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ON JOSEPH'S CONSTRUCTION OF WEYL GROUP REPRESENTATIONS

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One of the purposes of this paper is to identify a new construction of representations of Weyl groups due to Joseph [6] with Springer's construction (see [10], [11]). More precisely, let O be a nilpotent orbit of a complex semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ the Lie algebra of a maximal unipotent subgroup. To each irreducible component U of $O \cap \mathfrak{n}$, Joseph has attached a certain polynomial p_U on the dual of a Cartan subalgebra and has shown that the p_U's span a W-submodule (W denotes the Weyl group) in the space of polynomials when U runs through the irreducible components of $O \cap \mathfrak{n}$. On the other hand, for a nilpotent $A \in O$, let $\mathscr{B}^A = \{gB \mid g^{-1}A \in \mathfrak{n}\}$ be the S3 variety, where $B = N_G(\mathfrak{n})$ is the Borel subgroup whose unipotent radical has the Lie algebra $\mathfrak{n}$. Springer [10] defined W-module structures on the rational homology groups $H_*(\mathscr{B}^A, \boldsymbol{Q})$ on which also the finite group $C(A) = Z_G(A)/Z_G(A)^\circ$ acts compatibly. The $C(A)$-fixed subspace $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})^{C(A)}$ of the top homology ($d(A) = \dim \mathscr{B}^A$) is known to be W-irreducible. In this note, it will be proved (Theorem 3) that this irreducible W-module is isomorphic (up to the sign representation) to the previous W-module $\sum_U \boldsymbol{Q} p_U$ defined by Joseph. As Joseph has pointed out, it follows from the above identification that the polynomials p_U are harmonic. Furthermore, Spaltenstein [9] has shown that there is a natural surjection σ from the set $I(\mathscr{B}^A)$ of the irreducible components of $\mathscr{B}^A$ onto the set $I(O \cap \mathfrak{n})$ of those of $O \cap \mathfrak{n}$. The above isomorphism is given by the correspondence

$$p_U \mapsto \sum_{C \in \sigma^{-1}(U)} [C] \in H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})^{C(A)}$$

where $[C]$ is the fundamental cycle for an irreducible component C of $\mathscr{B}^A$.

In order to prove the above identification, in §2, we shall rather extend Joseph's idea to obtain a W-module isomorphic to the full homology group $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$. For this, we take the universal covering space $\tilde{O}$ of O and consider $\rho^{-1}(O \cap \mathfrak{n}) \subset \tilde{O}$ ($\rho: \tilde{O} \to O$). We define a W-module

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structure on the $\boldsymbol{Q}$-vector space with basis consisting of the symbols $[V]$, where V runs through the irreducible components of $\rho^{-1}(O \cap \mathfrak{n})$, and show that this W-module is naturally isomorphic to Springer's $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$ (Theorem 2).

In order to prove Theorem 2, we use the local formulas for Springer's representations described in an earlier paper [4]. In §1, we recall these formulas in slightly different forms from those given in [4] (Theorem 1). Since the proof given in [4] uses l-adic cohomology, we shall give an easier proof of Theorem 1 in §4, using Kazhdan-Lusztig's construction [7]. It seems to be known to experts that this construction in [7] coincides with Springer's [10], but there have been no references which give a proof of this fact. In Appendix, we shall prove this.

We have often discussed these subjects with T. Tanisaki, who has strongly helped our understanding. K. Watanabe and M.-N. Ishida have shown us a proof of Lemma 7 in 2.7, which is crucial in this paper, and have taught us some elements of commutative algebras. We here express a great gratitude to them and to the referee whose suggestion motivated us to rewrite Lemma 3 in the first draft.

1. Review of the local formulas of Springer's representations.

1.1. Let G be a connected complex reductive algebraic group, B a Borel subgroup of G and $\mathscr{B} = G/B$ the flag variety. Take a maximal torus T in B. Thus $W = N_G(T)/T$ is a Weyl group. Denote respectively by $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{n}$ the Lie algebras of G, B and the unipotent radical of B. Fix a nilpotent $A \in \mathfrak{g}$ and let $\mathscr{B}^A$ be the S3 variety for A, i.e.,

$$\mathscr{B}^A = \{gB \in G/B \mid A \in \operatorname{Lie}(gBg^{-1}) = g\mathfrak{b}\} = \{gB \mid g^{-1}A \in \mathfrak{n}\} .$$

Here we simply write $xX = \operatorname{Ad}(x)X$ for $x \in G$, $X \in \mathfrak{g}$.

It is known that $\mathscr{B}^A$ is connected and equidimensional of dimension $d(A) = (\dim Z(A) - r)/2$, where $Z(A) = \{g \in G \mid gA = A\}$, $r = \dim T$ (see [9]). Denote by $I(\mathscr{B}^A)$ the set of all irreducible components of $\mathscr{B}^A$. Through the action of $Z(A)$ on $\mathscr{B}^A$, the finite group $C(A) = Z(A)/Z(A)^\circ$ acts on the set $I(\mathscr{B}^A)$ ($Z(A)^\circ$ is the identity component of $Z(A)$). $C(A)$ also acts on the homology group $H_*(\mathscr{B}^A, \boldsymbol{Q})$.

Springer [10] defined an action of the Weyl group W on $H_*(\mathscr{B}^A, \boldsymbol{Q})$ which commutes with that of $C(A)$ and showed that the isotypic components $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})_\xi$ exhaust all the irreducible representations of W when A and ξ run through nilpotent elements of $\mathfrak{g}$ and irreducible representations of $C(A)$, respectively.

1.2. Let $s \in W$ be a simple reflection for the fixed Borel subgroup B, i.e., $P_s = B \cup BsB$ forms a subgroup of G. Since the simple reflections generate W, it is desirable to describe the explicit action of s on a given W-module. The top homology group $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$ has a distinguished basis consisting of the fundamental cycles $[C]$ for irreducible components $C \in I(\mathscr{B}^A)$. Considering

$$H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q}) = \bigoplus_{C \in I(\mathscr{B}^A)} \boldsymbol{Q}[C]$$

as Springer's W-module, we want to describe the action of a simple reflection s with respect to this basis.

We fix a simple reflection s once and for all. Let $P = P_s = B \cup BsB$ be the corresponding parabolic subgroup. Put $\mathscr{P} = G/P$ and

$$\mathscr{P}^A = \{gP \in \mathscr{P} \mid A \in \mathrm{Lie}\,(gPg^{-1})\} .$$

Through the $\boldsymbol{P}^1$-bundle $\mathscr{B} \to \mathscr{P}$, $\mathscr{B}^A$ is mapped surjectively onto $\mathscr{P}^A$. We denote this surjection $\mathscr{B}^A \to \mathscr{P}^A$ by ϕ. It is shown that a fiber of ϕ is isomorphic to $\boldsymbol{P}^0$ (the point) or $\boldsymbol{P}^1$ (the projective line).

DEFINITION 1.

(i) For $C \in I(\mathscr{B}^A)$, an irreducible component of $\mathscr{B}^A$, we call C *s-horizontal* if $\dim C = \dim \phi(C)$. We call C *s-vertical* otherwise, i.e., if $\dim \phi(C) = \dim C - 1$.

(ii) For $C \neq C'$ in $I(\mathscr{B}^A)$, we call (C, C') an *s-joining pair* if $\phi(C') \subset \phi(C)$.

1.3. When C is s-horizontal, $C \to \phi(C)$ is birational since the generic fibers are $\boldsymbol{P}^0$ and we are working over $\boldsymbol{C}$. When C is s-vertical, $C \to \phi(C)$ is a $\boldsymbol{P}^1$-bundle.

If (C, C') is an s-joining pair, then C is s-horizontal, C' is s-vertical and $\phi(C')$ is of codimension 1 in $\phi(C)$. Then C and C' intersect in codimension 1 and $C \cap C'$ is mapped surjectively onto $\phi(C')$. In fact, if both C and C' are s-horizontal (or s-vertical) and $\phi(C') \subset \phi(C)$, then $\phi(C) = \phi(C')$. When s-horizontal, a generic fiber of $C \cup C' \to \phi(C)$ is a point, which implies $C = C'$. When s-vertical, C and C' are the restrictions to $\phi(C)$ of the $\boldsymbol{P}^1$-bundle $\mathscr{B} \to \mathscr{P}$, which implies $C = C'$. Therefore, $\phi(C')$ is of codimension 1 in $\phi(C)$ and C' is a $\boldsymbol{P}^1$-bundle over $\phi(C')$. $C \cap C'$ is the inverse image of $\phi(C')$ under the map $C \to \phi(C)$; hence the required statement.

1.4. We assume to have an s-joining pair (C, C'). We call an irreducible component X of $C \cap C'$ *effective* if $X \to \phi(C')$ is surjective. Clearly we then have $\dim X = \dim \phi(C') = \dim C - 1$. We denote by $d(X, \phi(C'))$ this degree of the map $X \to \phi(C')$.

Let $h: \widetilde{C} \to C$ be the normalization of an s-horizontal component C.

For an irreducible component X' of $h^{-1}(C \cap C')_{\mathrm{red}}$, the reduced inverse image of $C \cap C'$, we also call X' *effective* if $h(X')$ is an effective component of $C \cap C'$. We denote by $E(\tilde{C}, C')$ the set of effective components in $h^{-1}(C \cap C')_{\mathrm{red}}$. An element X' of $E(\tilde{C}, C')$ is a simple divisor of the normal variety $\tilde{C}$ and we have the degree $d(X', \phi(C'))$ of the dominant map $\phi \circ h_{|X'}: X' \to \phi(C')$.

1.5. To an effective component $X' \in E(\tilde{C}, C')$ in $\tilde{C}$, we attach a map $g_{X'}: D \to G$, where D is a unit disk in $\mathbf{C}$. Since X' is a simple divisor in $\tilde{C}$, we can take an imbedding $D \hookrightarrow \tilde{C}$ which intersects X' transversally at a generic point η of X'. Then $D \times X'$ is locally isomorphic to $\tilde{C}$ at η. We may assume $h(x) \in C \cap C'$ for $x \in D$ if and only if $x = 0$. Since the B-bundle $\pi: G \to \mathscr{B}$ is locally trivial in the Zariski topology, we can take a local trivialization near $h(D) \subset C$,

$$h(D) \to G \qquad (h(D) \times B \xrightarrow{\sim} \pi^{-1}(h(D)))\ .$$

Define the analytic map

$$g_{X'}: D \to G$$

to be the composite map $D \to h(D) \to G$.

