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**ON JUNG'S CONSTANT AND RELATED CONSTANTS IN  
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# ON JUNG'S CONSTANT AND RELATED CONSTANTS IN NORMED LINEAR SPACES

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In this paper several results on certain constants related to the notion of Chebyshev radius are obtained. It is shown in the first part that the Jung constant of a finite-codimensional subspace of a space  $C(T)$  is 2, where  $T$  is a compact Hausdorff space which is not extremally disconnected. Several consequences are stated, e.g. the fact that every linear projection from a space  $C(T)$ ,  $T$  a perfect compact Hausdorff space, onto a finite-codimensional proper subspace has norm at least 2.

The second discusses mainly the "self-Jung constant" which measures "uniform normal structure." It is shown that this constant, unlike Jung's constant, is essentially determined by the finite subsets of the space.

**1. Jung constant in  $C(T)$  spaces.** For a bounded subset  $A$  of a normed linear space  $E$  and a subset  $Y$  of  $E$  we denote by  $\text{diam } A$  the diameter of  $A$  ( $\sup_{x,y \in A} \|x - y\|$ ), by  $r_Y(A)$  the relative Chebyshev radius of  $A$  with respect to  $Y$  ( $\inf_{y \in Y} \sup_{x \in A} \|x - y\|$ ), and by  $Z_Y(A)$  the relative Chebyshev center set of  $A$  in  $Y$  ( $\{y \in Y; \sup_{x \in A} \|x - y\| = r_Y(A)\}$ ). The Jung constant of  $E$  is  $J(E) = \sup\{2r_E(A); A \subset E, \text{diam } A = 1\}$ . It is easily seen that  $1 \leq J(E) \leq 2$ . For  $n$ -dimensional spaces  $E_n$ , it was shown by Jung [12] that  $J(l_2^n) = (2n/(n+1))^{1/2}$  and  $J(E_n) = 1$  if and only if  $E_n = l_\infty^n$ . Bohnenblust [2] showed that  $J(E_n) \leq 2n/(n+1)$ , and Leichtweiss [14] characterized the extremal case (in the 2-dimensional case it is the hexagonal plane). In the infinite-dimensional case, it was shown that  $J(l_2) = \sqrt{2}$  (Routledge [20]), and that  $J(E) = 1$  if and only if  $E = C(T)$  for a Stonian  $T$ , i.e. if  $E \in \mathcal{P}_1$  (Davis [5]) (cf. also [10], pages 91–92 in [11] and §6 in [4]).

Studying intersections of balls with subspaces, Franchetti [6] deduced that for every finite-codimensional subspace  $E$  of  $C[a, b]$  we have  $J(E) \geq 3/2$ . A stronger and more general result is true.

**1.1. PROPOSITION.** *If the compact Hausdorff space  $T$  is not extremally disconnected, then for every finite-codimensional subspace  $E$  of  $C(T)$  we have  $J(E) = 2$ .*

We need the following

1.2. LEMMA. *Let  $E$  be a finite-codimensional subspace of  $C(T)$ ,  $T$  compact Hausdorff. Then for every  $\varepsilon > 0$  and every infinite open  $V \subset T$  there is  $f \in E$  with  $\|f\| = 1$ ,  $f(T \setminus V) = 0$  and  $f \geq -\varepsilon$ .*

*Proof of the lemma.* In the case where  $V$  contains no isolated points, the proof is quite short: Since  $V$  is infinite,  $\{f \in E; f(T \setminus V) = 0\}$  is infinite dimensional and there are  $f_1 \in E$ ,  $t_1 \in V$  with  $\|f_1\| = 1 = f_1(t_1)$ ,  $f_1(T \setminus V) = 0$ . For  $V_1 = \{t \in V; f_1(t) > 1 - \varepsilon\}$ , which is infinite too, find in the same way  $f_2 \in E$ ,  $t_2 \in V_1$  with  $\|f_2\| = 1 = f_2(t_2)$ ,  $f_2(T \setminus V_1) = 0$ , etc.  $g = \sum_{j=1}^n f_j$  satisfies  $\|g\| \geq g(t_n) > n(1 - \varepsilon)$ , while, since  $f_j(t) < 0$  happens only when  $f_{j-1}(t) > 1 - \varepsilon$  and  $f_{j+1}(t) = 0$ ,  $g(t) > -\varepsilon$ . Normalize to get  $f$ .  $\square$

For the general case we apply

1.3. SUBLEMMA. *Given an infinite matrix  $(x^j(k))_{j=1, \dots, n; k=1, 2, \dots}$  such that  $x^j(k) \rightarrow 0$  as  $k \rightarrow \infty$  for  $j = 1, \dots, n$  and  $\varepsilon > 0$ , there are  $k$  and  $(\varepsilon_i)_{i=1}^\infty$  such that  $|\varepsilon_i| \leq \varepsilon$  for all  $i$  and  $x^j(k) = \sum_{i \neq k} \varepsilon_i x^j(i)$  for  $j = 1, \dots, n$ .*

*Proof of the sublemma.* We may assume that the rows  $x^1, \dots, x^n$  are linearly independent. Therefore there are also  $n$  independent columns, which we may assume to be the first  $n$  ones. Let  $(\gamma_{r,s})_{r,s=1}^n$  be the inverse of the matrix  $(x^j(k))_{j,k=1}^n$  and  $c = \sum_{r,s} |\gamma_{r,s}|$ . There is  $k$  such that  $|x^j(k)| < \varepsilon/c$  for  $j = 1, \dots, n$ . Represent the  $k$ th column as a linear combination of the first  $n$  ones.  $\square$

*Proof of the lemma in the general case.* Take a sequence  $(f_k)_{k=1}^\infty$  of disjointly supported nonnegative norm-one functions sitting in  $V$ . Apply the sublemma to  $x^j(k) = \mu_j(f_k)$ , where  $\mu_1, \dots, \mu_n \in C(T)^*$  are such that  $E = \{\mu_1, \dots, \mu_n\}_\perp$ . Take  $f = f_k - \sum_{i \neq k} \varepsilon_i f_i$ .  $\square$

