma-$semiring with the following operations: $s[/]A + s'[/]A = (s + s')[/]A$ and $(s[/]A)\alpha(s'[/]A) = (s\alpha s')[/]A$ for all $\alpha \in \Gamma$. We call this $\Gamma-$semiring the *Izuka factor* $\Gamma-$*semiring* of S by A.

If in Theorem 2.1, the $\Gamma-$semiring S_1 is additively cancellative, then K is an $h-$ideal of S_1 and the Izuka factor $\Gamma-$semiring $S_1[/]K$ is semi-isomorphic to S_2.

Let $\{S_i\}_{i\in I}$ be a family of $\Gamma-$semirings indexed by the nonempty set I. Then the Cartesian product $\prod_{i\in I} S_i$ is the set of all functions $x : I \to \bigcup_{i\in I} S_i$ such that the value of x at $i \in I$ is $x_i \in S_i$, $i \in I$. We identify x with $(x_i)_{i\in I}$. Now we define addition (+) and multiplication (.) on $\prod_{i\in I} S_i$ as follows: $(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$ and $(x_i)_{i\in I}\alpha(y_i)_{i\in I} = (x_i\alpha y_i)_{i\in I}$ for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and for all $\alpha \in \Gamma$. With these operations $\prod_{i\in I} S_i$ is a $\Gamma-$semiring. We call this $\Gamma-$semiring the complete direct sum of the family $\{S_i\}_{i\in I}$ of $\Gamma-$semirings. If for all $i \in I$, S_i is with the zero element 0_i then the complete direct sum $\prod_{i\in I} S_i$ is also with the zero element $(0_i)_{i\in I}$. (ii) If each S_i is additively regular then so is $\prod_{i\in I} S_i$. Let $S = \prod_{i\in I} S_i$. We associate with each $k \in I$ a pair of mappings (θ_k, τ) on the $\Gamma-$semiring $\prod_{i\in I} S_i$ onto the the $\Gamma-$semiring S_k as follows: $\theta_k((x_i)_{i\in I}) = x_k$ and $\tau(\alpha) = \alpha$ for all $(x_i)_{i\in I} \in \prod_{i\in I} S_i$ and for $\alpha \in \Gamma$. Clearly, (θ_k, τ) is a homomorphism of S onto S_k. We call (θ_k, τ), for all $k \in I$, the $k-th$ *canonical projection* of S onto S_k. If T is a sub$\Gamma-$semiring of $\prod_{i\in I} S_i$ then $\theta_k(T)$ is a sub$\Gamma-$semiring of S_k for all $k \in I$. A sub$\Gamma-$semiring T of the complete direct sum $\prod_{i\in I} S_i = S$ of the family $\{S_i\}_{i\in I}$ of $\Gamma-$semirings is said to be a *subdirect sum* of the family $\{S_i\}_{i\in I}$ if for each $k \in I$ the $k-$the canonical projection (θ_k, τ) of S restricted to T is such that $\theta_k(T) = S_k$.

Theorem 2.2. *A $\Gamma-$semiring S with zero is semi-isomorphic to a subdirect sum T of (additively cancellative) $\Gamma-$semirings S_i, $i \in I$, with zero elements 0_i if and only if for each $i \in I$ there exists a $k-$ideal ($h-$ideal) P_i of S such that $\bigcap_{i\in I} P_i = \{0\}$.*

Proof. Let S be semi–isomorphic to T and (f, g) be the semi–isomorphism of S onto T. Since T is a subdirect sum of $\Gamma-$semirings $(S_i)_{i\in I}$, then for each

$i \in I$, the $i-$th projection (θ_i, τ) of $\prod_{i\in I} S_i$ is such that $\theta_i(T) = S_i$. Then $(\theta_i of, \tau) = (\phi_i, \tau)$ (say) is an epimorphism of S onto S_i for all $i \in I$. Let P_i =kerϕ_i for all $i \in I$. Then P_i is a $k-$ideal of S for all $i \in I$. Now let $x \in \bigcap_{i\in I} P_i$. Then $\phi_i(x) = 0$ for all $i \in I$ implies that $\theta_i(f(x)) = 0$ for all $i \in I$ and so $f(x) = 0$. Hence $x \in$ker$f$. Since f is a semi–isomorphism, $x = 0$ whence $\bigcap_{i\in I} P_i = \{0\}$.