Lemma 1. *Let $g_{X'}: D \to G$ be the above map. Then $g_{X'}(x)^{-1}A \in \mathfrak{n}$. Let $\mathfrak{m}$ be the Lie algebra of the unipotent radical of P. Then $g_{X'}(x)^{-1}A \in \mathfrak{m}$ if and only if $x = 0$ in D.*

Proof. The first statement follows from $h(x) \in \mathscr{B}^A$. For the second statement, we have $h(x) \in C$ where C is s-horizontal and $h(x) \in C'$ if and only if $x = 0$, where C' is s-vertical. Note that $gB \in C'$, s-vertical, if and only if $gpB \in \mathscr{B}^A$ for any $p \in P$. But then the last condition is equivalent to $p^{-1}g^{-1}A \in \mathfrak{n}$ for any $p \in P$, which is also equivalent to $g^{-1}A \in \mathfrak{m}$. The statement is now clear. q.e.d.

Since $\mathfrak{m}$ is a hyperplane in $\mathfrak{n}$, we have the analytic map

$$\gamma_{X'}: D \to \mathfrak{n}/\mathfrak{m} \simeq \mathbf{C} \qquad (\gamma_{X'}(x) = g_{X'}(x)^{-1}A \bmod \mathfrak{m})$$

such that $\gamma_{X'}(x) = 0$ if and only if $x = 0$. We denote by $m(\gamma_{X'})$ the mapping degree of $\gamma_{X'}$, i.e., the order of $\gamma_{X'}$ at $0 \in D$.

Definition 2. For an s-joining pair (C, C'), we define the positive integer

$$n_C^{C'} = \sum_{X' \in E(\tilde{C}, C')} m(\gamma_{X'}) d(X', \phi(C'))$$

where $m(\gamma_{X'})$ is the above mapping degree and $d(X', \phi(C'))$ is the degree of the map $X' \to \phi(C')$ for an effective component X' in $E(\tilde{C}, C')$ (1.4).

1.6. We normalize Springer's action of the Weyl group W on $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$ so that for $A = 0$ it gives the sign representation and for A regular, the trivial representation. Hence they give the W-modules in [10] multiplied with the sign representation.

THEOREM 1. *The action of a simple reflection s on $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$ with respect to the distinguished basis $\{[C] \mid C \in I(\mathscr{B}^A)\}$ is given as follows;*

(i) $s[C] = -[C]$ *if and only if C is s-vertical.*

(ii) $s[C] = [C] + \sum_{(C,C')} n_C^{C'}[C']$ *if C is s-horizontal. The summation is over all s-joining pairs (C, C') for the C, and $n_C^{C'}$ is the number defined in* Definition 2 (1.5), *which depends on s.*

Though this theorem is slightly different from Theorem 2 in [4], the proof using l-adic cohomology is almost the same. We shall later (§4) give a direct proof relying on Kazhdan-Lusztig's construction [7].

2. An extended version of Joseph's construction.

2.1. In the B-bundle $\pi: G \to \mathscr{B} = G/B$, put

$$G^A = \pi^{-1}(\mathscr{B}^A) = \{g \in G \mid g^{-1}A \in \mathfrak{n}\} .$$

We also write the pulled back B-bundle

$$\pi: G^A \to \mathscr{B}^A .$$

On G^A, B acts on the right, and $Z(A)$, on the left. Let $O(A)$ be the nilpotent orbit in $\mathfrak{g}$ to which A belongs. We consider the covering space

$$\rho: \tilde{O}(A) = Z(A)^\circ \backslash G \to O(A) \subset \mathfrak{g} \qquad (\rho(Z(A)^\circ g) = g^{-1}A) .$$

We then have

$$\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n} \simeq \rho^{-1}(\mathfrak{n}) = Z(A)^\circ \backslash G^A ,$$
$$O(A) \cap \mathfrak{n} \simeq Z(A) \backslash G^A ,$$

the smooth $Z(A)^\circ$-boundle

$$\psi: G^A \to \tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n} ,$$

and the etale map

$$\rho_{|\rho^{-1}(\mathfrak{n})}: \tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n} \to O(A) \cap \mathfrak{n} .$$

We denote by $I(Z)$ the set of all irreducible components of a variety Z. We then have the natural bijections

$$I(\mathscr{B}^A) \to I(G^A) \to I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n}) ,$$

sending C to $\pi^{-1}(C)$ and then to $\psi(\pi^{-1}(C))$, as well as the surjection

$$I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n}) \to I(O(A) \cap \mathfrak{n}) ,$$

sending V to $\rho(V)$. Here the component group $C(A) = Z(A)/Z(A)^{\circ}$ acts on $I(\mathscr{B}^A) \simeq I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$ by permutations, and the last surjection is the quotient by this action of $C(A)$. Note that $\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n}$ and $O(A) \cap \mathfrak{n}$ are also equidimensional.

2.2. We shall interpret the notions related to $I(\mathscr{B}^A)$ in terms of those related to $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$, under the above bijective correspondence

$$\Phi: I(\mathscr{B}^A) \overset{\sim}{\to} I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n}) \qquad (\Phi(C) = \psi(\pi^{-1}(C))) .$$

Especially, we shall find a simpler expression of the numbers $n_C^{C'}$ in terms of the corresponding components $\Phi(C)$, $\Phi(C')$ in $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$. For this we need some notions analogous to those discussed in §1.

We have fixed a simple reflection s, and the parabolic $P = B \cup BsB$. The Lie algebra of the unipotent radical of P was denoted by $\mathfrak{m}$, which is a hyperplane in $\mathfrak{n}$. We first note the following:

LEMMA 2. *For $C \in I(\mathscr{B}^A)$, C is s-vertical if and only if $\rho(\Phi(C)) \subset \mathfrak{m}$.*

PROOF. As in the proof of Lemma 1 (1.5), C is s-vertical if and only if for any $gB \in C$, $g^{-1}A \in \mathfrak{m}$. Thus the lemma.

2.3.

DEFINITION 3. For $V, V' \in I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$, we call (V, V') an *s-joining pair* if $\rho(V) \not\subset \mathfrak{m}$, $\rho(V') \subset \mathfrak{m}$ and if V and V' intersect in codimension 1.

We easily see that (V, V') in $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$ is s-joining if and only if so is $(\Phi^{-1}(V), \Phi^{-1}(V'))$ in $I(\mathscr{B}^A)$.

DEFINITION 4. For an s-joining pair (V, V') in $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$, we call an irreducible component Y of $V \cap V'$ *effective* if the corresponding component $\pi(\psi^{-1}(Y))$ is so for $(\Phi^{-1}(V), \Phi^{-1}(V'))$ in $I(\mathscr{B}^A)$, i.e., $\pi(\psi^{-1}(Y)) \to \phi(\Phi^{-1}(V'))$ is surjective. $(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n} \overset{\psi}{\leftarrow} G^A \overset{\pi}{\to} \mathscr{B}^A \overset{\phi}{\to} \mathscr{P}^A.)$ Denote by $E(V, V')$ the set of all effective components in $V \cap V'$.

In the parabolic subgroup $P = B \cup BsB$, let U_s be the unipotent 1-dimensional subgroup of P such that $U_s \not\subset B$ and $sU_s s^{-1} \subset B$ (the root subgroup corresponding to $-\alpha$ when α is the simple root corresponding to s). Let (V, V') be an s-joining pair in $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$. Since $V' \subset \rho^{-1}(\mathfrak{m})$, P acts on V' (on the right, V' being regarded as a subset of $Z(A)^{\circ} \backslash G^A$).

LEMMA 3. *In the above situation, there exists a non-empty open subset V'_{reg} of V' satisfying*:

(i) *V'_{reg} has a good quotient $V'_{\text{reg}} \to V'_{\text{reg}}/U_s$ under the U_s-action whose fibers are 1-dimensional.*

(ii) *For an irreducible component* Y *of* $V \cap V'$, *the composite map*

$$Y \cap V'_{\mathrm{reg}} \to V'_{\mathrm{reg}} \to V'_{\mathrm{reg}}/U_s$$

is dominant if and only if Y *is effective.*

Takig this V'_{reg} *and an effective* $Y \in E(V, V')$, *denote by* $d(Y, (V'/U_s))$ *the degree of the dominant map*

$$Y \cap V'_{\mathrm{reg}} \to V'_{\mathrm{reg}}/U_s .$$

Then

$$d(Y, (V'/U_s)) = d(\pi(\psi^{-1}Y)), \phi(\Phi^{-1}(V')) ,$$

and hence the right hand side is independent of the choice of the V'_{reg}.

PROOF. We first show the existence of V'_{reg}. Put $C = \Phi^{-1}(V)$, $C' = \Phi^{-1}(V')$ in $I(\mathscr{B}^A)$. Then (C, C') is s-joining. Take an effective component $Y \in E(V, V')$ and put $X = \pi(\psi^{-1}(Y))$, an effective component in $C \cap C'$; hence $X \to \phi(C')$ is surjective. Then $\pi^{-1}(X)P = \pi^{-1}(C') = \psi^{-1}(V')$ in G^A. But then $P = BU_s \cup Bs$ and $\pi^{-1}(X)$ is B-stable. Therefore we have

$$\psi^{-1}(V')/U_s = (\pi^{-1}(X)U_s/U_s) \cup (\pi^{-1}(X)/sU_s s^{-1})s ,$$

where $\pi^{-1}(X)U_s/U_s$ is open dense in $\psi^{-1}(V')/U_s$. Hence

$$\pi^{-1}(X) \to \pi^{-1}(X)U_s/U_s \subset \psi^{-1}(V')/U_s$$

is dominant and the both have the same dimension. Hence this map is generically finite. Using the theorem of Rosenlicht on existence of generic quotients, we can thus take a non-empty open subset $\mathscr{U}_Y$ in $\pi^{-1}(X)U_s/U_s$ such that $\mathscr{U}_Y$ has a good quotient $Z(A)^\circ\backslash\mathscr{U}_Y$ with fiber dimension equal to $\dim Z(A)$. If we take the inverse image $\mathscr{U}'_Y$ of $\mathscr{U}_Y$ in $\psi^{-1}(V')$, then $\psi(\mathscr{U}'_Y) = Z(A)^\circ\backslash\mathscr{U}'_Y$ is open in V' and

$$Y \cap \psi(\mathscr{U}'_Y) \to \psi(\mathscr{U}'_Y)/U_s \simeq Z(A)^\circ\backslash\mathscr{U}_Y$$

is dominant and generically finite. Thus if we put

$$V'_{\mathrm{reg}} = \bigcup_{Y \in E(V,V')} \psi(\mathscr{U}'_Y) \subset V' ,$$

then V'_{reg} satisfies the requirement.