*Proof of the proposition.* Choose disjoint open subsets  $V_1, V_2$  and  $w \in \bar{V}_1 \cap \bar{V}_2$  (such  $w, V_1, V_2$  exist since  $T$  is not extremally disconnected). Fix  $\varepsilon > 0$ . Let  $A \subset E$  consist of all  $f_1 - f_2$ , when  $f_i$  run over all the functions  $f$  satisfying the conclusions of the lemma with respect to  $V_i$ . Then  $f^* = \sup_{f \in A} f$  is 1 on  $V_1$  and  $\leq \varepsilon$  on  $V_2$ , while  $f_* = \inf_{f \in A} f$  is  $-1$  on  $V_2$  and  $\geq -\varepsilon$  on  $V_1$ . Thus the diameter of  $A$  is  $\leq 1 + \varepsilon$ . The radius of  $A$ , however, is  $\geq 1$  since  $\max_{t_1, t_2 \in V} |f^*(t_1) - f_*(t_2)| = 2$  in every neighborhood  $V$  of  $w$ .  $\square$

**REMARK.** Proposition 1.1 verifies also a conjecture of Franchetti ([7]): If  $J(C(T)) < 2$  then  $T$  is extremally disconnected (and then  $J(C(T)) = 1$  by Davis' result). This last result has been proved independently by C. Franchetti [8].

Lemma 1.2 can be applied also to improve Proposition 2 in [6], giving an alternative proof of our Proposition 1.1 in the perfect case.

**1.4. PROPOSITION.** *Let  $F$  be a finite-codimensional subspace of  $C(T)$ ,  $T$  perfect compact Hausdorff space. Then for every  $x \in C(T)$  and every  $s > d \equiv d(x, F)$  we have*

$$Z_F(B(x, s) \cap F) = P_F x \quad \text{and} \quad r_F(B(x, s) \cap F) = s + d,$$

where  $B(x, s)$  is the closed  $s$ -ball centered at  $x$  ( $\{y; \|y - x\| \leq s\}$ ) and  $P_F x$  is the best approximation to  $x$  in  $F$ .

*Proof.* Given any  $y_0 \in F$  with  $\|x - y_0\| > d$ , we want to show that there is a  $y \in F$  with  $\|x - y\| \leq s$  and  $\|y - y_0\| > s + d$ . This will establish both claims, since if  $\|x - y_1\| < d + \varepsilon$  then clearly  $\|y - y_1\| \leq \|y - x\| + \|x - y_1\| < s + d + \varepsilon$  for every such  $y$ .

Without loss of generality we may assume  $y_0 = 0$ ,  $\|x\| = x(t_0)$  for some  $t_0 \in T$ . If  $\|x\| < s$ , let

$$0 < \varepsilon < \min\left(\frac{s - \|x\|}{s + d + 1}, \frac{\|x\| - d}{2}, 1\right),$$

$V = \{t; |x(t) - x(t_0)| < \varepsilon\}$ . Apply Lemma 1.2 to get  $z \in F$  with  $\|z\| = 1$  and  $z \geq -\varepsilon$  which vanishes off  $V$ . Let  $y = (s + d + \varepsilon)z$ . Clearly  $\|y\| > s + d$ . If  $t \notin V$ , then  $|(x - y)(t)| = |x(t)| \leq \|x\| < s$ . If  $t \in V$  then

$$-s < \|x\| - \varepsilon - (s + d + \varepsilon) \leq (x - y)(t) \leq \|x\| + \varepsilon(s + d + \varepsilon) < s.$$

If  $\|x\| \geq s$ , let  $y_1 \in F$  satisfy  $d < \|x - y_1\| < s$ . Let

$$0 < \varepsilon < \min\left(\frac{s - \|x - y_1\|}{s + d}, \frac{\|s - y_1\| - d}{2}, y_1(t_0)\right),$$

$V = \{t: |x(t) - x(t_0)| + |y_1(t) - y_1(t_0)| < \varepsilon\}$ . Apply Lemma 1.2 to get  $z \in F$  with  $\|z\| = 1 = z(t_1)$ ,  $z \geq -\varepsilon$  which vanishes off  $V$ . Let  $y = y_1 + (s + d)z$ .

$$\|y\| \geq y_1(t_1) + s + d > y_1(t_0) - \varepsilon + s + d > s + d.$$

If  $t \notin V$ , then  $|(x - y)(t)| = |(x - y_1)(t)| < s$ . If  $t \in V$ , then

$$\begin{aligned} -s &< (x - y_1)(t_0) - \varepsilon - s - d \\ &\leq (x - y)(t) \leq (x - y_1)(t) + (s + d)\varepsilon < s. \end{aligned}$$

□

1.5. COROLLARY. *If  $F$  is a subspace of  $C(T)$ ,  $T$  any compact Hausdorff space with no isolated points, and  $1 \leq \text{codim } F < \infty$ , then  $J(F) = 2$ .*

Thus, for perfect  $T$ , the restriction in Proposition 1.1 that  $T$  be non-Stonian is necessary (for  $J(F) = 2$ ) only in the case  $F = C(T)$ . Further concessions are impossible — since if  $t_0 \in T$  is isolated in the Stonian space  $T$ , then  $F = \{x \in C(T); x(t_0) = 0\}$  is isometric to  $C(T')$ , where  $T' = T \setminus \{t_0\}$  is Stonian too, hence  $J(F) = 1$ .

