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ON JAŚKOWSKI – TYPE SEMANTICS FOR THE INTUITIONISTIC PROPOSITIONAL LOGIC

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One of remarkable results of Jaśkowski is the construction of a sequence of finite matrices adequate for the intuitionistic propositional logic. This result was published in 1936 in the paper [2] where only a very condensed sketch of proof is to be found. It was 17 years before a more detailed proof was published in [4] by Rose who worked out some modification of the strategy suggested by Jaśkowski for eluding the lemma which he was unable to prove. A detailed proof following closely the Jaśkowski's strategy is presented in [6]. The sequence of finite matrices adequate for the intuitionistic propositional logic (INT) was obtained by Jaśkowski as a result of alternate application the operation of direct power an s.c. Γ-operation to the two-element Boolean algebra. This Γ-operation of Jaśkowski can be considered as a special case of the sum operation for pseudo-Boolean algebras introduced later by Troelastra [7]. Suppose we are given the pseudo-Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ with the sum $\mathcal{A} \oplus \mathcal{B}$ is the pseudo-Boolean algebra with the universe $A \cup B$ and the lattice ordering $\leqslant_{\mathcal{A}\oplus\mathcal{B}} = \leqslant_{\mathcal{A}} \cup \leqslant_{\mathcal{B}} \cup (A \times B)$. The result of Jaśkowski's Γ-operation performed on the pseudo-Boolean algebra $\mathcal{A}$ is isomorphic to $\mathcal{A} \oplus \mathcal{H}$ where $\mathcal{H}$ is the two-element Boolean algebra. Thus, the diagram of the lattice ordering of $\Gamma(\mathcal{A})$ can be obtained from that of $\mathcal{A}$ by adding the new greatest element. The sequence $\{\mathcal{F}_n : n = 1, 2, \ldots\}$ constructed in [2] is given by the conditions: $\mathcal{F}_1 = \mathcal{H}$, $\mathcal{F}_{n+1} = \Gamma(\mathcal{F}_n^n)$. Denoting the content of $\mathcal{F}_n$ by $E(\mathcal{F}_n)$ one can express the main result of Jaśkowski as follows:

Theorem. $\bigcap(E(\mathcal{F}_n) : n = 1, 2, \ldots) = INT$.

It should be mentioned that the theorem above immediately yields finite approximability of INT.

Let N be the set of positive integers. By Jaśkowski-type sequence determined by a mapping $f : N \to N$ and a pseudo-Boolean algebra $\mathcal{A}$ we mean the sequence $\mathcal{A}^f = \{\mathcal{A}^f_n : n \in N\}$ such that $\mathcal{A}^f_1 = \mathcal{A}$, $\mathcal{A}^f_{n+1} = \Gamma((\mathcal{A}^f_n)^{f(n)})$. Let us note the following generalization of Jaśkowski's theorem:

Proposition 1. *Let $\mathcal{A}^f$ be a Jaśkowski-type sequence. If the algebra $\mathcal{A}$ is finite then the following conditions are equivalent:*

(i) $\bigcap(E(\mathcal{A}^f_n) : n \in N) = INT$;

(ii) for every $n \in N$ there exists $m \in N$ such that $n < f(m)$.

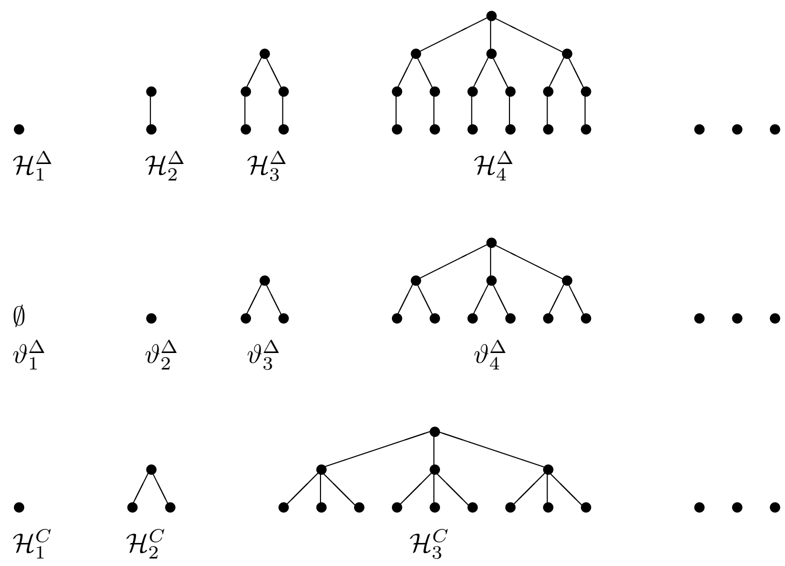
Note that Jaśkowski's theorem follows by Proposition 1 and the observation that the sequence $\{\mathcal{F}_n : n \in N\}$ can be described as $\mathcal{H}^\Delta$ where Δ is the identity mapping of N. Proposition 1 shows that it is impossible to improve the result of Jaśkowski in the direction suggested by McKay [3] who stated erroneously that the sequence $\mathcal{H}^f$ such that $f(n) = 2$ for every $n \in N$ is adequate for INT.

Suppose we are given a sequence of mappings $F = \{f^n : n \in N\}$ such that $f^n : N \to N$ for every $n \in N$. By diagonal sequence determined by F and a pseudo-Boolean algebra $\mathcal{A}$ we mean the sequence $\mathcal{A}^F = \{\mathcal{A}^F_n : n \in N\}$ such that for every $n \in N$, $\mathcal{A}^F_n$ is the n-th algebra of the Jaśkowski-type sequence $\mathcal{A}^{f^n}$ (i.e. $\mathcal{A}^F_n = \mathcal{A}^{f^n}_n$). Let us note the following fact about diagonal sequences (it can be generalized in various ways):

Proposition 2. *Let $\mathcal{A}^F$ be a diagonal sequence. If $n < m$ implies that $f^n(i) < f^m(i)$ for every $n, m, i \in N$ then $\bigcap(E(\mathcal{A}^F_n) : n \in N) = INT$.*

By the propositions stated here to construct a sequence of pseudo-Boolean algebras adequate for INT is very easy. In particular, one can obtain a simple sequence ϑ^Δ where ϑ is the degenerate pseudo-Boolean algebra and a diagonal sequence $\mathcal{H}^C$ where $C = \{c^n : n \in\}$ consists of the constant mappings : $N \to N c^n(i) = n$ for every $n, i \in N$).

It is known (see [1]) that every finite pseudo-Boolean algebra is determined (up to isomorphism) by the ordered set of join-irreducible elements. Moreover, the ordered set of join-irreducible elements considered as a Kripke-frame (see [5]) has the content equal to that of the algebra. Thus, having a sequence of finite pseudo-Boolean algebras adequate for INT one can construct the corresponding sequence of Kripke-frames which also is adequate. The sequences of frames corresponding to $\mathcal{H}^{\Delta}$, ϑ^{Δ} and $\mathcal{H}^{C}$ are described in Smorynski [5] where a very elegant Kripke-style proof of Jaśkowski's theorem is to be found. The frames corresponding to the successive algebras of the sequences $\mathcal{H}^{\Delta}$, ϑ^{Δ} and $\mathcal{H}^{C}$ can be visualized by means of the following diagrams (the empty frame corresponds to the degenerate algebra ϑ):



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