Secondly we show the last statement. For $Y \in E(V, V')$, take $\mathscr{U}_Y \subset \psi^{-1}(V')/U_s$, $\mathscr{U}'_Y \subset \psi^{-1}(V')$ $(\mathscr{U}'_Y/U_s = \mathscr{U}_Y)$ and $\psi(\mathscr{U}'_Y) \subset V'_{\mathrm{reg}}$ as above. We then have the commutative diagram:

$$\begin{array}{ccccc}
Y \cap \psi(\mathscr{U}'_Y) & \xleftarrow[\psi_{\mathrm{rest}}]{} & \psi^{-1}(Y) \cap \mathscr{U}'_Y & \xrightarrow[\pi_{\mathrm{rest}}]{} & X \\
\downarrow & & \downarrow & & \downarrow \\
V'_{\mathrm{reg}}/U_s \supset \psi(\mathscr{U}'_Y)/U_s & \xleftarrow[\bar{\psi}]{} & \mathscr{U}'_Y/U_s = \mathscr{U}_Y & \xrightarrow[\bar{\pi}]{} & \phi(C') .
\end{array}$$

It is enough to show that all three vertical maps have the same degrees. As noticed above, $\bar{\psi}$ is smooth with fibers isomorphic to $Z(A)^\circ$ and so is ψ_{rest}, by definition. The map π_{rest} has fibers isomorphic to B, while $\bar{\pi}$ has fibers isomorphic to $P/U_s \simeq B \amalg (B/sU_s s^{-1})s$ which is birationally isomorphic to B. Applying the following Lemma 4, we immediately see the required statement. q.e.d

The following lemma will be used later again.

LEMMA 4. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & Z_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ T & \xrightarrow{\;g'\;} & T_1 \end{array}$$

be a commutative diagram of irreducible varieties. Assume that f and f_1 are dominant and have finite degrees, and that the fibers of g and g' are irreducible and the restrictions of f to the fibers of g birationally isomorphic to those of g'. Then the degrees of f and f_1 coincide.

PROOF. Easy.

2.4. Let (V, V') be an s-joining pair in $I(\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n})$ and $Y \in E(V, V')$. Then Y is an irreducible reduced component of the scheme $V \times_{\mathfrak{n}} \mathfrak{m}$. Denote by $m(V \times_{\mathfrak{n}} \mathfrak{m}, Y)$ the multiplicity of the component Y in the scheme $V \times_{\mathfrak{n}} \mathfrak{m}$.

We shall prove the following theorem.

THEOREM 2. *For an s-joining pair (V, V') in irreducible components of $\tilde{O}(A) \times_{\mathfrak{g}} \mathfrak{n}$, define the positive number*

$$n_V^{V'} = \sum_{Y \in E(V,V')} m(V \times_{\mathfrak{n}} \mathfrak{m}, Y) d(Y, (V'/U_s))$$

where $d(Y, (V'/U_s))$ is as in Lemma 3, (iii). *Let $(C, C') = (\Phi^{-1}(V), \Phi^{-1}(V'))$ be the corresponding s-joining pair for $\mathscr{B}^A$. Then*

$$n_V^{V'} = n_C^{C'} ,$$

where the right hand side is as in Definition 2 (1.5).

2.5. The number $n_C^{C'}$ involves the normalization $\tilde{C}$ of C. We compare this situation with that involving the normalization $\tilde{V}$ of V. We first note the following general fact.

LEMMA 5. *Let*

$$\begin{array}{ccc} \tilde{Z} & \longrightarrow & \tilde{T} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

be a Cartesian diagram of irreducible varieties. Assume that $Z \to T$ is smooth and $\tilde{T} \to T$ is the normalization. Then $\tilde{Z}$ is the normalization of Z.

PROOF. Since $\tilde{Z} \to \tilde{T}$ is smooth, $\tilde{Z}$ is normal variety. Since $\tilde{T} \to T$ is finite and birational, so is $\tilde{Z} \to Z$. Hence $\tilde{Z} \to Z$ is the normalization. q.e.d.

By this lemma, in the Cartesian products

$$\begin{array}{ccccccc} \tilde{V} & \xleftarrow{\tilde{\psi}} & \psi^{-1}(V)^{\sim} & \simeq & \pi^{-1}(C)^{\sim} & \xrightarrow{\tilde{\pi}} & \tilde{C} \\ {\scriptstyle k}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\ V & \xleftarrow{\psi} & \psi^{-1}(V) & = & \pi^{-1}(C) & \xrightarrow{\pi} & C \end{array}$$

we have an isomorphism $\psi^{-1}(V)^{\sim} \simeq \pi^{-1}(C)^{\sim}$, which is the normalization of $\psi^{-1}(V) = \pi^{-1}(C)$. Let $E(C, C')$, $E(V, V')$ be the sets of effective components in $C \cap C'$, $V \cap V'$ respectively. By Lemma 3, there is a bijective correspondence

$$E(C, C') \ni X \mapsto \psi(\pi^{-1}(X)) \in E(V, V') .$$

The set of effective components $E(\tilde{C}, C')$ consists of irreducible components of $h^{-1}(C \cap C')$ which are mapped surjectively onto effective components in $C \cap C'$. We also call an irreducible component of $k^{-1}(V \cap V')$ in $\tilde{V}$ (k: $\tilde{V} \to V$) *effective* if it is mapped surjectively onto a component in $E(V, V')$. We denote by $E(\tilde{V}, V')$ the set of all effective components in $k^{-1}(V \cap V')$. We then have the natural maps

$$\begin{array}{ccc} E(\tilde{C}, C') & \xrightarrow{\sim} & E(\tilde{V}, V') \\ \downarrow & & \downarrow \\ E(C, C') & \xrightarrow{\sim} & E(V, V') \end{array}$$

where $E(\tilde{C}, C') \ni X' \mapsto \tilde{\psi}(\tilde{\pi}^{-1}(X')) \in E(\tilde{V}, V')$.

2.6. We are now going back to the situation in 1.5. For each effective component $X' \in E(\tilde{C}, C')$ we have taken a unit disk $D \hookrightarrow \tilde{C}$ which intersects X' at a generic point of X'. Then we have defined the map $g_{X'}: D \to G^A$ and

$$\gamma_{X'}: D \to \mathfrak{n}/\mathfrak{m} = \boldsymbol{C} \qquad (\gamma_{X'}(x) = g_{X'}(x)^{-1}A)$$

such that $\gamma_{X'}(x) = 0$ if and only if $x = 0$. The number $n_C^{C'}$ involves the mapping degree $m(\gamma_{X'})$ of $\gamma_{X'}$. Let $Y' = \tilde{\psi}(\tilde{\pi}^{-1}(X')) \in E(\tilde{V}, V')$ be the corresponding effective component in $\tilde{V}$. We note the following:

LEMMA 6. *Keep the above situation. For the maps*

$$\tilde{V} \to V \to \mathfrak{n} \hookleftarrow \mathfrak{m} ,$$

let $\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}$ *be the fiber product scheme. Let* Y' *be a reducible component of* $\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}$. *Denote by* $m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y')$ *the multiplicity of* Y' *in* $\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}$. *Then we have*

$$m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y') = m(\gamma_{X'}) .$$

PROOF. Recall the diagram in 2.5,

$$\begin{array}{ccccc} \tilde{V} & \xleftarrow{\tilde{\psi}} & \psi^{-1}(V)^{\sim} & \xrightarrow{\tilde{\pi}} & \tilde{C} \\ {\scriptstyle k}\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\ V & \xleftarrow{\psi} & \psi^{-1}(V) & \xrightarrow{\pi} & C . \end{array}$$

The map $g_{X'}\colon D \to \psi^{-1}(V) = \pi^{-1}(C) \subset G^A$ factors as

$$\tilde{g}_{X'}\colon D \to \psi^{-1}(V)^{\sim}(\to \psi^{-1}(V))$$

and define the map

$$\iota_{X'}\colon D \overset{\tilde{g}_{X'}}{\hookrightarrow} \psi^{-1}(V)^{\sim} \overset{\tilde{\psi}}{\to} \tilde{V} .$$

By definition, subvarieties D and Y' of $\tilde{V}$ intersects transversally, and the composite map

$$D \overset{\iota_{X'}}{\to} \tilde{V} \to \mathfrak{n} \to \mathfrak{n}/\mathfrak{m}$$

coincides with $\gamma_{X'}$. Now $Y' \subset \tilde{V}$ is the component in $(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m})_{\mathrm{red}}$ which intersects D transversally. By definition, $m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y')$ is the length of the Artinian ring $(\mathcal{O}_{\tilde{V}} \otimes_{\mathcal{O}_{\mathfrak{n}}} \mathcal{O}_{\mathfrak{m}})_{Y'}$, the stalk of the structure sheaf of $\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}$ at the generic point of Y', which is isomorphic to $\mathcal{O}_{\tilde{V},Y'}/f\mathcal{O}_{\tilde{V},Y'}$, where $\mathcal{O}_{\tilde{V},Y'}$ is the local ring of the divisor Y' in $\tilde{V}$ and $f \in C[\mathfrak{n}]$ is the linear form defining the hyperplane $\mathfrak{m}$. It is easy to see that this length equals the mapping degree $m(\gamma_{X'})$ of $\gamma_{X'}$ in the above. q.e.d.

2.7. In order to identify quantities involving multiplicities, we use the following lemma, which may be more or less known. Since we could find no references which includes its proof, we here cite the proof communicated by Watanabe and Ishida.