Applying Franchetti's observation on the relation between projection constants of hyperplanes and radii of hypercircles [8], we get:

1.6. COROLLARY. *If  $F$  is a finite-codimensional proper subspace of  $C(T)$ ,  $T$  perfect compact Hausdorff space, then every linear projection of  $C(T)$  onto  $F$  has norm  $\geq 2$ .*

*Proof.* Let  $F = \{\mu_1, \dots, \mu_n\}_\perp$ ,  $\mu_i \in C(T)^*$ ,  $\|\mu_i\| = 1$ ,  $E = \{\mu_1, \dots, \mu_{n-1}\}_\perp$  such that  $F$  is a maximal subspace of  $E$ . A linear projection of  $E$  onto  $F$  has the form  $Px = x - \mu_n(x)z$ , where  $z \in E$  and  $\mu_n(z) = 1$ . But

$$\begin{aligned} \|P\| &\geq \sup_{0 \leq \alpha < 1} \sup_{\|x\| \leq 1} \|Px\| = \sup_{0 \leq \alpha < 1} \sup_{\substack{y \in F \\ \|y + \alpha z\| \leq 1}} \|y\| \\ &\geq \sup_{0 \leq \alpha < 1} r_F(B(-\alpha z, 1) \cap F). \end{aligned}$$

By Proposition 1.4, since  $d(-\alpha z, F) = \alpha$ ,  $r_F(B(-\alpha z, 1) \cap F) = 1 + \alpha$ , so that  $P = \sup_{0 \leq \alpha < 1} (1 + \alpha) = 2$ .

Thus, every projection of  $E$  onto  $F$ , and therefore also every projection of  $C(T)$  onto  $f$ , has norm  $\geq 2$ .  $\square$

**2. Jung constants and normal structure coefficients.** By a classical result of Garkavi and Klee (cf., e.g. [13])  $r_A(A) = r(A)$  for all convex closed and bounded  $A \subset E$  is equivalent to  $E$  having dimension  $\leq 2$  or being an inner product space. Therefore, besides the Jung constant  $J(E)$ , one may study also the “self-Jung constant”  $J_s(E) = \sup\{2r_A(A); A \subset E \text{ convex, diam } A = 1\}$ . Clearly  $J_s(E) \geq J(E)$ .  $E$  is said to have “normal structure” if for every such  $A$  we have  $r_A(A) < \text{diam } A$ . Thus  $J_s(E)$  measures to what extent  $E$  has “uniform normal structure”. Bynum [3] introduced the “normal structure coefficient”;  $N(E) = 2/J_s(E)$ , and two other coefficients,  $BS(E)$  and  $WCS(E)$ , analogously defined by the “asymptotic diameter” and the “asymptotic radius” of bounded, or

weakly convergent, sequences in  $E$ , respectively, i.e.

$$\inf \left\{ \frac{\lim_k \sup_{m,n>k} \|x_n - x_m\|}{\inf \lim_k \sup_{n \geq k} \|y - x_n\|; y \in \overline{\text{conv}}(x_k)_{k=1}^\infty} \right\},$$

where the infimum is taken over all bounded nonconvergent sequences  $(x_n) \subset E$  in the  $BS(E)$  case, and over all weakly convergent, non-norm-convergent sequences in the  $WCS(E)$  case. Clearly  $1 \leq N(E) \leq BS(E) \leq WCS(E)$  and  $WCS(E) \leq 2$  unless  $E$  has the Schur property (i.e. unless in  $E$  norm and weak sequential convergence coincide).

It is easy to see, and hinted in [3], that  $BS(E) = \sup\{N(F); F \subset E \text{ separable}\}$  and  $WCS(E) = \sup\{WCS(F); F \subset E \text{ separable}\}$ .

In [15], Lim shows that  $J_s(E) = \sup\{2r_A(A); A \subset E \text{ convex and separable, diam } A = 1\}$ , hence  $N(E) = BS(E)$  for every normed  $E$ . This can be further improved, using the following observations:

2.1. PROPOSITION. (a) *If  $E$  is a dual Banach space, then*

$$J(E) = \sup\{2r_E(K); K \subset E \text{ finite, diam } K = 1\}.$$

(b) *If  $E$  is a reflexive Banach space, then*

$$J_s(E) = \sup\{2r_{\text{conv } K}(K); K \subset E \text{ finite, diam } K = 1\}.$$

*Proof.* (a) Let  $A \subset E$  be any with  $\text{diam } A = 1$ ,  $r < r_E(A)$  any. Then  $\bigcap_{x \in A} B(x, r) = \emptyset$  and by  $w^*$ -compactness of the balls there is a finite  $K = \{x_1, \dots, x_n\} \subset A$  with  $\bigcap_{x \in K} B(x, r) = \emptyset$ , i.e.  $r < r_E(K)$ .

(b) Let  $A \subset E$  be convex closed with  $\text{diam } A = 1$ ,  $r < r_A(A)$  any. Then  $\bigcap_{x \in A} B(x, r) \cap A = \emptyset$  and by  $w$ -compactness of the balls and of  $A$  there is a finite  $K \subset A$  with  $\bigcap_{x \in K} B(x, r) \cap \text{conv } K \subset \bigcap_{x \in K} B(x, r) \cap A = \emptyset$ , i.e.  $r < r_{\text{conv } K}(K)$ .  $\square$

2.2. PROPOSITION. (Maluta, [16].) *If  $E$  is a non reflexive Banach space, then  $J_s(E) = 2$ .*

*Proof.* By a theorem of D. P. Milman and V. D. Milman [18] there is, in every nonreflexive Banach space  $E$  and for every  $\varepsilon > 0$ , a sequence  $(x_n)_{n=1}^\infty$  in  $E$  such that for every  $m \geq 1$  and every  $y' \in \text{conv}(x_1, \dots, x_m)$ ,  $y'' \in \text{conv}(x_{m+1}, x_{m+2}, \dots)$  we have  $1 - \varepsilon < \|y' - y''\| < 1 + \varepsilon$ . Taking  $A = \text{conv}(x_n)_{n=1}^\infty$ , one has  $\text{diam } A \leq 1 + \varepsilon$  while  $r_A(A) \geq 1 - \varepsilon$ .  $\square$

2.3. COROLLARY. (a)  $(\text{Lim}) J_s(E) = \sup\{2r_{\text{conv } A}A; A \subset E \text{ separable, diam } A = 1\} = \max\{J_s(F); F \text{ a separable subspace of } E\}$ .

