On (*k*, *d*)-Colorings and Fractional Nowhere-Zero Flows

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Abstract: The concepts of (k, d)-coloring and the star chromatic number, studied by Vince, by Bondy and Hell, and by Zhu are shown to reflect the cographic instance of a wider concept, that of fractional nowhere-zero flows in regular matroids. © 1998 John Wiley & Sons, Inc. J Graph Theory 28: 155–161, 1998

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1. INTRODUCTION

Vince [12] introduced the following generalization of chromatic number.

Definition 1.1. *A* (k, d)-coloring *of a graph G is a function $c : V(G) \to Z_k$ such that for every $xy \in E(G), |c(x) - c(y)| \geq d$. (Here, Z_k denotes the cyclic group of residues mod k, and $|a|$ is the smaller of the two integers a and $k - a$.) The* star chromatic number, *$\chi^*(G)$, is the infimum of k/d over all (k, d)-colorings of G.*

Vince proved, by means of analytical arguments, that this infimum is a minimum (and hence rational). He also proved that for every k, d such that $k/d \geq \chi^*(G)$, there exists a (k, d)-coloring of G. Setting $d = 1$ we have that the chromatic number of G is $\chi(G) = \lceil \chi^*(G) \rceil$. Later, Bondy and Hell [1] improved Vince's result by giving a purely combinatorial proof. A further study and an alternate definition of $\chi^*(G)$ in terms of homomorphisms into intervals of a unit circle appear in [14]. The purpose of this note is to show that (k, d)-colorings are instances of the more general concept of fractional nowhere-zero flows in regular matroids.

2. FRACTIONAL FLOWS IN GRAPHS

It is helpful to introduce the notion of fractional flows in graphs before considering the general matroidal case. Let k be a positive integer. A *k-flow* in a graph G is an orientation $\omega(G)$ together with a function $f : E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm(k-1)\}$ such that the net flow $\sum_{vu \in \delta^+(v)} f(vu) - \sum_{uv \in \delta^-(v)} f(uv)$ is zero for each $v \in V(G)$. The *flow index*, $\xi(G)$ is the least k for which G has a *nowhere-zero k*-flow (that is, $f(e) \neq 0$, for all $e \in E(G)$). This parameter has been studied by many authors (see [8] for a thorough review). We generalize this notion with the following.

Definition 2.1. *A* (k, d)-flow *in a graph G is a k-flow $(\omega(G), f)$ such that the range of f is contained in $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$. The* star flow index *$\xi^*(G)$ is the infimum of k/d over all (k, d)-flows in G.*

Thus, a $(k, 1)$-flow is the same as a nowhere-zero k-flow. We shall see that, analogously to (k, d)-colorings, the infimum in Definition 2.1 is a minimum, and G has a (k, d)-flow whenever $k/d \geq \xi^*(G)$, and, thus, that $\xi(G) = \lceil \xi^*(G) \rceil$.

It is well known that, in the setting of matroids, vertex colorings and nowhere-zero flows are dual concepts. In particular, if G is a plane graph and H its planar dual, then $\chi(G) = \xi(H)$. We shall see that a similar correspondence holds between the concepts of star chromatic number and star flow index.

3. FLOWS IN MATROIDS

The proper setting for the study of flows and colorings is that of regular matroids. We assume familiarity with the circuit/cocircuit axioms of basic matroid theory such as in [13]. Let $\mathcal{C}(\mathcal{B})$ denote the $\{0, 1\}$-valued circuit-element (cocircuit-element) incidence matrix of a matroid M. If M is binary then, over GF(2), we have $\mathcal{C}\mathcal{B}^T = 0$. An *orientation* $\omega(M)$ of M is a signing $(1 \mapsto \pm 1)$ of the elements of $\mathcal{C}$ and $\mathcal{B}$ such that $\mathcal{C}\mathcal{B}^T = 0$ as rational matrices. It is well known that a binary matroid is orientable if and only if it is *regular*. (See [13] for terminology and a proof.) It is a good exercise to find the relationship between orientations of a graph G and of the graphic matroid $M(G)$. For any circuit C in M, let $C^+(C^-)$ denote the set of elements in C that are positively (negatively) oriented with respect to $\omega(M)$. For any cocircuit B in M, we define B^+ and B^- similarly.