LEMMA 7. *Let* A *be a geometric local domain of dimension* 1 *and* $\tilde{A}$ *the normalization of* A. *Denote by* $\mathfrak{p}$ *the maximal ideal of* A *and by*

$\kappa(\mathfrak{p}) \simeq A/\mathfrak{p}$ the residue field of $\mathfrak{p}$. Then, for $f \neq 0$ in A,

$$\operatorname{length}_A (A/fA) = \sum_{\mathfrak{p}' \in \operatorname{Max} \tilde{A}} \operatorname{length}_{\tilde{A}_{\mathfrak{p}'}} (\tilde{A}_{\mathfrak{p}'}/f\tilde{A}_{\mathfrak{p}'})[\kappa(\mathfrak{p}'): \kappa(\mathfrak{p})]$$

where $\operatorname{length}_B M$ *denotes the length of a B-module M,* $\operatorname{Max} \tilde{A}$ *the set of maximal ideals in $\tilde{A}$ and $\kappa(\mathfrak{p}')$ the residue field of $\mathfrak{p}'$.* [:] *denotes the degree of a field extension.*

PROOF. Since

$$\begin{aligned}\operatorname{length}_A (\tilde{A}/f\tilde{A}) &= \sum_{\mathfrak{p}' \in \operatorname{Max} \tilde{A}} \operatorname{length}_A (\tilde{A}_{\mathfrak{p}'}/f\tilde{A}_{\mathfrak{p}'}) \\ &= \sum_{\mathfrak{p}' \in \operatorname{Max} \tilde{A}} \operatorname{length}_{\tilde{A}_{\mathfrak{p}'}} (\tilde{A}_{\mathfrak{p}'}/f\tilde{A}_{\mathfrak{p}'})[\kappa(\mathfrak{p}'): \kappa(\mathfrak{p})] ,\end{aligned}$$

it is enough to show the equality

$$\operatorname{length}_A (A/fA) = \operatorname{length}_A (\tilde{A}/f\tilde{A}) .$$

But then the A-module $\tilde{A}/A$ has finite length as an A-module. Hence for the map

$$\bar{f}: \tilde{A}/A \to \tilde{A}/A \qquad (\bar{f}(x + A) = fx + A)$$

we have $\operatorname{length}_A (\operatorname{Ker} \bar{f}) = \operatorname{length}_A (\operatorname{Coker} \bar{f})$. In the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \longrightarrow & \tilde{A}/A & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \longrightarrow & \tilde{A}/A & \longrightarrow & 0 , \end{array} \qquad (f'(x) = fx, (x \in \tilde{A}))$$

f and f' are injective by assumption. By the snake lemma, we have an exact sequence

$$0 \to \operatorname{Ker} \bar{f} \to \operatorname{Coker} f \to \operatorname{Coker} f' \to \operatorname{Coker} \bar{f} \to 0 ,$$

which implies

$$\operatorname{length}_A (\operatorname{Coker} f) = \operatorname{length}_A (\operatorname{Coker} f') .$$

Since $\operatorname{Coker} f = A/fA$ and $\operatorname{Coker} f' = \tilde{A}/f\tilde{A}$, we have the required equality. q.e.d.

Using Lemma 7, we can prove the following lemma.

LEMMA 8. *Let the situation be as in 2.5 and 2.6. Take an effective component Y of $V \cap V'$, $Y \in E(V, V')$. Then*

$$m(V \times_{\mathfrak{n}} \mathfrak{m}, Y) = \sum_{k(Y')=Y, Y' \in E(\tilde{V}, V')} m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y')d(Y', Y)$$

where $k: \tilde{V} \to V$ *is the normalization of* V *and* $d(Y', Y)$ *is the degree of the map* $Y' \to Y$ *under* k.

PROOF. Let $A = \mathcal{O}_{V,Y}$ be the local ring of the divisor Y in V, and $f \in \mathcal{O}_{V,Y}$ the image of the defining form of $\mathfrak{m}$ by the map $C[\mathfrak{n}] \to \mathcal{O}_{V,Y}$. Then, by definition,

$$m(V \times_{\mathfrak{n}} \mathfrak{m}, Y) = \text{length}_A (A/fA) ,$$
$$m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y') = \text{length}_{\tilde{A}_{\mathfrak{p}'}} (\tilde{A}_{\mathfrak{p}'}/f\tilde{A}_{\mathfrak{p}'})$$
$$(\mathfrak{p}' \text{ is the maximal ideal defining } Y') ,$$
$$d(Y', Y) = [\kappa(\mathfrak{p}'): \kappa(\mathfrak{p})] .$$

Hence Lemma 8 follows from Lemma 7. q.e.d.

2.8. We are now in a position to prove Theorem 2 stated in 2.4. Let the assumption be as in Theorem 2. By Lemma 8, we have

$$n_V^{V'} = \sum_{Y \in E(V,V')} m(V \times_{\mathfrak{n}} \mathfrak{m}, Y) d(Y, (V'/U_s))$$
$$= \sum_{Y' \in E(\tilde{V},V')} m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y') d(Y', Y) d(Y, (V'/U_s)) .$$

Under the bijective correspondence (2.5)

$$E(\tilde{C}, C') \xrightarrow{\sim} E(\tilde{V}, V') ,$$

let $Y' = \tilde{\psi}(\tilde{\pi}^{-1}(X'))$, $X' \in E(\tilde{C}, C')$ and $X = h(X') \in E(C, C')$. Using Lemma 5, we also have

$$d(Y', Y) = d(X', X) .$$

By Lemma 3, we have

$$d(Y, (V'/U_s)) = d(X, \phi(C')) .$$

By Lemma 6, we have

$$m(\tilde{V} \times_{\mathfrak{n}} \mathfrak{m}, Y') = m(\gamma_{X'}) .$$

Since $d(X', \phi(C')) = d(X', X) d(X, \phi(C'))$, we have

$$n_V^{V'} = \sum_{X' \in E(\tilde{C},C')} m(\gamma_{X'}) d(X', \phi(C'))$$

which equals $n_C^{C'}$ by Definition 2. The proof of Theorem 2 has thus been completed.

2.9. Thanks to Theorem 2, we have obtained a new construction of Springer's representations. Let $O \subset \mathfrak{g}$ be a nilpotent orbit and $\rho: \tilde{O} \to O$ be the universal covering of O. Fix a Borel subalgebra with nilpotent radical $\mathfrak{n}$ and consider the Weyl group W as a Coxeter system with simple reflections determined by $\mathfrak{n}$. For a simple reflection s, we have the notion: the hyperplane $\mathfrak{m}$ of $\mathfrak{n}$, s-joining pairs in the set $I(\tilde{O} \times_{\mathfrak{g}} \mathfrak{n})$ of irreducible components of $\tilde{O} \times_{\mathfrak{g}} \mathfrak{n}$ etc. ... . On the vector space

$$\mathscr{V}_O = \sum \boldsymbol{Q}[V]$$

with basis $\{[V] \mid V \in I(\tilde{O} \times_{\mathfrak{g}} \mathfrak{n})\}$, define the action of s as follows;

(i) $s[V] = -[V]$ if $\rho(V) \subset \mathfrak{m}$,

(ii) $s[V] = [V] + \sum_{(V,V')} n_V^{V'} [V']$ if $\rho(V) \not\subset \mathfrak{m}$, where $n_V^{V'}$ is the number defined in Theorem 2 and the last summation is over the s-joining pairs for V fixed. From Theorem 2, it follows that these actions of the simple reflections extend to the action of the Weyl group W on $\mathscr{V}_O$ and under the correspondence

$$\mathscr{V}_O \ni [V] \mapsto [C] \in H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$$
$$(A \in O,\ C = \pi(\psi^{-1}(V)) \in I(\mathscr{B}^A))$$

the W-module $\mathscr{V}_O$ is isomorphic to the Springer W-module $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$. Both spaces have natural actions of $\pi_1(O)$ ($\simeq Z(A)/Z(A)^\circ$ for G simply connected) which commute with the W-actions.

3. Identification with Joseph's construction.

3.1. In this section, we shall show that Joseph's representation on the space spanned by $I(O \cap \mathfrak{n})$ coincides with Springer's W-module $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})^{C(A)}$, the subspace fixed by $C(A)$ which is irreducible ($A \in O$).

We keep the situation and the notations as in §2. In the space $\mathscr{V}_O$, we have the W-action such that, for a simple reflection s,

$$s[V] = [V] + \sum_{(V,V')} n_V^{V'}[V'] \quad \text{if} \quad \rho(V) \not\subset \mathfrak{m}\,.$$

We rewrite this expression. Fix s and consider $V \in I(\tilde{O} \times_{\mathfrak{g}} \mathfrak{n})$ such that $\rho(V) \not\subset \mathfrak{m}$. Let $P = B \cup BsB$ be the parabolic subgroup corresponding to s. For a reduced irreducible component Y of $V \times_{\mathfrak{n}} \mathfrak{m}$, put

$$YP = \{yp \mid y \in Y,\ p \in P\} \subset V \times_{\mathfrak{n}} \mathfrak{m}\,.$$

LEMMA 9. *The following three conditions are equivalent.*

(i) $\dim YP = \dim V$.

(ii) *YP is dense in some $V' \in I(\tilde{O} \times_{\mathfrak{g}} \mathfrak{n})$ such that $\rho(V') \subset \mathfrak{m}$.*

(iii) *$Y \in E(V, V')$ for some s-joining pair (V, V').*

PROOF. Interpretation of Lemma 3 in 2.3. q.e.d.

We put

$$E(V \times_{\mathfrak{n}} \mathfrak{m}) = \{Y \in I(V \times_{\mathfrak{n}} \mathfrak{m}) \mid \dim YP = \dim V\}\,.$$

Then by Lemma 9, we have the surjection

$$E(V \times_{\mathfrak{n}} \mathfrak{m}) \to \{V' \mid (V, V') \text{ is } s\text{-joining pair}\}\,,$$

which we write as $Y \mapsto \overline{YP} \in I(\tilde{O} \times_{\mathfrak{p}} \mathfrak{n})$. The s-action on $[V]$ can then be rewritten as

$$s[V] = [V] + \sum_{Y \in E(V \times_{\mathfrak{n}} \mathfrak{m})} m(V \times_{\mathfrak{n}} \mathfrak{m}, Y) d(Y, (\overline{YP}/U_s))[\overline{YP}]\,.$$

3.2. We now consider the etale map (with Galois group $C(A)$)

$$\rho\colon \tilde{O} \times_{\mathfrak{g}} \mathfrak{n} \to O \cap \mathfrak{n}\,.$$

We write the adjoint action of G on $\mathfrak{g}$ on the right, in this section, so that it is compatible with the notations in §2. (For instance, $(\operatorname{Ad} g^{-1})X = Xg$, $X \in \mathfrak{g}$, $g \in G$.) For an irreducible component $U \in I(O \cap \mathfrak{n})$ such that $U \not\subset \mathfrak{m}$, we put

$$E(U \times_{\mathfrak{n}} \mathfrak{m}) = \{Z \in I(U \cap \mathfrak{m}) \mid \dim ZP = \dim U\}\,.$$

Then it is clear that, for $Y \in I(V \times_{\mathfrak{n}} \mathfrak{m})$ $(\rho(V) = U)$,

$$Y \in E(V \times_{\mathfrak{n}} \mathfrak{m}) \quad \text{if and only if} \quad \rho(Y) \in E(U \times_{\mathfrak{n}} \mathfrak{m})\,.$$

For $Z \in E(U \times_{\mathfrak{n}} \mathfrak{m})$, let $\overline{ZP}$ be the irreducible component of $O \cap \mathfrak{n}$ which contains ZP as a dense subset. As in Lemma 3, the U_s-action on $\overline{ZP}$ has the generic quotient $(\overline{ZP})_{\text{reg}}/U_s$ of the same dimension as that of Z. We denote by $d(Z, (\overline{ZP}/U_s))$ the degree of the map $Z \cap (\overline{ZP})_{\text{reg}} \to (\overline{ZP})_{\text{reg}}/U_s$. We also denote by $m(U \times_{\mathfrak{n}} \mathfrak{m}, Z)$ the multiplicity of the reduced component Z in the scheme $U \times_{\mathfrak{n}} \mathfrak{m}$.