Conversely, suppose for all $i \in I$ there exists a $k-$ideal P_i of S such that $\bigcap_{i\in I} P_i = \{0\}$. We prove that S is semi–isomorphic to a subdirect sum of the family $\{S/P_i\}_{i\in I}$ of $\Gamma-$semirings. Let us define a pair of mapping (f, τ) from the $\Gamma-$semiring S to the complete direct sum $\prod_{i\in I}(S/P_i)$ by $f(x)(i) = x/P_i$ for all $x \in S$, for all $i \in I$ and τ is as usual the identity semigroup isomorphism on Γ. Clearly, (f, τ) is a homomorphism of the $\Gamma-$semiring S into the $\Gamma-$semiring $\prod_{i\in I}(S/P_i)$. Let $x \in$ker$f$. Then, $f(x)(i) = 0/P_i$ for all $i \in I$ implies that $x/P_i = 0/P_i$ for all $i \in I$, whence $x \in Pi$ for all $i \in I$. So $x = 0$. Hence ker$f = \{0\}$. Also, $f(S) = T$ (say) is a subring of $\prod_{i\in I}(S/P_i)$. Hence (f, τ) is a semi–isomorphism of S onto T. Now, for the $i-$th projection map (θ_i, τ), $\theta_i(T) = \theta_i(f(S)) = \{f(s)(i) : s \in S\} = \{s/P_i : s \in S\} = S/P_i$, implying that T is a subdirect sum of the family $\{S/P_i\}_{i\in I}$ of $\Gamma-$semirings. This completes the proof. □

Similarly, we can prove the theorem when $\Gamma-$semirings S_i, $i \in I$, are aditively cancellative. Then $k-$ideals will be replaced by $h-$ideals and the Bourne factor $\Gamma-$semirings S/P_i by Izuka factor $\Gamma-$semirings $S[/]P_i$.

Theorem 2.3. *Let S and S' be two $\Gamma-$semirings with right operator semirings R and R', respectively. Suppose that there exists a homomorphism (f, τ) of the $\Gamma-$semiring S onto the $\Gamma-$semiring S'. Then R' is semi–isomorphic to $R/(\ker f)^{*'}$.*

Proof. Let us define a mapping $\overline{f} : R \to R'$ by $\overline{f}(\sum_i[\alpha_i, x_i]) = \sum_i[\alpha_i, f(x_i)]$ for $\sum_i[\alpha_i, x_i] \in R$. If $\sum_i[\alpha_i, x_i] = \sum_j\lfloor\gamma_j, y_j\rfloor$ in R then $\sum_i s\alpha_i x_i = \sum_j s\gamma_j y_j$ for all $s \in S$, whence $\sum_i f(s)\alpha_i f(x_i) = \sum_j f(s)\gamma_j f(y_j)$ for all $s \in S$. Since $f : S \to S'$ is surjective, it follows that $\sum_i y\alpha_i f(x_i) = \sum_j y\gamma_j f(y_j)$ for all $y \in S'$, implying that $\sum_i[\alpha_i, f(x_i)] = \sum_j\lfloor\gamma_j, f(y_j)\rfloor$. Thus $\overline{f}$ is well–defined. Clearly, $\overline{f}$ is a semiring homomorphism of R to R'. Let $\sum_{i=1}^m[\alpha_i, y_i] \in R'$. Then there exists $x_i \in S$ such that $f(x_i) = y_i$ for all $i = 1, 2, \ldots, m$ (since f is onto). So $\sum_{i=1}^m[\alpha_i, y_i] = \sum_{i=1}^m[\alpha_i, f(x_i)] = \overline{f}(\sum_{i=1}^m[\alpha_i, x_i])$ where $\sum_{i=1}^m[\alpha_i, x_i] \in R$. Hence, $\overline{f}$ is surjective and so $R/\ker\overline{f}$ is semi–isomorphic to R' (by the fundamental homomorphism theorem of semiring).