(b) If  $J_s(E) < 2$ , then  $J_s(E) = \sup\{2r_{\text{conv } K}; K \subset E \text{ finite, } \text{diam } K = 1\} = \sup\{J_s(F); F \text{ a finite dimensional subspace of } E\}$ .

(c) If  $E$  has “uniform normal structure”, so does every reflexive  $G$  which is finitely representable in  $E$  (i.e. such that for every finite dimensional subspace  $F$  of  $G$  and every  $\varepsilon > 0$  there is an isomorphism  $T$  of  $F$  onto a subspace of  $E$  with  $\|T\| \|T^{-1}\| < 1 + \varepsilon$ ).

*Proof.* Immediate from Propositions 2.1(b) and (2.2) and from the fact that every non reflexive Banach space contains a separable non reflexive subspace.  $\square$

REMARK. It is not clear, however, from the above whether “uniform normal structure” is a superproperty, i.e. whether “reflexive” can be dropped in (c) or, equivalently, whether “uniform normal structure” implies superreflexivity.

We observe here that the (absolute) Jung constant  $J(E)$  cannot be estimated from either side by the Jung constants of its subspaces in a similar way. Any space  $E$  is a subspace of some  $\mathcal{P}_1$ -space  $F = l_\infty(\Gamma)$  for some  $\Gamma$  (e.g. the dual ball) and  $J(F) = 1$  while  $J(E)$  can be any. Thus we may have  $J(E) > J(F)$  when  $E \subset F$ . We cannot also get lower bounds for  $J(E)$  by considering finite or separable subsets, as shown by:

2.4. EXAMPLES. (a)  $J(c_0) = 2$  by Proposition 1.1 (e.g. take  $A = \{(-1)^n e_n; n = 1, 2, \dots\}$ , then  $\text{diam } A = 1 = r_{c_0}(A)$ ). However, for every finite  $A = \{x_1, \dots, x_n\} \subset c_0$ ,  $\bar{x} = \frac{1}{2}(\max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i) \in c_0$  satisfies  $r(\bar{x}, A) = \frac{1}{2} \text{diam } A$ .

(b) Let  $\Gamma$  be an uncountable set,  $E = \{x \notin m(\Gamma); \text{spt } x \text{ countable}\}$  (where  $\text{spt } x = \{\gamma; x(\gamma) \neq 0\}$ ).  $E$  is a closed subspace of  $m(\Gamma)$ , hence Banach. Every separable subset of  $E$  is contained in a subspace of  $m(\Gamma_0)$ , where  $\Gamma_0 \subset \Gamma$  is countable (the union of the supports of a dense sequence).  $m(\Gamma_0)$  is a subspace of  $E$  isometric to  $m = l_\infty$  which has Jung constant  $J(M) = 1$ . On the other hand, let  $\Gamma = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1, \Gamma_2$  are uncountable and disjoint,  $A_i = \{x \in E; 0 \leq x \leq x_{\Gamma_i}\}$  ( $i = 1, 2$ ),  $A = A_1 \cup (-A_2)$ . It is easily seen that  $\text{diam } A = 1$  but  $r(A) = 1$ . Thus  $J(E) = 2$ .

On the other hand, we have:

2.5. PROPOSITION. Let  $(E_\alpha)_{\alpha \in D}$  be a net of linear subspaces of the Banach space  $E$ , directed by inclusion, such that  $\bigcup_{\alpha \in D} E_\alpha = E$ . Then: (a) If  $E$  is reflexive, then  $J_s(E) = \sup_\alpha J_s(E_\alpha) = \lim_{\alpha \in D} J_s(E_\alpha)$ .

(b) If  $E$  is a dual space and each  $E_\alpha$  admits a norm-1 linear projection  $P_\alpha$ , then  $J(E) = \sup_\alpha J(E_\alpha) = \lim_{\alpha \in D} J(E_\alpha)$ .

*Proof.* If  $P$  is a norm-1 projection of  $E$  onto  $F$ , then for every  $A \subset F$ ,  $x \in E$  we have  $r(Px, A) \leq r(x, A)$ , hence  $r_F(A) = r_E(A)$ , thus  $J(F) \leq J(E)$ . Therefore for every  $\alpha \leq \beta$  we have  $J_s(E_\alpha) \leq J_s(E_\beta) \leq J_s(E)$  or  $J(E_\alpha) \leq J(E_\beta) \leq J(E)$ , respectively. In either case it is enough to consider  $A = \text{conv}(x_1, \dots, x_n) \in E$  with

$$\frac{r_A(A)}{\text{diam } A} > \frac{1}{2}J_s(E) - \varepsilon \quad \text{or} \quad \frac{r_E(A)}{\text{diam } A} > \frac{1}{2}J(E) - \varepsilon,$$

respectively. But taking  $x'_1, \dots, x'_n \in E_\alpha$  with  $\|x_i - x'_i\| < \varepsilon$  for  $i = 1, \dots, n$  we get  $A' = \text{conv}\{x'_1, \dots, x'_n\} \subset E_\alpha$  (for some  $\alpha$ ) satisfying, respectively,

$$\frac{r_{A'}(A')}{\text{diam } A'} > \frac{r_A(A) - \varepsilon}{\text{diam } A + \varepsilon} \quad \text{or} \quad \frac{r_E(A')}{\text{diam } A'} > \frac{r_E(A) - \varepsilon}{\text{diam } A + \varepsilon}. \quad \square$$

**2.6. COROLLARY.** *For every  $1 \leq p < \infty$  and every infinite dimensional  $L_p(\mu)$  space we have*

- (a)  $J_s(L_p(\mu)) = J_s(l_p) = \sup_n J_s(l_p^n) = \lim_n J_s(l_p^n)$  and
- (b)  $J(L_p(\mu)) = J(l_p) = \sup_n J(l_p^n) = \lim_n J(l_p^n)$ .