LEMMA 10. *Assume that $U \in I(O \cap \mathfrak{n})$ is not contained in $\mathfrak{m}$. Then in the W-module $\mathscr{V}_O$, we see that $s(\sum_{\rho(V)=U} [V])$ equals*

$$\sum_{\rho(V)=U} [V] + \sum_{Z \in E(U \times_{\mathfrak{n}} \mathfrak{m})} m(U \times_{\mathfrak{n}} \mathfrak{m}, Z) d(Z, (\overline{ZP}/U_s))\Big(\sum_{\rho(V')=\overline{ZP}} [V']\Big)\,.$$

3.3. We prove Lemma 10. Let $U \in I(O \cap \mathfrak{n})$ be not contained in $\mathfrak{m}$ and take $Z \in E(U \times_{\mathfrak{n}} \mathfrak{m})$. Then we have the Cartesian diagrams

$$\begin{array}{ccccc} \rho^{-1}(U) & \hookleftarrow & \rho^{-1}(Z) & \hookrightarrow & \rho^{-1}(\overline{ZP}) \\ \downarrow & & \downarrow & & \downarrow \\ U & \hookleftarrow & Z & \hookrightarrow & \overline{ZP} \end{array}\,.$$

Let $\rho^{-1}(Z) = \coprod_{i=1}^{n} Y_i$, $\rho^{-1}(\overline{ZP}) = \coprod_{j=1}^{m} V'_j$ and $\rho^{-1}(U) = \coprod_{k=1}^{l} V_k$ the irreducible decompositions of $\rho^{-1}(Z)$, $\rho^{-1}(\overline{ZP})$ and $\rho^{-1}(U)$ respectively. Note that $Y_i \in E(V_k \times_{\mathfrak{n}} \mathfrak{m})$ for some V_k and (V_k, V'_j) is s-joining if $V_k \cap V'_j \neq \varnothing$.

Since the components Y_i is isomorphic to each other by the covering transformation group $C(A)$ and since $Y_i \to Z$ is etale, we have

$$m(U \times_{\mathfrak{n}} \mathfrak{m}, Z) = m(V_k \times_{\mathfrak{n}} \mathfrak{m}, Y_i)$$

for every Y_i and V_k which contains Y_i. In order to see this, let $\mathcal{O}_{U,Z}$ (resp. $\mathcal{O}_{V_k,Y_i}$) be the local ring of U at Z (resp. V_k at Y_i), and $f \in \mathcal{O}_{U,Z}$ be the defining form of $\mathfrak{m}$ in $\mathfrak{n}$. Hence $\mathcal{O}_{U \times_{\mathfrak{n}} \mathfrak{m}, Z} = \mathcal{O}_{U,Z}/f\mathcal{O}_{U,Z}$ and

$$\mathcal{O}_{U,Z}/f\mathcal{O}_{U,Z} \to \mathcal{O}_{V_k,Y_i}/f\mathcal{O}_{V_k,Y_i}$$

is an etale homomorphism of Artinian local rings. It is therefore enough to prove that, if $A \to B$ is a finite flat homomorphism of Artinian local rings, then $\text{length}_A A = \text{length}_B B$. In fact, then B is a free A-module of rank equal to $\dim_{\kappa(A)}\kappa(B)$; hence $\text{length}_A B = (\dim_{\kappa(A)}\kappa(B))\ (\text{length}_A A)$, where $\kappa(A)$, $\kappa(B)$ denote the residue fields of A, B respectively. But then we also have $\text{length}_A B = (\dim_{\kappa(A)} \kappa(B))\ (\text{length}_B B)$, which implies the required equality.

We next consider the degrees involved. Put $Z_{\text{reg}} = (\overline{ZP})_{\text{reg}} \cap Z$, $(Y_i)_{\text{reg}} = Y_i \cap (V'_j)_{\text{reg}}$ $(V'_j \supset Y_i)$. We then have the commutative diagram of generically finite maps

$$\begin{array}{ccc} \coprod_{i=1}^{n} (Y_i)_{\text{reg}} & \longrightarrow & \coprod_{j=1}^{m} (V'_j)_{\text{reg}}/U_s \\ \downarrow & & \downarrow \\ Z_{\text{reg}} & \longrightarrow & (\overline{ZP})_{\text{reg}}/U_s\,. \end{array}$$

Since the group $C(A)$ acts transitively on all diagrams

$$\begin{array}{ccc} (Y_i)_{\text{reg}} & \longrightarrow & (V'_j)_{\text{reg}}/U_s \\ \downarrow & & \downarrow \\ Z_{\text{reg}} & \longrightarrow & (\overline{ZP})_{\text{reg}}/U_s \end{array} \qquad (Y_i \subset V'_j)\,,$$

all the $d(Y_i, (V'_j/U_s))$ are the same for $Y_i \subset V'_j$ and we have

$$d(Z, (\overline{ZP}/U_s)) \sharp C(A) = n\delta d(Y_i, V'_j/U_s))\,,$$

where n is the cardinality of the Y_i's and δ is the degree of $(V'_j)_{\text{reg}}/U_s \to (\overline{ZP})_{\text{reg}}/U_s$ (constant for every j).

As to δ, we have

$$\delta m = \sharp C(A)$$

where m is the cardinality of the V'_j's. In fact, consider the commutative diagram

$$\begin{array}{ccc} \coprod_{j=1}^{m} (V_j')_{\rm reg} & \longrightarrow & \coprod_{j=1}^{m} (V_j')_{\rm reg}/U_s \\ \downarrow & & \downarrow \\ (\overline{ZP})_{\rm reg} & \longrightarrow & (\overline{ZP})_{\rm reg}/U_s \,. \end{array}$$

Since the fibers of both horizontal maps are isomorphic to U_s, we have by Lemma 4

$$\sharp C(A) = md(V_j', (\overline{ZP}/U_s)) = m\delta \,.$$

We now consider the s-action on $\sum_{\rho(V)=U}[V] = \sum_{k=1}^{l}[V_k]$. By the formula given in 3.1,

$$s\Big(\sum_{k=1}^{l}[V_k]\Big) = \sum_{k=1}^{l}[V_k] + \sum_{Y \in E(\rho^{-1}(U) \times_{\mathfrak{n}} \mathfrak{m})} m(\rho^{-1}(U) \times_{\mathfrak{n}} \mathfrak{m}, Y)d(Y, (\overline{YP}/U_s))[\overline{YP}] \,.$$

In the summation $\sum_Y$ on the right hand side, we pick up the part of the expression

$$\sum_{\rho(Y)=Z} m(\rho^{-1}(U) \times_{\mathfrak{n}} \mathfrak{m}, Y)d(Y, (\overline{YP}/U_s))[\overline{YP}]$$
$$= \sum_{i=1}^{n} m(\rho^{-1}(U) \times_{\mathfrak{n}} \mathfrak{m}, Y_i)d(Y_i, (\overline{Y_iP}/U_s))[\overline{Y_iP}] \,.$$

We know $m(\rho^{-1}(U) \times_{\mathfrak{n}} \mathfrak{m}, Y_i) = m(U \times_{\mathfrak{n}} \mathfrak{m}, Z)$, and $d(Y_i, (\overline{Y_iP}/U_s))$ is constant for all $i = 1, \cdots, n$. Therefore the above sum equals

$$m(U \times_{\mathfrak{n}} \mathfrak{m}, Z)d(Y_i, (V_j'/U_s))\Big((n/m) \sum_{j=1}^{m}[V_j']\Big) \,.$$

Using the equalities $d(Z, (\overline{ZP}/U_s)) \sharp C(A) = n\delta d(Y_i, (V_j'/U_s))$ and $\delta m = \sharp C(A)$, we see

$$md(Z, (\overline{ZP}/U_s)) = nd(Y_i, (V_j'/U_s)) \,.$$

Hence the above equals

$$m(U \times_{\mathfrak{n}} \mathfrak{m}, Z)d(Z, (\overline{ZP}/U_s))\Big(\sum_{j=1}^{m}[V_j']\Big) \,.$$

We thus see $s(\sum_{k=1}^{l}[V_k])$ equals

$$\sum_{k=1}^{l}[V_k] + \sum_{Z \in E(U \times_{\mathfrak{n}} \mathfrak{m})} m(U \times_{\mathfrak{n}} \mathfrak{m}, Z)d(Z, (\overline{ZP}/U_s))\Big(\sum_{\rho(V')=\overline{ZP}}[V']\Big) \,,$$

which completes the proof of Lemma 10.

3.4. We call Joseph's construction [6] of W-modules. To an irreducible component U of $O \cap \mathfrak{n}$, Joseph attaches a certain polynomial p_U on $\mathfrak{h}^*$, the dual of the Cartan subalgebra $\mathfrak{h}$. He shows that space

$$\mathscr{H}_O = \sum_{U \in I(O \cap \mathfrak{n})} \boldsymbol{C}\, p_U$$

spanned by the p_U's in the space of polynomials is W-stable. Furthermore, there are formulas for the action of a simple reflection s;

(i) If $U \subset \mathfrak{m}$, then $sp_U = -p_U$.

(ii) If $U \not\subset \mathfrak{m}$, then

$$sp_U = p_U + \sum_{Z \in E(U \times_{\mathfrak{n}} \mathfrak{m})} m(U \times_{\mathfrak{n}} \mathfrak{m}, Z) d(Z, \overline{(ZP}/U_s)) p_{\overline{ZP}}$$

(see Theorem and its proof in [6, §3.1]).
By Theorem 2 (2.4), (2.9) and Lemma 10 (3.3), the following theorem is now clear.