Now

$$\begin{aligned}
\ker\overline{f} &= \{\sum_i[\alpha_i, x_i] \in R : \sum_i[\alpha_i, f(x_i)] = 0_{R'}\} \\
&= \{\sum_i[\alpha_i, x_i] \in R : \sum_i y\alpha_i f(x_i) = 0_{S'} \quad \text{for all} \quad y \in S'\} \\
&= \{\sum_i[\alpha_i, x_i] \in R : \sum_i f(x)\alpha_i f(x_i) = 0_{S'} \quad \text{for all} \quad x \in S\}
\end{aligned}$$

$$= \{\sum_i [\alpha_i, x_i] \in R : f(\sum_i x\alpha_i x_i) = 0_{S'} \quad \text{for all} \quad x \in S\}$$
$$= \{\sum_i [\alpha_i, x_i] \in R : \sum_i x\alpha_i x_i \in \ker f \quad \text{for all} \quad x \in S\} = (\ker f)^{*'} .$$

This completes the proof. □

Using Lemma 3.13, [9], we can prove that $R_{S/\ker f}$ is isomorphic to $R/(\ker f)^{*'}$ but later we need the very method of proof employed above.

Theorem 2.4. *Let S be a $\Gamma-$semiring and R be its right operator semiring. Then S is additively cancellative if and only if so is R.*

Proof. Let S be additively cancellative and let $\sum_i[\alpha_i, x_i] + \sum_j\lfloor\beta_j, y_j\rfloor = \sum_i[\alpha_i, x_i] + \sum_k[\gamma_k, z_k]$ in R then $\sum_i s\alpha_i x_i + \sum_j s\beta_j y_j = \sum_i s\alpha_i x_i + \sum_k s\gamma_k z_k$ for all $s \in S$. Since S is additively cancellative, $\sum_j s\beta_j y_j = \sum_k s\gamma_k z_k$ for all $s \in S$. Hence $\sum_j\lfloor\beta_j, y_j\rfloor = \sum_k[\gamma_k, z_k]$, which implies that R is cancellative. Conversely, suppose R is additively cancellative and $x + y = x + z$ in S. This implies that $[x, \alpha] + [y, \alpha] = [x, \alpha] + [z, \alpha]$ for all $\alpha \in \Gamma$ i.e. $[y, \alpha] = [z, \alpha]$ for all $\alpha \in \Gamma$. So $y\alpha s = z\alpha s$ for all $\alpha \in \Gamma$. In particular, $\sum_{j=1}^n y\gamma_j f_j = \sum_{j=1}^n z\gamma_j f_j$ where $\sum_{j=1}^n[\gamma_j, f_j]$ is the right unity of S. This implies that $y = z$. Hence S is additively cancellative. □

Lemma 2.1. *Let S be a $\Gamma-$semiring and R be its right operator semiring. Then $\{0_S\}^{*'} = \{0_R\}$ and $\{0_R\}^* = \{0_S\}$.*

Theorem 2.5. *Let $\{S_i\}_{i\in I}$ be a family of additively cancellative $\Gamma-$semirings and let R_i be the right operator semiring of S_i. Suppose that the $\Gamma-$semiring S is semi–isomorphic to a subdirect sum of $\{S_i\}_{i\in I}$. Then the right operator semiring R of S is semi–isomorphic to a subdirect sum of $\{R_i\}_{i\in I}$.*