Let Γ be an abelian group. A Γ-*flow* in a regular matroid M is an orientation $\omega(M)$ and a function $f : M \to \Gamma$ such that, for every cocircuit $B, \sum_{e \in B^+} f(e) = \sum_{e \in B^-} f(e)$. A flow f is said to be *nowhere-zero* if $f(e) \neq 0$, for all $e \in M$. An *integer flow* is a Γ-flow where $\Gamma = \mathbb{Z}$, the ring of integers. For integers $0 < d < k$, a (k, d)-*flow* is an integer flow with values in the set $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$, and a *nowhere-zero* k-*flow* is a $(k, 1)$-flow. As with graphs, the *star flow index* $\xi^*(M)$ is the infimum of k/d over all (k, d)-flows in M, and the *flow index* $\xi(M)$ is the minimum k for which M has a nowhere-zero k-flow.

The following facts about nowhere-zero flows are well known and can be found in [11].

Proposition 3.1. *Let $\omega(M)$ be an oriented regular matroid.*

1. *If M has no coloops* (*one-element cocircuits*), *then M has a nowhere-zero k-flow for some integer k, and, hence, $\xi(M)$ and $\xi^*(M)$ are bounded.*
2. *For any abelian group Γ of order k, M has a nowhere-zero Γ-flow if and only if M has a nowhere-zero k-flow. Furthermore, if f is a Z_k-flow in M, then M has a k-flow f' such that $f'(e) \equiv f(e) \pmod{k}$, for all $e \in E$.*

Our starting point is the following lemma, due to Hoffman [7].

Lemma 3.1 (Hoffman's Lemma)**.** *Let M be an oriented regular matroid. Given a pair of non-negative rational functions $l, u : M \to \mathbb{Q}$ such that $0 \leq l(e) \leq u(e)$ for $e \in M$, there exists a rational flow $f : M \to \mathbb{Q}$ such that $l(e) \leq f(e) \leq u(e)$ for every $e \in M$ if and only if, for every cocircuit B,*

$$\sum_{e \in B^+} l(e) \leq \sum_{e \in B^-} u(e) \text{ and } \sum_{e \in B^-} l(e) \leq \sum_{e \in B^+} u(e). \tag{1}$$

Additionally, f can be chosen to be integer valued provided that l and u are integer valued.

In case M is graphic, Hoffman's Lemma is just the Ford–Fulkerson flow theorem [3]. If M is cographic, then this is the Potential Differences Existence Theorem of Ghouila–Houri [5]. If $l(e) \equiv l$ and $u(e) \equiv u$ are constant, then (1) becomes

$$\frac{l}{u} \leq \frac{|B^+|}{|B^-|} \leq \frac{u}{l}.$$

Thus, by Hoffman's Lemma with $l \equiv d$ and $u \equiv k - d$, we obtain the following.

Theorem 3.1. *A regular matroid M has a (k, d)-flow if and only if there exists an orientation $\omega(M)$ such that, for any cocircuit $B, d/(k-d) \leq |B^+|/|B^-| \leq (k-d)/d$.*

Corollary 3.1. *The star flow index $\xi^*(M)$ of a regular matroid M is the minimum over all orientations $\omega(M)$ of*

$$1 + \max\left\{\frac{|B^+|}{|B^-|}, \frac{|B^-|}{|B^+|} : B \text{ is a cocircuit in } M\right\}.$$

$$= \max\left\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|} : B \text{ is a cocircuit in } M\right\}.$$

This maximum is unbounded (and, hence, $\xi^*(M) := \infty$) if and only if M has a coloop. Putting $d = 1$, we have that, for any regular matroid M,

$$\xi(M) = \lceil \xi^*(M) \rceil.$$

A (k, d)-coloring $c : V(G) \to Z_k$ of an (arbitrarily oriented) graph G induces a Z_k-nowhere-zero flow f in the cographic matroid $M^*(G)$ by letting $f(xy) = c(x) - c(y)$ for every arc $xy \in G$. By 2. of Proposition 3.1, this is equivalent to the existence of an integer flow in $M^*(G)$ whose values range in absolute value between d and $k - d$, that is, a (k, d)-flow in $M^*(G)$. This process can be reversed to obtain a (k, d)-coloring of G from a (k, d)-flow of $M^*(G)$. Thus, from Theorem 3.1 we have the following.