THEOREM 3. *The W-module* $\mathscr{H}_O$ *is irreducible and isomorphic to the fixed subspace* $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{C})^{C(A)}$ *of* $C(A)$ *in the Springer module* $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{C})$ *for* $A \in O$. *Moreover, the correspondences*

$$p_U \mapsto \sum_{\rho(V)=U, V \in I(O \times_{\mathfrak{g}} \mathfrak{n})} [V] \mapsto \sum_{\rho(\Phi(C))=U, C \in I(\mathscr{B}^A)} [C]$$

($U \in I(O \cap \mathfrak{n})$) *give the W-isomorphisms*

$$\mathscr{H}_O \xrightarrow{\sim} (\mathscr{V}_O \otimes_{\boldsymbol{Q}} \boldsymbol{C})^{C(A)} \xrightarrow{\sim} H_{2d(A)}(\mathscr{B}^A, \boldsymbol{C})^{C(A)} .$$

COROLLARY. $\mathscr{H}_O$ *is a subspace of the space of harmonic polynomials.*

PROOF. From the results of Borho-MacPherson [1], it follows that the W-module $\mathscr{H}_O$ has multiplicity one in the space of homogeneous polynomials of degree $d(A)$. Hence the corollary.

For the application to enveloping algebras, see Joseph's papers [6], [12]. Very recently, Kashiwara and Tanisaki have found nice applications of our results [13].

4. Proof of Theorem 1.

4.1. Using Kazhdan-Lusztig's construction of Springer's representation [7], we shall prove the local formulas in Theorem 1 (1.6). We first recall their construction.

Let $\bar{\phi}: E \to X$ be a locally trivial $\boldsymbol{P}^1$-bundle of algebraic varieties over $\boldsymbol{C}$. Assume that the additive group $\boldsymbol{C}$ acts on $E \to X$ as the bundle isomorphism

$$A: \boldsymbol{C} \times E \to E \qquad (\bar{\phi}(A(t, x)) = \bar{\phi}(x)) .$$

Let $E^A = \{x \in E \mid A(t, x) = x \ (t \in \boldsymbol{C})\}$ be the fixed point subvariety of E. Kazhdan and Lusztig define an involution σ on $H_*(E^A, \boldsymbol{Q})$, the rational homology group of E^A with arbitrary supports (the Borel-Moore homology group), as follows. Choose a Riemann metric on each fiber $\boldsymbol{P}^1$ (2-sphere) of E as a topological fiber bundle, and let

$$\alpha: E \to E$$

be the antipodal involution on each fiber. Choose a closed neighborhood U of E^A in E such that the inclusion $i: E^A \hookrightarrow U$ is a proper homotopy equivalence (U contracts properly to E^A). Then $i_*: H_*(E^A, \boldsymbol{Q}) \xrightarrow{\sim} H_*(U, \boldsymbol{Q})$ is an isomorphism. Since the action of $\boldsymbol{C}$ on each fiber is algebraic, one can choose a continuous function $\mu: X \to \boldsymbol{R}_+$ such that $A(t, x) \in U$ for $(t, x) \in \boldsymbol{C} \times \alpha(E^A)$ satisfying $|t| \geqq \mu(\bar{\phi}(x))$. We then have the map

$$\beta: \alpha(E^A) \to U \qquad (\beta(x) = A(\mu(\bar{\phi}(x)), x))$$

and thus have the map

$$E^A \xrightarrow{\alpha} \alpha(E^A) \xrightarrow{\beta} U \xhookleftarrow{i} E^A .$$

These maps induce

$$H_*(E^A, \boldsymbol{Q}) \xrightarrow[(\beta\circ\alpha)_*]{} H_*(U, \boldsymbol{Q}) \xleftarrow[i_*]{\sim} H_*(E^A, \boldsymbol{Q})$$

and give

$$\sigma = i_*^{-1} \circ (\beta \circ \alpha)_* \in \operatorname{End} H_*(E^A, \boldsymbol{Q}) .$$

Working in the proper equivalence category, we see σ is an involution and independent of the choice of metrics, U and μ.

4.2. Let the situation and the notations be as in §1. Let $p: \mathscr{B} = G/B \to \mathscr{P} = G/P$ be the $\boldsymbol{P}^1$-bundle for the choice of a simple reflection s of the Weyl group W. A nilpotent element A gives rise to the $\boldsymbol{C}$-actions on $\mathscr{B}$ and $\mathscr{P}$ by the left translation of $\exp tA$ ($t \in \boldsymbol{C}$). $\mathscr{P}^A$ is the fixed point subvariety of $\mathscr{P}$ by this $\boldsymbol{C}$-action. Then $E = p^{-1}(\mathscr{P}^A) \to \mathscr{P}^A (\bar{\phi} = p_{|E})$ is our $\boldsymbol{P}^1$-bundle. The $\boldsymbol{C}$-action on E satisfies the condition of 4.1 and gives rise to the involution σ on $H_*(\mathscr{B}^A, \boldsymbol{Q})$ ($\mathscr{B}^A = E^A$). Kazhdan and Lusztig claim that this involution coincides (up to the sign representation) with the action of the s in Springer's representation of w on $H_*(\mathscr{B}^A, \boldsymbol{Q})$. We shall later give a proof of this fact in Appendix and we shall now take it for granted, i.e.,

$$\sigma = s \quad \text{on} \quad H_*(\mathscr{B}^A, \boldsymbol{Q}) .$$

We now have the maps

$$\begin{array}{ccc} \mathscr{B}^A & \hookrightarrow & E \\ & \searrow^{\phi} \quad \swarrow_{\bar{\phi}} & \\ & \mathscr{P}^A . & \end{array}$$

The map ϕ induces the surjection

$$\phi_*: H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q}) \twoheadrightarrow H_{2d(A)}(\mathscr{P}^A, \boldsymbol{Q})$$

on the top homology groups such that the fundamental cycles $[C]$ of s-horizontal components $C \in I(\mathscr{B}^A)$ are mapped to $[\phi(C)]$. The kernel $\mathrm{Ker}\,\phi_*$ is then spanned by the fundamental cycles of s-vertical components. Restricting the bundle to $\phi(C)$ for C s-vertical, we easily see that s acts on $\mathrm{Ker}\,\phi_*$ as -1 and on $H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})/\mathrm{Ker}\,\phi_*$ as 1. Hence we know (i) in Theorem 1.

4.3. We fix C which is an s-horizontal component of $\mathscr{B}^A$. Then the subspace $H_{2d(A)}(\phi^{-1}(\phi(C)), \boldsymbol{Q}) \subset H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q})$ is s-stable and

$$H_{2d(A)}(\phi^{-1}(\phi(C)), \boldsymbol{Q}) \simeq \sum_{C' \in I(\mathscr{B}^A), \phi(C') \subset \phi(C)} \boldsymbol{Q}[C'] .$$

Hence we care only for the s-vertical components C' such that (C, C') are s-joining pairs for C, i.e.,

$$s[C] = [C] + \sum_{(C,C')} N_C^{C'}[C']$$

for some number $N_C^{C'}$, where the summation is over C' such that (C, C') are s-joining. We have to prove $N_C^{C'} = n_C^{C'}$ where the right hand side is the number defined in Definition 2 (1.5).

Take an s-vertical C' such that (C, C') is s-joining. Since $\phi(C') \neq \phi(C'')$ for $C' \neq C''$, the $d(A)$-dimensional components of $E_1^A = \phi^{-1}(\phi(C)_1)$ consist of certain open subsets C_1 and C_1' of C and C' respectively. Here we have put

$$\phi(C)_1 = \phi(C) \backslash \bigcup_{C'' \neq C'} \phi(C'') .$$

Under the natural surjection

$$H_{2d(A)}(\mathscr{B}^A, \boldsymbol{Q}) \twoheadrightarrow H_{2d(A)}(E_1^A, \boldsymbol{Q}) ,$$

$[C]$ (resp. $[C']$) is mapped to $[C_1]$ (resp. $[C_1']$) and the kernel is spanned by those $[C'']$ $(C'' \neq C, C')$. Furthermore, applying the involution s on the $\boldsymbol{P}^1$-bundle

$$E_1 = \bar{\phi}^{-1}(\phi(C)_1) \rightarrow \phi(C)_1 ,$$

we see that the above map is s-equivalent. Hence, considering the s-action on $H_{2d(A)}(E_1^A, \boldsymbol{Q})$, we have

$$s[C_1] = [C_1] + N_C^{C'}[C_1'] .$$

We have thus reduced the problem to the situation in the $\boldsymbol{P}^1$-bundle $E_1 \rightarrow \phi(C)_1$, where the fixed point subvariety E_1^A has two components (if necessary, we may disregard the lower dimensional components).

4.4. We are going back to 1.4 and 1.5. Let $\tilde{C}$ be the normalization of C $(\tilde{C} \xrightarrow{h} C \xrightarrow{\phi} \phi(C))$ and put $\tilde{C}_1 = (\phi \circ h)^{-1}(\phi(C)_1)$. Pull back the bundle

$E_1 \to \phi(C)_1$ to obtain the Cartesian square

$$\begin{array}{ccc} \widetilde{E}_1 & \xrightarrow{\bar{h}} & E_1 \\ {\scriptstyle\tilde{\phi}}\downarrow & & \downarrow{\scriptstyle\bar{\phi}} \\ \widetilde{C}_1 & \xrightarrow{\phi\circ h} & \phi(C)_1 \end{array} .$$

The C-action on E_1 can also be pulled back to one on $\widetilde{E}_1$ and we denote the fixed point subvariety by $\widetilde{E}_1^A = \bar{h}^{-1}(E_1^A)$.

We shall now notice the following:

(1) An irreducible component X_1' of $(\phi \circ h)^{-1}(\phi(C_1'))$ is mapped surjectively onto $\phi(C_1')$ if and only if $X_1' = \widetilde{C}_1 \cap X'$ for some effective $X' \in E(\widetilde{C}, C')$ and the degree $d(X_1', \phi(C_1'))$ of the map $X_1' \to \phi(C_1')$ equals $d(X', \phi(C'))$.

(2) Assuming that the components of E_1^A are C_1 and C_1', we see that the components of $\widetilde{E}_1^A$ are $\bar{h}^{-1}(C_1) = \widetilde{C}_1 \times_{\phi(C)_1} C_1$ and $\tilde{\phi}^{-1}(X_1')$ where $X_1' = \widetilde{C}_1 \cap X'$ runs through $X' \in E(\widetilde{C}, C')$.