Proof. By the proof of Theorem 2.2, for each $i \in I$, there exists a homomorphism (ϕ_i, τ) of S onto S_i such that $\bigcap_{i\in I} \ker\phi_i = \{0\}$, where each $\ker\phi_i$ is an $h-$ideal of S. Now, let us define a mapping $\overline{\phi_i} : R \to R_i$ by $\overline{\phi_i}(\sum_j\lfloor\alpha_j, x_j\rfloor) = \sum_j\lfloor\alpha_j, \phi_i(x_j)\rfloor$ for all $i \in I$ and for all $\sum_j\lfloor\alpha_j, x_j\rfloor \in R$. Then for each $i \in I$, $\overline{\phi_i}$ is a surjective semiring homomorphism of R onto R_i and $\ker\overline{\phi_i} = (\ker\phi_i)^{*'}$ (by the proof of Theorem 2.3). This implies that $\bigcap_{i\in I} \ker\overline{\phi_i} = \bigcap_{i\in I}(\ker\phi_i)^{*'} = (\bigcap_{i\in I} \ker\phi)^{*'} = \{0_S\}^{*'} = \{0_R\}$ (by Lemma 2.1). Hence by Theorem 2.2, [7], R is semi–isomorphic to a subdirect sum of the family $\{R_i\}_{i\in I}$ of semirings. □

Theorem 2.6. *Let $\{S_i\}_{i\in I}$ be a family of additively cancellative primitive $\Gamma-$semirings. If a $\Gamma-$semiring S is semi–isomorphic to a subdirect sum of $\{S_i\}_{i\in I}$ then the right operator semiring R of S is semi–isomorphic to a subdirect sum of a family of additively cancellative primitive semirings.*

Proof. Let R_i be the right operator semiring of S_i for all $i \in I$. Then by Theorem 3.17, [9], and Theorem 2.17, R_i is a primitive and additively cancellative semiring for all $i \in I$. Now, by Theorem 2.5, R is semi–isomorphic to a subdirect sum of $\{R_i\}_{i\in I}$. This completes the proof. □

3. Jacobson radical of $\Gamma-$semiring

Let S be a $\Gamma-$semiring and I be the set of all irreducible $\Gamma S-$semimodules. Then $J(S) = \bigcap_{M\in I} A_S(M)$ is called the *Jacobson radical* of S. If I is empty, then S itself is considered as $J(S)$ and in that case we say that S is a radical $\Gamma-$semiring. The zeroid $Z(S)$ of a $\Gamma-$semiring S is contained in $J(S)$ since $Z(S) \subseteq A_S(M)$ for all $\Gamma S-$semimodule M.

Proposition 3.1. *Let S be a $\Gamma-$semiring. Then $J(S)$ is an $h-$ideal of S and also a $k-$ideal of S.*

Proof. The proposition follows from the fact that $A_S(M)$ is an $h-$ideal of S (Proposition 3.9, [9]). Since every $h-$ideal is also a $k-$ideal, $J(S)$ is also a $k-$ideal of S. □

Theorem 3.1. *Let S be a $\Gamma-$semiring and R be its right operator semiring. Then $J(S) = J(R)^*$ and $J(R) = J(S)^{*'}$ where $J(R) = \bigcap A_R(M)$, intersection runs over all irreducible $R-$semimodules ([5]) M and $A_R(M) = \{x \in R : xM = \{0\}\}$.*

Proof. Since "M is an irreducible $\Gamma S-$semimodule if and only if M is an irreducible $R-$semimodule" (Proposition 3.8, [9]) and $A_S(M)^{*'} = A_R(M)$ and $A_R(M)^* = A_S(M)$, where M is an irreducible $\Gamma S-$semimodule or $R-$semimodule (Proposition 3.10, [9]), $J(S)^{*'} = (\bigcap_{M\in I} A_S(M))^{*'} = \bigcap_{M\in I} A_S(M)^{*'} = \bigcap_{M\in I} A_R(M) = J(R)$, where I is the set of all irreducible $\Gamma S-$semimodules and hence the set of all irreducible $R-$semimodules. Since $J(S)$ is an $h-$ideal of S (Theorem 3.1), so by Theorem 6.14 ([1]) $J(R)^* = (J(S)^{*'})^* = J(S)$. □