Corollary 3.2. *The star chromatic number $\chi^*(G) = \xi^*(M^*(G))$ of a graph G equals*

$$\min_{\omega(G)} \max_{C} \left\{\frac{|C|}{|C^+|}, \frac{|C|}{|C^-|}\right\},$$

where the minimum is over all orientations of G and the maximum is over all circuits of G.

We note that the characterization of the (integer) chromatic number $\chi = \lceil \chi^* \rceil$ of a graph via the formula of Corollary 3.2 was proved independently of Hoffman's Lemma by Minty [9].

4. SOME OBSERVATIONS REGARDING χ^* AND ξ^*

(1) Vince's results [12] regarding the star-chromatic number of a graph immediately follow from Corollary 3.2. For example, in the case of the odd circuit C_{2k+1}, at least $k + 1$ edges must be similarly oriented in any orientation and, hence, $\chi^*(C_{2k+1}) = (2k + 1)/k = 2 + 1/k$.

(2) Let $c : V \to Z_k$ be a (k, d)-coloring of a graph $G = (V, E)$. For each $a \in Z_k$, let $I(a)$ denote the independent set $\{v \in V : c(v) \in \{a, a + 1, \ldots, a + d - 1\} (\text{mod } k)\}$. The k independent sets $\{I(a) : a \in Z_k\}$ together cover every vertex

exactly d times. Let us call such a collection a (k, d)-*independent cover*. Since any graph with a (k, d)-independent cover has an independent set of size at least $|V|d/k$, it follows that $\alpha(G) \geq |V|/\chi^*(G)$, an improvement on the well-known bound $|V|/\chi(G)$.

Although a (k, d)-coloring always provides a (k, d)-independent cover, the two concepts are not equivalent. Take, for example, the graph G_{10} on 10 vertices and 35 edges obtained by adding all edges joining two disjoint circuits of length five. Each "side" of G_{10} induces a C_5 subgraph and, hence, has a (5, 2)-independent cover. Two such covers, one from each "side," form a (10, 2)-independent cover of G_{10}. On the other hand, G_{10} does not admit a (10, 2)-coloring as $\chi(G_{10}) = 6$.

(3) A *weighted independent cover* is a collection of independent sets, each of which is assigned a positive rational weight, such that the total weight of the sets containing each vertex is at least 1. The *fractional chromatic number* $\chi^f(G)$ is defined to be the least total weight of any weighted independent cover of G. This parameter has been studied in several articles (see [4], [6], for example). As the existence of a (k, d)-independent cover of G implies $\chi^f(G) \leq k/d$, we have the following.

Observation 4.1. *For any graph G, $\chi^f(G) \leq \chi^*(G)$.*

Equality does not always hold here; for instance, $\chi^f(G_{10}) = 5$, while $\chi^*(G_{10}) = 6$. (We leave these for the reader to check!)

(4) Let the graph $G = (V, E_1 \cup E_2)$ be the union of two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Obviously, $\chi(G) \leq \chi(G_1)\chi(G_2)$. Such a product formula also holds for the flow index—a fact utilized in Seymour's proof [10] that $\xi(G) \leq 6 = 2 \times 3$ for any 2-edge connected graph G. Unfortunately, analogous statements, where χ and ξ are replaced by χ^* and ξ^*, are false. A counterexample for χ^* is provided again by the graph G_{10}; the star chromatic number of the disjoint union of two C_5's is 2.5 and $\chi^*(K_{5,5}) = 2$, whereas $\chi^*(G_{10}) = 6$. Using a similar construction, one can find, for any pair of rational numbers $a, b \geq 2$, a graph G consisting of two subgraphs G_1 and G_2, such that $\chi^*(G_1) = a, \chi^*(G_2) = b$, and $\chi^*(G) = \lceil a \rceil \lceil b \rceil$. Analogous examples exist for ξ^*.