*Proof.* For every measurable partition  $D = \{D_0, D_1, \dots, D_n\}$  of the measure space, with  $0 < \mu(D_i) < \infty$  for  $i = 1, \dots, n$ , the characteristic functions  $\{\chi_{D_1}, \dots, \chi_{D_n}\}$  span in  $L_p(\mu)$  a subspace  $F_D$  isometric to  $l_p^n$ , and admitting the norm-1 projection  $P_D f = \sum_{i=1}^n (\int_{D_i} f d\mu) \chi_{D_i} / \mu(D_i)$ . The  $F_D$  clearly form a net directed by inclusion whose union is dense in  $L_p(\mu)$ , so that we can apply Proposition 2.5. □

In order to give lower bounds for  $J$  and  $J_s$  in  $n$ -dimensional spaces, consider “ $(n, m, r)$ -symmetric block designs”, i.e. 0-1 symmetric  $n \times n$  matrices  $A = (a_{ij})_{i,j=1}^n$  such that

$$\sum_{j=1}^n a_{ij} a_{kj} = \begin{cases} m & \text{if } i = k, \\ r & \text{if } i \neq k, \end{cases}$$

where  $n > m > 0$  and  $r$  is, necessarily,  $m(m-1)/(n-1)$ .

**2.7. LEMMA.** *If  $E$  is an  $n$ -dimensional space with a symmetric basis  $(e_k)_{k=1}^n$  (i.e. such that  $\|\sum_{k=1}^n |\alpha_k| e_k\| = \|\sum_{k=1}^n \alpha_{\pi(k)} e_k\|$  for all scalars  $\alpha_1, \dots, \alpha_n$  and all permutations  $\pi$  of  $\{1, \dots, n\}$ ), and if there is an  $(n, m, r)$ -symmetric block design  $(a_{ij})_{i,j=1}^n$ , then*

$$J(E) \geq 2 \min_{0 \leq \alpha \leq 1} \left\| \left( (1 - \alpha) \sum_{i=1}^m e_i + \alpha \sum_{i=m+1}^n e_i \right) \left\| \sum_{i=1}^{2(m-r)} e_i \right\|^{-1} \right\|$$



and

$$J_s(E) \geq 2 \left\| \left( \left( 1 - \frac{m}{n} \right) \sum_{i=1}^m e_i + \frac{m}{n} \sum_{i=m+1}^n e_i \right) \right\| \left\| \sum_{i=1}^{2(m-r)} e_i \right\|^{-1}.$$

If there is an  $(n, m, m/2)$ -symmetric block design (hence, necessarily,  $m = (n + 1)/2$ ), then also

$$J_s(E) \geq \left\| \sum_{i=1}^n e_i \right\| \left\| \sum_{i=1}^m e_i \right\|^{-1}.$$

*Proof.* Consider the points  $x_i = \sum_{j=1}^n a_{ij} e_j$  and the sets  $A = \text{conv}(x_1, \dots, x_n)$  or  $A_0 = \text{conv}(0, x_1, \dots, x_n)$ , respectively. By symmetry, center points are multiples of  $\sum_{i=1}^n e_i$ . Also,

$$\min_{0 \leq \alpha \leq m/n} \max \left( \left\| (1 - \alpha) \sum_{i=1}^m e_i + \alpha \sum_{i=m+1}^n e_i \right\|, \alpha \left\| \sum_{i=1}^n e_i \right\| \right) = \frac{1}{2} \left\| \sum_{i=1}^n e_i \right\|. \quad \square$$

**2.8. COROLLARY.** *If there is an  $(n, m, r)$ -symmetric block design then, for every  $1 \leq p \leq \infty$ , we have*

$$J(l_p^n) \geq \left( \frac{2^{p-1}(n-1)}{\|(m, n-m)\|_{q-1}} \right)^{1/p} \quad \left( \text{where } \frac{1}{p} + \frac{1}{q} = 1 \right)$$

(since the minimizing  $\alpha$  for  $p > 1$  is  $m^{1/p-1}/(m^{1/p-1} + (n-m)^{1/p-1})$  and for  $p = 1$  it is 1 if  $m \geq 2n$  and 0 if  $m \leq 2n$ ), and

$$J_s(l_p^n) \geq \frac{(2^{p-1}(n-1)\|(m, n-m)\|_{p-1}^{p-1})^{1/p}}{n}.$$

If there is an  $(n, m, m/2)$ -symmetric block design, then

$$J_s(l_p^n) \geq (2n/(n+1))^{1/p}.$$

**2.9. LEMMA.** *There are  $(n, m, r)$ -symmetric block designs in each of the following cases:*

- (a)  $n$  is any,  $m = 1, r = 0$ , or  $m = n - 1, r = n - 2$ .
- (b)  $n = 2^{2^t}, m = 2^{t-1}(2^t - 1), r = 2^{t-1}(2^{t-1} - 1)$  or  $m = 2^{t-1}(2^t + 1), r = 2^{t-1}(2^{t-1} + 1)$ .
- (c)  $n = 2^t - 1, m = 2^{t-1}, r = 2^{t-2}$ , or  $m = 2^{t-1} - 1, r = 2^{t-2} - 1$ .