Now we have the following characterization of the Jacobson radical of a $\Gamma-$semiring:

Theorem 3.2. *The Jacobson radical of a $\Gamma-$semiring S is the intersection of all primitive $h-$ideals of S.*

Proof. Let S be a $\Gamma-$semiring. We know that an $h-$ideal P of S is primitive if and only if $P = A_S(M)$ for some irreducible $\Gamma S-$semimodule M (Theorem 3.18, [9]). Now $A_S(M)$ is an ideal of S for any ΓS-semimodule M (by Proposition 3.9, [9]). So the theorem follows from the definition of Jacobson radical. □

Theorem 3.3. *If P is an ideal of a $\Gamma-$semiring S, then $J(P) = P \cap J(S)$, where $J(P)$ is the Jacobson radical of P considered as a $\Gamma-$semiring.*

Proof. We first observe that $P^{*'}$ is the right operator semiring of P ([3]) considered as a $\Gamma-$semiring (in fact $R = S^{*'}$). Hence, by Theorem 3.1, $J(P^{*'}) = J(P)^{*'}$ and so $(J(P^{*'}))^* = (J(P)^{*'})^* = J(P)$ (by Theorem 6.6, [1]). Now, by Proposition 6.5 ([1]), $P^{*'}$ is an ideal of the right operator semiring R of S. So by Theorem 2 ([5]), $J(P^{*'}) = P^{*'} \cap J(R)$, which implies that $(J(P^{*'}))^* = (P^{*'} \cap J(R))^*$, which implies that $J(P) = (P^{*'})^* \cap J(R)^* = P \cap J(S)$ (using Theorem 6.6 [1] and Theorem 3.1). This completes the proof. □

Corollary 3.1. *Let S be a $\Gamma-$semiring. Then $J(S)$, considered as a $\Gamma-$semiring, is a radical $\Gamma-$semiring, i.e. $J(J(S)) = J(S)$.*

Proof. Follows immediately from Theorem 3.3. □

Theorem 3.4. *Let S be a $\Gamma-$semiring and R be its right operator semiring. Let Q be an ideal of R. Then $(J(Q^*))^{*'} = J(Q)$.*

Proof. By Proposition 6.4 ([1]), Q^* is an ideal of S. So by Theorem 3.3, $J(Q^*) = Q^* \cap J(S)$ which implies that $(J(Q^*))^{*'} = (Q^*)^{*'} \cap (J(S))^{*'} = Q \cap J(R)$ (using Theorem 6.6 [1] and Theorem 3.1) $= J(Q)$ (by Theorem 2 [5]). □

Corollary 3.2. *$J(Q^*) = J(Q)^*$ for any ideal Q of R, where R is the right operator semiring of a $\Gamma-$semiring S.*

Proof. $J(Q) = (J(Q^*))^{*'}$ (using Theorem 3.4). So $J(Q)^* = ((J(Q^*))^{*'})^* = J(Q^*)$. □

Theorem 3.5. *Let S be a $\Gamma-$semiring. If $S\Gamma x\Gamma S \subseteq J(S)$ then $x \in J(S)$.*

Proof. Let $S\Gamma x\Gamma S \subseteq J(S)$. Then $[\Gamma, S\Gamma x\Gamma S] \subseteq [\Gamma, J(S)] \subseteq J(S)^{*'}$ (since $S\Gamma J(S) \subseteq J(S)$, $J(S)$ being an ideal) $= J(R)$ (using Theorem 3.1) implying that $[\Gamma, S][\Gamma, x][\Gamma, S] \subseteq J(R)$ implying that $R[\alpha, x]R \subseteq J(R)$ for all $\alpha \in \Gamma$. So by Theorem 5 ([5]), $[\alpha, x] \in J(R)$ for all $\alpha \in \Gamma$. This implies that $x \in J(R)^* = J(S)$ (using Theorem 3.1). This completes the proof. □