(3) Since $\tilde{\phi}^{-1}(X_1')$ (resp. C_1') is a $\boldsymbol{P}^1$-bundle over X_1' (resp. $\phi(C_1')$), the degree $d(\tilde{\phi}^{-1}(X_1'), C_1')$ equals $d(X_1', \phi(C_1')) = d(X', \phi(C'))$.

(4) Under the map

$$\bar{h}_*: H_{2d(A)}(\widetilde{E}_1^A, \boldsymbol{Q}) \to H_{2d(A)}(E_1^A, \boldsymbol{Q}) ,$$

$[\bar{h}^{-1}(C_1)]$ is mapped to $[C_1]$ while $[\tilde{\phi}^{-1}(X_1')]$ is mapped to $d(\tilde{\phi}^{-1}(X_1'), C_1')\,[C_1']$.

(5) Assume that we have

$$s[\bar{h}^{-1}(C_1)] = [\bar{h}^{-1}(C_1)] + \sum_{X' \in E(\widetilde{C}, C')} m(X')[\tilde{\phi}^{-1}(X_1')]$$

in the s-module $H_{2d(A)}(\widetilde{E}_1^A, \boldsymbol{Q})$. Then by (3) and (4), we have

$$s[C_1] = [C_1] + \Big(\sum_{X' \in E(\widetilde{C}, C')} m(X')d(X', \phi(C'))\Big)[C_1']$$

in $H_{2d(A)}(E_1^A, \boldsymbol{Q})$, and hence

$$N_C^{C'} = \sum_{X' \in E(\widetilde{C}, C')} m(X')d(X', \phi(C')) .$$

Thus, in order to prove Theorem 1, (ii), i.e., $n_C^{C'} = N_C^{C'}$, it suffices to prove the equality

$$(*) \qquad m(X') = m(\gamma_{X'}) \quad \text{for} \quad X' \in E(\widetilde{C}, C')$$

where the right hand side is as in 1.5. Let $\eta \in X_1' = X' \cap \widetilde{C}_1$ be a generic point of the simple divisor X_1' in $\widetilde{C}_1$. As in 1.5, we take a unit disk $D \hookrightarrow \widetilde{C}_1$ transversally intersecting X_1' at η. Taking a small neighborhood X_2' of η in X_1', we have an open neighborhood $D \times X_2'$ of η in $\widetilde{C}_1$. Restricting $\widetilde{E}_1 \to \widetilde{C}_1$ to $D \times X_2'$, we have $\widetilde{E}_2 = \tilde{\phi}^{-1}(D \times X_2') \to D \times X_2'$

such that the fixed point subvariety $\tilde{E}_2^A$ of the $\boldsymbol{C}$-action through A has the two components $\tilde{C}_2 = \bar{h}^{-1}(C_1) \cap \tilde{E}_2$ and $C_2' = \tilde{\phi}^{-1}(X_2')$. Here under the natural map

$$H_{2d(A)}(\tilde{E}_1^A, \boldsymbol{Q}) \to H_{2d(A)}(\tilde{E}_2^A, \boldsymbol{Q}) ,$$

$[\bar{h}^{-1}(C_1)]$ is mapped to $[\tilde{C}_2]$ and $[\tilde{\phi}^{-1}(X_1')]$ is mapped to $[C_2']$. Thus it suffices to see the equality $(*)$ in the local model $\tilde{E}_2 \to D \times X_2'$.

We now consider the inclusion

$$D \simeq D \times \eta \subset D \times X_2'$$

and pull back the $\boldsymbol{P}^1$-bundle $\tilde{E}_2$ over D. Denote this $\boldsymbol{P}^1$-bundle by $E_D \simeq D \times \boldsymbol{P}^1 \to D$. The following facts can been seen by inspection of the definitions.

LEMMA 11. *Let $\gamma_{X'}: D \to \boldsymbol{C}$ be as in 1.5. Then the resulting $\boldsymbol{C}$-action on E_D is given by $(x, z) \mapsto (x, z + t\gamma_{X'}(x))$ $(t \in \boldsymbol{C}, (x, z) \in D \times \boldsymbol{P}^1)$ where we take the coordinate $z \in \boldsymbol{C} \cup \{\infty\} = \boldsymbol{P}^1$ of $\boldsymbol{P}^1$.*

LEMMA 12. *The fixed point subvariety E_D^A consists of the two components $D \times \{\infty\}$ and $\{0\} \times \boldsymbol{P}^1$ in E_D. Under the Gysin isomorphism*

$$H_2(E_D^A, \boldsymbol{Q}) \simeq H_{2d(A)}(\tilde{E}_2^A, \boldsymbol{Q}) ,$$

which is also s-equivariant, the fundamental cycle $[D \times \{\infty\}]$ (resp. $[\{0\} \times \boldsymbol{P}^1]$) is mapped to $[\tilde{C}_2]$ (resp. $[C_2']$).

4.5. In order to finish the proof of Theorem 1, it suffices to prove the following lemma.

LEMMA 13. *Let $\gamma: D \to \boldsymbol{C}$ be a holomorphic map such that $\gamma(0) = 0$. Let $\boldsymbol{C}$ act on $E_D = D \times \boldsymbol{P}^1$ by $(x, z) \mapsto (x, z + t\gamma(x))$ $(t \in \boldsymbol{C})$. Then in the homology group $H_2(E_D^A, \boldsymbol{Q})$ of the fixed point subvariety E_D^A, the s-action on the cycle $[D \times \{\infty\}]$ is described as*

$$s[D \times \{\infty\}] = [D \times \{\infty\}] + m(\gamma)[\{0\} \times \boldsymbol{P}^1]$$

where $m(\gamma)$ is the mapping degree of γ.

PROOF. Let $\alpha(x, z) = (x, -1/\bar{z})$ $((x, z) \in E_D)$ be an antipodal involution. Then $\alpha(E_D^A) = D \times \{0\} \cap \{0\} \times \boldsymbol{P}^1$. Put

$$U = \{(x, z) \in D \times \boldsymbol{P}^1 \,|\, |z| \geqq R \quad \text{or} \quad |x| \leqq r\}$$

for fixed $R > 0$ and $0 < r < 1$. Then U is a closed neighborhood of E_D^A which properly contracts to E_D^A. For sufficiently large $t \gg 0$, $\beta(\alpha(E_D^A)) = \{(x, -1/\bar{z} + t\gamma(x)) \,|\, (x, z) \in E_D^A\} \subset U$. The s-action is, by definition, the composite map

$$H_2(E_D^A, \boldsymbol{Q}) \xrightarrow{(\beta\circ\alpha)_*} H_2(U, \boldsymbol{Q}) \xleftarrow{\sim} H_2(E_D^A, \boldsymbol{Q}) .$$

The cycle $s[D \times \{\infty\}]$ is represented by $[\beta(\alpha(D \times \{\infty\}))] \in H_2(U, \boldsymbol{Q})$ where $\beta(\alpha(D \times \{\infty\})) = \{(x, t\gamma(x)) | x \in D\}$. We have to show, in $H_2(U, \boldsymbol{Q})$,

$$[\beta(\alpha(D \times \{\infty\}))] = [D \times \{\infty\}] + m(\gamma)[\{0\} \times \boldsymbol{P}^1] .$$

For this, we take embeddings

$$\begin{array}{ccc} D \times \boldsymbol{P}^1 & \hookrightarrow & \boldsymbol{P}^1 \times \boldsymbol{P}^1 \\ \cup & & \cup \\ U & \hookrightarrow & \mathscr{U} \end{array}$$

such that $\mathscr{U}$ is a closed neighborhood of $K = \boldsymbol{P}^1 \times \{\infty\} \cup \{0\} \times \boldsymbol{P}^1$ which properly contracts to K and $\mathscr{U} \cap D \times \boldsymbol{P}^1 = U$. Thus we have the natural isomorphisms

$$H_2(E_D^A, \boldsymbol{Q}) \xrightarrow{\sim} H_2(U, \boldsymbol{Q}) \xleftarrow{\sim} H_2(\mathscr{U}, \boldsymbol{Q}) \xrightarrow{\sim} H_2(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \boldsymbol{Q}) .$$

In these isomorphisms, $[D \times \{\infty\}]$ corresponds to $[\boldsymbol{P}^1 \times \{\infty\}]$. We extend the cycle $\beta(\alpha(D \times \{\infty\}))$ in U to a homological cycle F in $\mathscr{U}$ such that $F \cap U = \beta(\alpha(D \times \{\infty\}))$. It suffices to see

$$[F] = [\boldsymbol{P}^1 \times \{\infty\}] + m(\gamma)[\{0\} \times \boldsymbol{P}^1]$$

in $H_2(\mathscr{U}, \boldsymbol{Q}) \simeq H_2(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \boldsymbol{Q})$. In the homology group $H_2(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \boldsymbol{Q})$, we have the intersection numbers

$$[\boldsymbol{P}^1 \times \{\infty\}] \cdot [\{0\} \times \boldsymbol{P}^1] = 1 ,$$
$$[\boldsymbol{P}^1 \times \{\infty\}] \cdot [\boldsymbol{P}^1 \times \{\infty\}] = 0 ,$$

and

$$[\{0\} \times \boldsymbol{P}^1] \cdot [\{0\} \times \boldsymbol{P}^1] = 0 .$$

We now let

$$[F] = a[\boldsymbol{P}^1 \times \{\infty\}] + b[\{0\} \times \boldsymbol{P}^1] .$$

Applying the intersections, we have

$$1 = [F] \cdot [\{0\} \times \boldsymbol{P}^1] = a \quad \text{and} \quad [F] \cdot [\boldsymbol{P}^1 \times \{\infty\}] = b .$$

But then,

$$[F] \cdot [\boldsymbol{P}^1 \times \{\infty\}] = [F] \cdot [\boldsymbol{P}^1 \times \{0\}]$$

which is equal to the intersection number of the local curves $\{(x, t\gamma(x)) | x \in D\}$ and $\{(x, 0) | x \in D\}$. This number is clearly equal to the mapping degree $m(\gamma)$ of γ. We have thus proved the lemma and hence have completed the proof of Theorem 1.

Appendix: IDENTIFICATION.