*Proof.* For (a) take the unit matrix,  $a_{ij} = \delta_{ij}$  or its complement  $a_{ij} = 1 - \delta_{ij}$ . For (b) define, inductively,  $A_0 = (1)$ ,  $B_0 = (0)$ ,

$$A_{t+1} = \begin{pmatrix} B_t & A_t & A_t & A_t \\ A_t & B_t & A_t & A_t \\ A_t & A_t & B_t & A_t \\ A_t & A_t & A_t & B_t \end{pmatrix}, \quad (B_{t+1})_{ij} = 1 - (A_{t+1})_{ij}.$$

For (c), let  $W_t = (w_{ij}^t)_{i,j=1}^{2^t}$  be the Walsh matrix, defined inductively by  $W_0 = (1)$ ,

$$W_{t+1} = \begin{pmatrix} W_t & W_t \\ W_t & -W_t \end{pmatrix},$$

and consider  $(\frac{1}{2}(1 - w_{ij}^t))_{i,j=2}^{2^t}$ . □

2.10. COROLLARY. (a)  $J(l_p^{2^t}) \geq ((2^t - 1)/2^{t-1})^{1/p}$ .

(b)  $J_s(l_p^n) \geq 2^{p-1/p}[(n-1) + (n-1)^p]^{1/p}/n$ .

(c)  $J_s(l_p^{2^t-1}) \geq ((2^t - 1)/2^{t-1})^{1/p}$ .

(d)  $J(l_p) \geq 2^{1/p}$ .

(e)  $J_s(l_p) \geq \max(2^{1/p}, 2^{p-1/p})$ .

((e) follows also from Corollary 2.6 and Bynum's estimate  $WCS(L_p) \leq \min(2^{p-1/p}, 2^{1/p})$ .)

2.11. COROLLARY. (a)  $J_s(E) \geq 2^{1/p_E}$ , where  $p_E = \inf\{p; l_p \text{ is finitely represented in } E\} = \text{the maximal "type" of } E \text{ in the sense of Maurey and Pisier [17]}$ . Thus, if  $E$  has uniform normal structure, it is "B-convex" ([13]). In fact, stronger conditions are imposed on  $E$  (cf. [1]).

(b) For every infinite-dimensional  $E$ ,  $J_s(E) \geq \sqrt{2}$  (Maluta, [16]) (since  $p_E \leq 2$  by Dvoretzky's theorem).

Now we observe some upper bounds.

2.12. PROPOSITION. If  $\dim E \leq n$ , then  $J_s(E) \leq 2n/(n+1)$ .

*Proof.* Given a convex  $A \subset E$  with  $\text{diam } A = 1$ , take any  $r < r_A(A)$ . Then  $\bigcap_{x \in A} B(x, r) \cap \bar{A} = \emptyset$  hence by Helly's theorem, there are  $x_0, \dots, x_n \in A$  with  $\bigcap_{i=0}^n B(x_i, r) \cap A = \emptyset$ . But, taking

$$\bar{x} = \frac{1}{n+1} \sum_{i=0}^n x_i \in A,$$

we have

$$\begin{aligned} \|\bar{x} - x_j\| &= \frac{1}{n+1} \left\| \sum_{i=0}^n (x_i - x_j) \right\| = \frac{1}{n+1} \left\| \sum_{i \neq j} (x_i - x_j) \right\| \\ &\leq \frac{1}{n+1} \max_{i \neq j} \|x_i - x_j\| \leq \frac{n}{n+1}, \end{aligned}$$

hence  $r < n/(n+1)$ . Since  $r < r_A(A)$  was arbitrary,  $r_A(A) \leq n/(n+1)$ .  $\square$

If  $(x_0, x_1, \dots, x_n) \in E$ , the “ $n$ -volume” of  $\text{conv}(x_0, \dots, x_n)$  is

$$V(x_0, \dots, x_n) = \sup \left\{ \det \begin{pmatrix} 1, f_1, \dots, f_n \\ x_0, x_1, \dots, x_n \end{pmatrix}; f_i \in B(E^*), i = 1, \dots, n \right\}.$$

Following Sullivan [21], we define the modulus of  $n$ -convexity of  $E$ ,

$$\delta_E^{(n)}(\varepsilon) = \inf \left\{ 1 - \frac{1}{n+1} \left\| \sum_{i=0}^n x_i \right\|; x_i \in B(E), \right. \\ \left. i = 0, \dots, n, V(x_0, \dots, x_n) \geq \varepsilon \right\}$$

(so that

$$\delta_E^{(1)}(\varepsilon) = \delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x_0 + x_1}{2} \right\|; x_0, x_1 \in B(E), \|x_0 - x_1\| \geq \varepsilon \right\}$$

is the ordinary modulus of convexity). Sullivan showed that if  $E$  is “ $n$ -uniformly convex”, i.e. if  $\delta_E^{(n)}(\varepsilon) > 0$  for all  $\varepsilon > 0$ , then  $E$  is superreflexive and has normal structure. Bynum [3] observed that  $J_s(E) \leq 2(1 - \delta_E(1))$ . One can push this argument one step further:

### 2.13. PROPOSITION.

$$J_s(E) \leq 2 \min_{\varepsilon} \max \left( 1 - \delta_E^{(2)}(\varepsilon), \frac{2}{3}\varepsilon + \frac{1}{2} \right).$$

*Proof.* Let  $A \subset E$  be convex with  $\text{diam } A = 1$ . Suppose  $r_A(A) > r > 1 - \delta_E^{(2)}(\varepsilon)$ . Take  $\eta > 0$  and  $x_0, x_1 \in A$  with  $\|x_1 - x_0\| > 1 - \eta$  and  $x_2 \in A$  with

$$\left\| x_2 - \frac{x_0 + x_1}{2} \right\| > r,$$

$x_3 \in A$  with

$$\left\| x_3 - \frac{x_0 + x_1 + x_2}{3} \right\| > r.$$

Translating, we may assume  $x_3 = 0$ . Take  $f_1 \in B(E^*)$  with

$$f_1(x_1 - x_0) > 1 - \eta \quad \text{and} \quad f_2 \in B(E^*)$$

with

$$f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) > r.$$

Then

$$\begin{aligned} V(x_0, x_1, x_2) &= \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_0) & f_1(x_1) & f_1(x_2) \\ f_2(x_0) & f_2(x_1) & f_2(x_2) \end{vmatrix} \\ &= \begin{vmatrix} f_1(x_1 - x_0) & f_1(x_2 - (x_0 + x_1)/2) \\ f_2(x_1 - x_0) & f_2(x_2 - (x_0 + x_1)/2) \end{vmatrix} \\ &> (1 - \eta)r - f_1\left(x_2 - \frac{x_0 + x_1}{2}\right)f_2(x_1 - x_0). \end{aligned}$$