4. Semisimple $\Gamma-$semiring

A $\Gamma-$semiring S is said to be *semisimple* if its Jacobson radical $J(S) = \{0\}$. Let S be a $\Gamma-$semiring and P be a (left, right) ideal of S. P is said to be *strongly seminilpotent* if there exists a positive integer n such that $(P\Gamma)^{n-1}P \subseteq Z(S)$, where $(P\Gamma)^{n-1} = (P\Gamma)(P\Gamma)\ldots\ldots$ $(n-1)$ times, $(P\Gamma)^0 P = P$ and $Z(S)$ is the zeroid of S. P is said to be *strongly nilpotent* if there exists a positive integer n such that $(P\Gamma)^{n-1}P = \{0\}$. A strongly nilpotent (left, right) ideal of a $\Gamma-$semiring is strongly seminilpotent.

Theorem 4.1. *Let S be a $\Gamma-$semiring and P be a strongly seminilpotent right ideal of S. Then $P \subseteq J(S)$.*

Proof. If possible, suppose $P \not\subset J(S) = \bigcap_{M\in I} A_S(M)$, where I is the set of all irreducible $\Gamma S-$semimodules. Then there exists an $M \in I$ such that $P \not\subset A_S(M)$. This implies that $M\Gamma P \neq \{0\}$. Since P is strongly seminilpotent, there exists a positive integer n such that $(P\Gamma)^{n-1}P \subseteq Z(S)$. This implies that for $p_j \in P$ $(j = 1, 2, \dots, n)$ and for $\gamma_j \in \Gamma$ $(j = 1, 2, \dots, n-1)$ $p_1\gamma_1p_2\gamma_2 \dots p_{n-1}\gamma_{n-1}p_n + z = z$ for some $z \in S$, which implies that $m\alpha(p_1\gamma_1p_2\gamma_2 \dots p_{n-1}\gamma_{n-1}p_n)+m\alpha z = m\alpha z$ for all $\alpha \in \Gamma$ and for all $m \in M$, which implies that $m\alpha(p_1\gamma_1p_2\gamma_2 \dots p_{n-1}$ $\gamma_{n-1}p_n) = \{0\}$ (since M is additively cancellative) for all $\alpha \in \Gamma$ and for all $m \in M$. This implies that $M\Gamma(P\Gamma)^{n-1}P = \{0\}$. If this relation is true for $n = 1$ then $M\Gamma P = \{0\}$ – contrary to $M\Gamma P \neq \{0\}$. Hence there exists $m \in M$ and a positive integer k such that $m\Gamma(P\Gamma)^{k-1}P \neq \{0\}$ and $m\Gamma(P\Gamma)^kP = \{0\}$. Let ν $(\neq 0)\in m\Gamma(P\Gamma)^{k-1}P \subseteq M$. Since M is irreducible, there exist $a_i, b_j \in S$, $\alpha_i, \beta_j \in \Gamma$, where $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, t$; r, t are positive integers; such that $m + \sum_{i=1}^{r} \nu\alpha_ia_i = \sum_{j=1}^{t} \nu\beta_jb_j$. So $m\delta p + \sum_{i=1}^{r} \nu\alpha_ia_i\delta p = \sum_{j=1}^{t} \nu\beta_jb_j\delta p$ for all $\delta \in \Gamma$ and for all $p \in P$. Since $\sum_{i=1}^{r} \nu\alpha_ia_i\delta p, \sum_{j=1}^{t} \nu\beta_jb_j\delta p \in m\Gamma(P\Gamma)^{k-1}P\Gamma S\Gamma P \subseteq m\Gamma(P\Gamma)^{k-1}P\Gamma P$ (since P is a right ideal of S) $= m\Gamma(P\Gamma)^kP = \{0\}$. Hence $m\delta p = 0$ for all $\delta \in \Gamma$ and for all $p \in P$ implies that $M\Gamma P = \{0\}$ – a contradiction. This completes the proof. □