A1. Let G be a connected reductive algebraic group over $\boldsymbol{C}$, $\mathfrak{g}$ its Lie algebra and $\mathscr{N}$ the closed subvariety of $\mathfrak{g}$ consisting of nilpotent elements in $\mathfrak{g}$. Considering $\mathscr{B}$ as the set of all Borel subalgebras of $\mathfrak{g}$, put

$$\tilde{\mathfrak{g}} = \{(A, \mathfrak{b}) \in \mathfrak{g} \times \mathscr{B} \mid A \in \mathfrak{b}\} .$$

By the projection to the first factor, we have

$$\rho: \tilde{\mathfrak{g}} \to \mathfrak{g} .$$

Here ρ has fibers

$$\rho^{-1}(A) = \{\mathfrak{b} \in \mathscr{B} \mid A \in \mathfrak{b}\} = \mathscr{B}^A \qquad (A \in \mathfrak{g}) ,$$

and if we denote by $\mathfrak{g}_{rs}$ the open subset of $\mathfrak{g}$ consisting of regular semisimple elements in $\mathfrak{g}$,

$$\rho_{rs} = \rho_{|\tilde{\mathfrak{g}}_{rs}}: \tilde{\mathfrak{g}}_{rs} = \rho^{-1}(\mathfrak{g}_{rs}) \to \mathfrak{g}_{rs}$$

turns out to be an etale covering. If we fix a Borel subalgebra and a Cartan subalgebra contained in it, the covering ρ_{rs} acquires the Weyl group W as its Galois group. We thus have the local system $(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$ on $\mathfrak{g}_{rs}$ which has the W-action under our choice.

Let ${}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$ be the DGM extension of $(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$ to $\mathfrak{g}$, i.e., the bounded complex of sheaves of $\boldsymbol{Q}$-vector spaces on $\mathfrak{g}$ whose cohomology sheaves are constructible such that

(i) ${}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}|\mathfrak{g}_{rs}} \simeq (\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$

(ii) $\operatorname{codim}_{\mathfrak{g}} \operatorname{Supp} \mathscr{H}^i({}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}) > i$ (for all $i > 0$),

(iii) $\boldsymbol{R}\mathscr{H}om_{\boldsymbol{Q}_{\mathfrak{g}}}({}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}, \boldsymbol{Q}_{\mathfrak{g}}) \simeq {}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$,

(iv) $\mathscr{H}^i({}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}) = 0$ for $i < 0$.

(See [3, §4]. Here we consider the middle perversity and take a shift of degrees.) Then ${}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}}$ has the W-action as an object in the derived category $D_c^b(\boldsymbol{Q}_{\mathfrak{g}})$ whose objects are bounded complexes of sheaves of $\boldsymbol{Q}$-vector spaces on $\mathfrak{g}$ whose cohomologies are constructible. Lusztig [8] observed that

$$\boldsymbol{R}\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}} \simeq {}^{\pi}(\rho_{rs})_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}_{rs}} \quad \text{in} \quad D_c^b(\boldsymbol{Q}_{\mathfrak{g}})$$

since the fibers of ρ are "small" in the sense of [3]. Thus the direct image sheaves $\boldsymbol{R}\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}}$ on $\mathfrak{g}$ acquires the W-action which gives rise to the W-module structures on $H^*(\mathscr{B}^A, \boldsymbol{Q}) = (R^*\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}})_A$.

A2. Since $\boldsymbol{R}\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}}$ has the W-action as an object of $D_c^b(\boldsymbol{Q}_{\mathfrak{g}})$, the restriction to $\mathscr{N}$

$$(\boldsymbol{R}\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}})_{|\mathscr{N}} = \boldsymbol{R}(\rho_{|\tilde{\mathscr{N}}})_*\boldsymbol{Q}_{\tilde{\mathscr{N}}} \qquad (\tilde{\mathscr{N}} = \rho^{-1}(\mathscr{N}))$$

also has the W-action as an object of $D_c^b(\boldsymbol{Q}_{\mathcal{N}})$. This W-action induces the "standard" W-action on $H^*(\mathscr{B}, \boldsymbol{Q}) = (R^*\rho_*\boldsymbol{Q}_{\tilde{\mathfrak{g}}})_0$ the special stalk at $0 \in \mathcal{N}$, which coincides with the classical W-action induced by the topological W-action on $\mathscr{B} = K/T_K$ where $K \subset G$ is a maximal compact subgroup and T_K is a maximal (topological) torus of K. The work of Borho-MacPherson [1], [2] gives the following criterion for the uniqueness of the W-action on $\boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$ on $\mathcal{N}$: (See the proof of the main theorem in [1].)

THEOREM A1 (Uniquness theorem). *A W-action on $\boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$ in $D_c^b(\boldsymbol{Q}_{\mathcal{N}})$ is unique if it induces the standard W-action on the special stalk $H^*(\mathscr{B}, \boldsymbol{Q})$,* i.e., *that induced by the standard topological action of W on $\mathscr{B}$.*

REMARK 1. The theorem is true for the l-adic cohomology in the fields of characteristic $p \neq l$. The proof of the identifications in [4] becomes simpler if we use the above uniquess theorem.

REMARK 2. Borho-MacPherson's result depends on the "deep" decomposition theorem of Deligne-Gabber-Beilinson-Bernstein for l-adic sheaves in positive characteristics. Recently, we recovered an analytic proof of their result, which uses the "Fourier transform" of holonomic systems ([5]).

A3. Using Theorem A1, we can rather easily identify Kazhdan-Lusztig's W-action with Springer's one. We choose a simple reflection $s \in W$ and the situation is assumed to be as in §4.

In the $\boldsymbol{P}^1$-bundle

$$\tilde{p} = \mathrm{Id}_{\mathcal{N}} \times p \colon \mathcal{N} \times \mathscr{B} \to \mathcal{N} \times \mathscr{P},$$

consider the restricted $\boldsymbol{P}^1$-bundle

$$E = \tilde{p}^{-1}(\tilde{p}(\tilde{\mathcal{N}})) \to \tilde{p}(\tilde{\mathcal{N}})(\subset \mathcal{N} \times \mathscr{P}).$$

There is a $\boldsymbol{C}$-action on E such that

$$\boldsymbol{C} \times E \to E \qquad ((t, A, \mathfrak{b}) \mapsto (A, e^{tA}(\mathfrak{b})))$$

whose fixed point subvariety is $\tilde{\mathcal{N}} \subset E$. We shall define an involution on $\boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$ on $\tilde{p}(\tilde{\mathcal{N}})$ which, after taking the direct image by $\rho' \colon \mathcal{N} \times \mathscr{P} \to \mathcal{N}$, will coincide with the s-action on $\boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$. Choose a closed neighborhood U of $\tilde{\mathcal{N}}$ in E such that U properly contracts to $\tilde{\mathcal{N}}$ fiber by fiber in the fibering $E \to \tilde{p}(\tilde{\mathcal{N}})$. The $\boldsymbol{C}$-action

$$\beta_t \colon (A, \mathfrak{b}) \mapsto (A, e^{tA}(\mathfrak{b})) \qquad (t \in \boldsymbol{C})$$

gives the $\boldsymbol{C}$-action on the $\boldsymbol{P}^1$-bundle $E \to \tilde{p}(\tilde{\mathcal{N}})$.

Taking the antipodal involution α of E, we have

$$\tilde{\mathcal{N}} \xrightarrow{\alpha} \alpha(\tilde{\mathcal{N}}) \xrightarrow{\beta_t} \beta_t(\alpha(\tilde{\mathcal{N}})) \subset U$$

for sufficiently large $t \gg 0$. This gives rise to the morphism

$$\boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_U \xrightarrow{(\beta_t \circ \alpha)^*} \boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}} ,$$

and, since U contracts to $\tilde{\mathcal{N}}$ fiber by fiber, we have a quasi-isomorphism

$$\boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_U \xrightarrow{\sim} \boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}} .$$

We thus have the morphism

$$\sigma\colon \boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}} \xleftarrow{\sim} \boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_U \to \boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$$

in $D^b_c(\boldsymbol{Q}_{\tilde{p}(\tilde{\mathcal{N}})})$. Taking the direct image by the projection $\rho'\colon \tilde{p}(\tilde{\mathcal{N}}) \subset \mathcal{N} \times \mathcal{P} \to \mathcal{N}$, we have the endomorphism

$$\boldsymbol{R}\rho'_*(\sigma)\colon \boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}} \to \boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$$

where $\boldsymbol{R}(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}} = \boldsymbol{R}\rho'_*\boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_{\tilde{\mathcal{N}}}$.

LEMMA A1. *The endomorphism $\boldsymbol{R}\rho'_*(\sigma)$ coincides with the s-action ($=\sigma$ in §4) on $H^*(\mathcal{B}^A, \boldsymbol{Q}) \simeq (R^*(\rho_{|\tilde{\mathcal{N}}})_*\boldsymbol{Q}_{\tilde{\mathcal{N}}})_A$ under Kazhdan-Lusztig's construcion for the homology group $H_*(\mathcal{B}^A, \boldsymbol{Q}) =$ the dual of $H^*(\mathcal{B}^A, \boldsymbol{Q})$.*

PROOF. In the maps

$$\tilde{\mathcal{N}} \subset U \xrightarrow{\tilde{p}_{|U}} \tilde{p}(\tilde{\mathcal{N}}) \xrightarrow{\rho'} \mathcal{N},$$

$U^A = (\rho' \circ \tilde{p}_{|U})^{-1}(A)$ is a closed neighborhood which properly contracts to $\mathcal{B}^A$. Kazhdan-Lusztig's action is defined, fiber by fiber, through a closed neighborhood U^A on the Borel-Moore homology $H_*(U^A, \boldsymbol{Q}) \simeq H_*(\mathcal{B}^A, \boldsymbol{Q})$. We have $H_*(U^A, \boldsymbol{Q}) \cong H^*(U^A, \boldsymbol{Q})^\vee$ (the dual). But then $\mathcal{H}^i(\boldsymbol{R}\rho'_*\boldsymbol{R}\tilde{p}_*\boldsymbol{Q}_U)_A \cong H^*(U^A, \boldsymbol{Q}) \cong H^*(\mathcal{B}^A, \boldsymbol{Q})$. Hence our construction of the s-action is the sheafification of Kazhdan-Lusztig's s-action. q.e.d.

LEMMA A2. *The endomorphism $\boldsymbol{R}\rho'_*(\sigma)$ coincides with the standard action of the s on $H^*(\mathcal{B}, \boldsymbol{Q})$ at the special stalk $A = 0$.*

PROOF. Easy.

It follows from Theorem A1 and the above lemmas that Kazhdan-Lusztig's W-action on $H_*(\mathcal{B}^A, \boldsymbol{Q})$ coincides with Springer's W-action.

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