But

$$\begin{aligned} f_1\left(x_2 - \frac{x_0 + x_1}{2}\right) &= f_1(x_1 - x_0) + f_1\left(\frac{2x_2 + x_0}{2} - \frac{3x_1}{2}\right) \\ &= f_1(x_1 - x_0) + \frac{3}{2}f_1\left(\frac{2x_2 + x_0}{3} - x_1\right) \\ &> (1 - \eta) - \frac{3}{2} = -\frac{1}{2} - \eta \end{aligned}$$

and also

$$f_1\left(x_2 - \frac{x_0 + x_1}{2}\right) = -f_1(x_1 - x_0) + f_1\left(\frac{2x_2 + x_1}{2} - \frac{3x_0}{2}\right) < \frac{1}{2} + \eta.$$

Similarly,

$$f_2(x_1 - x_0) = f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) + f_2\left(\frac{3x_1}{2} - \frac{2x_2 + x_0}{2}\right) > r - \frac{3}{2}$$

and

$$f_2(x_1 - x_0) = -f_2\left(x_2 - \frac{x_0 + x_1}{2}\right) + f_2\left(\frac{x_1 + 2x_2}{2} - \frac{3x_0}{2}\right) < \frac{3}{2} - r.$$

Thus

$$\varepsilon \geq V(x_0, x_1, x_2) > (1 - \eta)r - \left(\frac{1}{2} + \eta\right)\left(\frac{3}{2} - r\right).$$

Since  $\eta > 0$  was arbitrary,  $3r/2 - 3/4 \leq \varepsilon$  or  $r \leq 2\varepsilon/3 + 1/2$ .  $\square$

If we use this estimate for  $l_2$  we get  $J_s(l_2) \leq 1.61$  (while  $2(1 - \delta_{l_2}(1)) = \sqrt{3}$  and  $J_s(l_2) = \sqrt{2}$ ). In any  $E$ , if  $\delta_E^{(2)}(3/4) > 0$ , then  $J_s(E) < 2$ .

**2.14. PROPOSITION.** *For every  $n$  and every  $\varepsilon > 0$ , we have  $J_s(E) \leq 2 \max(1 - (1 - \varepsilon)/n!, 1 - \delta_E^{(n)}(\varepsilon))$ , so that if  $\delta_E^{(n)}(1) > 0$  then  $E$  has uniform normal structure.*

*Proof.* Let  $A \subset E$  be convex with  $\text{diam } A = 1$ . Take any  $r < r_A(A)$  and any  $\eta > 0$ . Find  $x_0, x_1 \in A$  with  $\|x_0 - x_1\| > 1 - \eta$  and  $x_k \in A$ ,  $k = 2, 3, \dots, n+1$  with  $\|x_k - k^{-1}\sum_{i=0}^{k-1}x_i\| > r$  (such  $x_k$  exist since  $k^{-1}\sum_{i=0}^{k-1}x_i \in A$  and  $r_A(A) > r$ ). Translate to get  $x_{n+1} = 0$ , so that  $x_i \in B(E)$ ,  $i = 0, \dots, n$ . Find  $f_1 \in B(E^*)$  with  $f_1(x_1 - x_0) > 1 - \eta$  and  $f_k \in B(E^*)$ ,  $k = 2, \dots, n$ , with  $f_k(x_k - k^{-1}\sum_{i=0}^{k-1}x_i) > r$ . Consider

$$\begin{aligned} V(x_0, \dots, x_n) &\geq \det \left( \begin{array}{c} 1, f_1, \dots, f_n \\ x_0, x_1, \dots, x_n \end{array} \right) \\ &= \det \left( \begin{array}{c} f_1, f_2, f_3, \dots, f_n \\ x_1 - x_0, x_2 - \frac{1}{2}(x_0 + x_1), x_3 - \frac{1}{3} \sum_{i=0}^2 x_i, \dots, x_n - \frac{1}{n} \sum_{i=0}^{n-1} x_i \end{array} \right). \end{aligned}$$

All the entries in the last determinant have absolute value  $\leq 1$ , but the subdiagonal ones,  $f_m(x_k - k^{-1}\sum_{i=0}^{k-1}x_i)$  for  $m > k$ , are small for  $r$  close to 1: since  $m^{-1}\sum_{i=0}^{m-1}f_m(x_m - x_i) > r$  and  $|f_m(x_m - x_i)| \leq 1$ , we have  $1 - m(1 - r) < f_m(x_m - x_i) \leq 1$  for  $i < m$ , hence

$$|f_m(x_k - x_i)| = |f_m(x_m - x_i) - f_m(x_m - x_k)| < m(1 - r)$$

and

$$\left| f_m \left( x_k - \frac{1}{k} \sum_{i=0}^{k-1} x_i \right) \right| = \left| \frac{1}{k} \sum_{i=0}^{k-1} f_m(x_k - x_i) \right| < m(1 - r),$$

too. Thus

$$V(x_0, \dots, x_n) > (1 - \eta)r^{n-1} - (n! - 1)n(1 - r) > \varepsilon$$

provided  $r > 1 - (1 - \varepsilon - \eta)/n!n$ . Therefore for such  $r$  we must have

$$r < \left\| 1/(n+1) \sum_{i=0}^n x_i \right\| < 1 - \delta_E^{(n)}(\varepsilon).$$

Since  $\eta > 0$  and  $r < r_A(A)$  were arbitrary, we get

$$r_A(A) \leq \max(1 - \delta_E^{(n)}(\varepsilon), 1 - (1 - \varepsilon)/n!). \quad \square$$

**REMARK.** The rough estimate we used above can be improved, but since the computation of  $\delta_E^{(n)}$  seems to be quite complicated, it is not clear whether finer estimates will yield more results.