(5) We finish with an extension of the notion of chromatic number to (general) orientable matroids. As explained in [2], orientable matroids need not be binary (as is tacitly assumed in some works such as [13]). The following definition is more general than—but consistent with—that given in Section 3. An *orientation* of an arbitrary matroid is a signing $1 \to \pm 1$ of $\mathcal{C}$ and $\mathcal{B}$ such that, for any row C of $\mathcal{C}$ and any row B of $\mathcal{B}$, if $C_e, B_e \neq 0$ for some $e \in E$, then there exists $f \in E \setminus \{e\}$ such that one of C_eB_e, C_fB_f equals $+1$ and the other equals -1. A matroid is *orientable* if it has at least one orientation. One can use Corollaries 3.1 and 3.2 to *define* $\xi^*(M)$ and $\chi^*(M)$ (and, hence, $\xi(M)$ and $\chi(M)$) for an arbitrary orientable matroid M. There are several natural questions one might ask. For example, the chromatic number of a (loop-free) orientable matroid of rank r is bounded by the size of its largest circuit, which is at most $r + 1$. However, we do not know whether the flow index of a (coloop-free) orientable matroid of bounded

rank is bounded. (This is true for regular matroids, since their underlying simple matroids have bounded size.)

Two orientations of M are said to belong to the same *reorientation class* if one is obtained from the other by multiplying a corresponding set of columns of $\mathcal{B}$ and $\mathcal{C}$ by -1. Although regular matroids have only one reorientation class, orientable matroids can have many reorientation classes. Winfried Hochstättler has pointed out that it may be more sensible to define ξ^* (and χ^*) for each reorientation class $\psi(M)$ of M by appropriately restricting the minimum in Corollary 3.2.

Definition 4.1. *The star flow index of a reorientation class $\psi(M)$ of an orientable matroid M is given by*

$$\xi^*(\psi(M)) = \min_{\omega \in \psi(M)} \max_{B} \left\{ \frac{|B|}{|B^+|}, \frac{|B|}{|B^-|} \right\},$$

where the maximum is taken over the cocircuits B of M.

References

[1] J. A. Bondy and P. Hell, On the star chromatic number, *J. Graph Theory* **14** (1990), 479–482.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented matroids, encyclopedia of mathematics and its applications* **46,** Cambridge University Press, Cambridge (1993).

[3] L. R. Ford and D. R. Fulkerson, Maximal flow through a network, *Canad. J. Math.* **8** (1956), 399–404.

[4] Z. Füredi, Matching and covering in hypergraphs, *Graphs and Combinatorics* **4** (1988), 115–206.

[5] A. Ghouila–Houri, Sur l'existence d'un flot ou d'une tension prenant ses valeurs dans un groupe abélien, *C. R. Acad. Sciences* **250** (1960), 3931–3932.

[6] P. Hell and F. Roberts, Analogues of the Shannon capacity of a graph, *Ann. Discrete Math.* **12** (1982), 155–186.

[7] A. J. Hoffman, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, *Combinatorial Analysis: Proceedings of the Tenth Symposium in Applied Mathematics of the American Mathematical Society*, R. Bellman and M. Hall Jr., Eds., American Math. Soc. (1960), 113–128.

[8] F. Jaeger, Nowhere-zero flow problems, in *Selected topics in graph theory*, Vol. 3, L. Beineke and R. Wilson, Eds., Academic Press, New York (1988), 91–95.

[9] G. J. Minty, A theorem on n-coloring the points of a linear graph, *Amer. Math. Monthly* **69** (1962), 623–624.

[10] P. D. Seymour, Nowhere-zero 6-flows, *J. Combin. Theory Ser. B* **30** (1981), 130–135.

[11] W. T. Tutte, A contribution to the theory of chromatic polynomials. *Canad. J. Math.* **6** (1955), 80–91.

[12] A. Vince, Star chromatic number, *J. Graph Theory* **12** (1988), 551–559.

[13] D. Welsh, *Matroid theory,* Academic Press, San Francisco (1976).

[14] X. Zhu, Star chromatic numbers and products of graphs, *J. Graph Theory* **16** (1992), 557–569.