Corollary 4.1. *If a $\Gamma-$semiring S is semisimple then it does not have any non-zero strongly seminilpotent right ideal and consequently, S does not have any strongly nilpotent right ideal.*

Proof. Follows easily from Theorem 4.1 and the remark made above the theorem. □

Theorem 4.2. *A $\Gamma-$semiring S is semisimple if and only if its right operator semiring R is semisimple.*

Proof. Let the $\Gamma-$semiring S be semisimple. Then its Jacobson radical $J(S) = \{0_S\}$ implies that $J(S)^{*'} = \{0_S\}^{*'}$ implies that $J(R) = \{0_R\}$ (using Theorem 3.1 and Lemma 2.1). Hence R is a semisimple semiring ([5]). Converse follows by reversing the above argument. □

Lemma 4.1. *Let S be a $\Gamma-$semiring and R be its right operator semiring. Let P be an ideal of S and $R_{S[/]P}$ be the right operator semiring of the Izuka factor $\Gamma-$semiring $S[/]P$. Then $R_{S[/]P}$ and $R[/]P^{*'}$ are isomorphic.*

Proof. Easy modification of the proof of Lemma 3.13 ([9]). □

Theorem 4.3. *Let S be a $\Gamma-$semiring. Then both $S/J(S)$ and $S[/]J(S)$ are semisimple, i.e. $J(S/J(S)) = \{J(S)\}$ and $J(S[/]J(S)) = \{J(S)\}$.*

Proof. Let S be a $\Gamma-$semiring and R, $R_{S/J(S)}$ be respectively the right operator semirings of S and $S/J(S)$. By Lemma 3.13 ([9]), $R_{S/J(S)}$ and $R/J(S)^{*'}$ are isomorphic. Now by Theorem 3.1, $J(S)^{*'} = J(R)$. So $R_{S/J(S)}$ and $R/J(R)$ are isomorphic. Again by Theorem 3 ([5]), $R/J(R)$ is a semisimple semiring and so $R_{S/J(S)}$ is a semisimple semiring. Hence, by Theorem 4.2, $S/J(S)$, is semisimple $\Gamma-$semiring. □

In a similar fashion, using Lemma 4.1 and Theorem 4.2, we can prove that $S[/]J(S)$ is semisimple $\Gamma-$semiring.

Theorem 4.4. *If a $\Gamma-$semiring S is semisimple then S is semi–isomorphic to a subdirect sum of primitive $\Gamma-$semirings. Conversely, if a $\Gamma-$semiring S is semi–isomorphic to a subdirect sum of additively cancellative primitive $\Gamma-$semirings, then S is semisimple.*

Proof. Let the $\Gamma-$semiring S be semisimple. Then $J(S) = \{0\}$. Since by Theorem 3.2, $J(S) = \bigcap_{k\in\Lambda} P_k$ where $\{P_k\}_{k\in\Lambda}$ is the family of all primitive $h-$ideals of S, $\bigcap_{k\in\Lambda} P_k = \{0\}$, where each P_k is a $k-$ideal of S (since each $h-$ideal is a $k-$ideal). Then by the proof of the converse part of Theorem 2.2, S is semi–isomorphic to a subdirect sum of $\Gamma-$semirings $\{S/P_k\}_{k\in\Lambda}$, each of which is primitive since each P_k is a primitive ideal.

Conversely, suppose that the $\Gamma-$semiring S is semi–isomorphic to a subdirect sum T of additively cancellative primitive $\Gamma-$semirings $\{S_i\}_{i\in I}$. Let R be the right operator semiring of S and R_i be the right operator semiring of S_i, $i \in I$. Then by the proof of Theorem 2.6, R is semi–isomorphic to a subdirect sum of additively cancellative semirings $\{R_i\}_{i\in I}$. Hence, by Theorem 3.3 ([7]), $J(R) = \{0\}$. Hence R is a semisimple semiring ([5]) and so by Theorem 4.2, S is a semisimple $\Gamma-$semiring. □

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