Lim [15] gave the following upper bound for  $J_s(l_p)$ ,  $p > 2$ :

$$J_s(l_p) \leq 2 \left( 1 + \frac{1 + t^{p-1}}{(1 + t)^{p-1}} \right)^{-1/p},$$

where  $0 \leq t \leq 1$  solves  $(p - 2)t^{p-1} + (p - 1)t^{p-2} = 1$ .

Maluta [16] defined another related constant for a normed  $E$ :

$$D(E) = \sup \left\{ \limsup_k \sup_{n \geq k} d(x_{n+1}, \text{conv}(x_i)_{i=1}^n); (x_n) \subset E, \text{diam}(x_n)_{n=1}^\infty = 1 \right\},$$

and showed that:

(i)  $D(E) = \sup\{D(F); F \subset E \text{ separable}\}$ ;

(ii)  $D(E) = 0$  if and only if  $E$  is finite-dimensional.

(iii) If  $D(E) < 1$  then the Banach space  $E$  is reflexive and has normal structure (but  $E = (\Sigma \oplus l_n)_2$  is reflexive and has normal structure although  $D(E) = 1$ ).

(iv)  $2D(E) \leq J_s(E)$  and, if  $E$  is reflexive,  $D(E) \leq 1/WCS(E)$ .

Maluta asked if  $D(E) = 1/WCS(E)$  for every reflexive  $E$ . She showed that this is the case for  $l_p$ , i.e.  $D(l_p) = 2^{-1/p}$  (Bynum showed  $WCS(l_p) = 2^{1/p}$ );  $D((\Sigma \oplus l_\infty)_2) = 2^{-1/2}$  (Bynum showed  $WCS((\Sigma \oplus l_\infty)_2) = \sqrt{2} > 1 = N((\Sigma \oplus l_\infty)_2)$ ). For the space  $l_{p,1}$ , i.e.  $l_p$  with the norm  $\|x\|_{p,1} = \|x^+\|_p + \|x^-\|_p$ , which is of special interest since in it  $\delta(1) = 0$ , one still has  $D(l_{p,1}) = 1/WCS(l_{p,1}) = 2^{-1/p}$ . We can give an affirmative answer to Maluta's question in the case that  $E$  satisfies the (weak) Opial condition:  $w_n \xrightarrow{w} 0 \Rightarrow \liminf \|x_n - x\| \geq \liminf \|x_n\| \forall x \neq 0$  [19]. The  $l_p$  spaces ( $1 < p < \infty$ ) satisfy this condition, but the  $L_p[0, 1]$  spaces do not, unless  $p = 2$ .

**2.15. PROPOSITION.** *If  $E$  satisfies Opial's condition, then  $D(E) \geq 1/WCS(E)$ .*

*Proof.* For any  $0 \leq r < 1/WCS(E)$ , we can find  $(x_n) \subset E$  with  $x_n \xrightarrow{w} 0$ ,  $\text{diam}(x_n) = 1$  and  $\limsup \|x_n - x\| > r$  for every  $x \in \text{conv}(x_n)$ . In particular,  $\limsup \|x_n\| > r + \varepsilon$  for some  $\varepsilon > 0$ , so that we can take a subsequence  $(x'_n)$  with  $\|x'_n\| > r + \varepsilon, \forall n$ . By Opial's condition we have  $\liminf \|x'_n - x\| \geq r + \varepsilon, \forall x$ . Let  $n_1 = 1$ . If  $n_1, \dots, n_k$  have been chosen, take a finite  $\varepsilon/2$ -net,  $(y_1, \dots, y_{m_k})$ , for  $\text{conv}(x'_{n_1}, \dots, x'_{n_k})$ , and find  $n_{k+1}$  so that  $\|x'_{n_k} - y_j\| > r + \varepsilon/2$  for every  $n \geq n_{k+1}, j \leq m_k$ . Then  $d(x'_{n_{k+1}}, \text{conv}(x'_i)_{i=1}^k) > r$ , so that  $D(E) \geq r$ .  $\square$

The parameters  $J(E)$ ,  $2D(E)$  and  $R(E)$  ([1]), although all of them between 1 and  $J_s(E)$ , are incomparable even for reflexive infinite dimensional spaces:

### 2.15. EXAMPLES.

(a)  $J(l_2) = 2D(l_2) = \sqrt{2} > 1 = R(l_2)$ .

(b)  $E = (\Sigma \oplus l_\infty^n)_2$ . Here  $J(E) = 1$ ;  $2D(E) = \sqrt{2}$  and  $R(E) = 2$ .

(c)  $E = (\Sigma \oplus l_1^n)_2$ . Here  $2D(E) = \sqrt{2}$  again, but  $J(E) = R(E) = 2$ .

In concluding, we remark that none of the convexity properties  $J(E) < 2$ ,  $J_s(E) < 2$ ,  $WCS(E) > 1$  or  $D(E) < 1$  is isomorphy invariant. In fact, the “best” spaces have “worst” equivalent renormings. For  $J$  this follows from Proposition 1.1 ( $m = C(\beta N)$  has a maximal subspace 2-isomorphic to it of the type  $C(T)$ ,  $T$  non-Stonian). For  $J_s$ ,  $WCS$  or  $D$ , it was observed by Maluta that  $D(l_2, \|\cdot\|_J) = 1$ , where  $\|x\|_J = \max(\|x\|_2, \sqrt{2}\|x\|_\infty)$ .

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