Gen. Math. Notes, Vol. 25, No. 2, December 2014, pp.68-77
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

On (k, μ)-Paracontact Metric Manifolds

D.G. Prakasha[1] and Kakasab K. Mirji[2]

[1,2]Department of Mathematics
Karnatak University, Dharwad - 580003, India
[1]E-mail: prakashadg@gmail.com
[2]E-mail: mirjikk@gmail.com

(Received: 7-8-14 / Accepted: 7-10-14)

Abstract

The object of this paper is to study (k, μ)-paracontact metric manifolds with qusi-conformal curvature tensor. It has been shown that, h-quasi conformally semi-symmetric and ϕ-quasi-conformally semi-symmetric (k, μ)-paracontact metric manifold with $k \neq -1$ cannot be an η-Einstein manifold.

Keywords: *(k, μ)-paracontact metric manifolds, Quasi-conformal curvature tensor, η-Einstein manifolds.*

1 Introduction

The study of paracontact geometry was initiated by Kaneyuki and Williams [7]. A systematic study of paracontact metric manifolds and their subclasses was started out by Zamkovay [16]. Since then several geometers studied paracontact metric manifolds and obtain various important properties of these manifolds ([1, 2, 3, 4, 5, 6, 11], etc). The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [10], the author introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (k, μ)-nullity condition (or distribution) for some real constant k and μ. Such manifolds are known as (k, μ)-paracontact metric manifolds. The class of (k, μ)-paracontact metric manifolds contains para-Sasakian manifolds.

As a generalization of locally symmetric spaces, many authors have studied semi-symmetric spaces and in turn their generalizations. A semi-Riemannian manifold (M^{2n+1}, g), $n \geq 1$, is said to be semi-symmetric if its curvature tensor

R satisfies $R(X,Y) \cdot R = 0$ for all vector fields X, Y on M^{2n+1}, where $R(X,Y)$ acts as a derivation on R ([9, 13]). In [15], Yildiz and De studied h-projectively semi-symmetric and ϕ-projectively semi-symmetric (k, μ)-contact metric manifolds.

In [14], Yano and Sawaki introduced the notion of quasi-conformal curvature tensor which is generalization of conformal curvature tensor. It plays an important role in differential geometry as well as in theory of relativity.

The present paper is organized as follows: Section 2 is devoted to preliminaries on (k, μ)-paracontact metric manifolds. In section 4 and 5, we study (k, μ)-paracontact metric manifold M^{2n+1} $(n > 1)$ with $k \neq -1$, satisfying h-quasi-conformally semi-symmetric and ϕ-quasi-conformally semi-symmetric conditions, respectively. It has been shown that, under both the conditions the manifold M^{2n+1} $(n > 1)$ cannot be an η-Einstein manifold.

2 Preliminaries

A contact manifold is an odd-dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given such a form η, there exists a unique vector field ξ, called the characteristic vector field or the Reeb vector field of η, satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M^{2n+1}. A semi-Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1,1) such that

$$\eta(X) = g(X, \xi), \ \ d\eta(X,Y) = g(X, \phi Y) \ \ and \ \ \phi^2 X = X - \eta(X)\xi, \tag{1}$$

for all vector field X, Y on M^{2n+1}. Then the structure (ϕ, ξ, η, g) on M^{2n+1} is called a paracontact metric structure and the manifold M^{2n+1} equipped with such a structure is said to be a paracontact metric manifold.

It can be easily seen that in a para-contact metric manifold the following relations hold:

$$\phi\xi = 0, \ \eta \cdot \phi = 0, \ g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2}$$

for any vector field X, Y on M^{2n+1}.

Given a paracontact metric manifold M^{2n+1} (ϕ, ξ, η, g) we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi g$, where $\pounds$ denotes the operator of Lie differentiation. Then h is symmetric and satisfies.

$$h\xi = 0, \ \ h\phi = -\phi h, \ \ Tr \cdot h = Tr \cdot \phi h = 0. \tag{3}$$

If $\bigtriangledown$ denotes the Levi-Civita connection of g, then we have the following relation

$$\bigtriangledown_X \xi = -\phi X + \phi h X. \tag{4}$$

A para-contact metric manifold M^{2n+1} (ϕ, ξ, η, g) for which ξ is a killing vector field or equivalently, $h = 0$ is called a K-paracontact manifold.

A paracontact metric structure (ϕ, ξ, η, g) is normal, that is, satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$. This is equivalent to

$$(\bigtriangledown_X \phi)Y = -g(X, Y)\xi + \eta(Y)X.$$

Any para-Sasakian manifold is K-paracontact, and the converse holds when $n = 1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y). \tag{5}$$

A paracontact metric manifolds M^{2n+1} (ϕ, ξ, η, g) is said to be a (k, μ)-space if its curvature tensor R satisfies

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{6}$$

for all tangent vector fields X, Y, where k, μ are smooth functions on M^{2n+1}.

Here, the characteristic vector field ξ belongs to the (k, μ)-nullity distribution. A paracontact metric manifold with ξ belongs to (k, μ)-nullity distribution is called a (k, μ)-paracontact metric manifold. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to k-nullity distribution. A paracontact metric manifold with ξ belongs to k-nullity distribution is called as $N(k)$-paracontact metric manifold.

The geometric behavior of the (k, μ)-paracontact metric manifold is different according as $k < -1$, $k = -1$ and $k > -1$. In particular, for the case $k < -1$ and $k > -1$, (k, μ)-nullity condition (6) determines the whole curvature tensor field completely. Fortunately, for both the case $k < -1$ and $k > -1$, same formula holds. For this reason, in this paper we consider the (k, μ)-paracontact metric manifolds with the condition $k \neq -1$(which is equivalent to take the case $k < -1$ and $k > -1$).

For a (k, μ)-paracontact metric manifold M^{2n+1} (ϕ, ξ, η, g) $(n > 1)$, the following identities hold:

$$\begin{aligned}
h^2 &= (1 + k)\phi^2, && (7)\\
(\bigtriangledown_X \phi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX), && (8)\\
S(X, Y) &= [2(1 - n) + n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) && (9)\\
&+ [2(n - 1) + n(2k - \mu)]\eta(X)\eta(Y), &&\\
S(X, \xi) &= 2nk\eta(X), && (10)\\
Q\xi &= 2nk\xi, && (11)\\
Q\phi - \phi Q &= 2[2(n - 1) + \mu]h\phi, && (12)
\end{aligned}$$

for any vector fields X, Y on M^{2n+1}, where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively.

A (k,μ)-paracontact metric manifold is called an η-Einstein manofold if it satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are two scalars.

For a $(2n+1)$-dimensional semi-Riemannian manifold, the quasi-conformal curvature tensor $\widetilde{C}$ is given by

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &- g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{13}$$

where a and b are two scalars, and r is the scalar curvature of the manifold. If $a = 1$ and $b = \frac{-1}{2n-1}$, then quasi-conformal curvature tensor reduces to conformal curvature tensor.

3 h-Quasi-conformally Semi-Symmetric (k,μ)-Paracontact Metric Manifold with $k \neq -1$

Definition 3.1 *A (k,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ $(n > 1)$ is said to be h-quasi-confomally semi-symmetric if the quasi-conformal curvature tensor $\widetilde{C}$ satisfies the condition*

$$\widetilde{C}(X,Y)\cdot h = 0, \tag{14}$$

for all X and Y on M^{2n+1}.

Let M^{2n+1} (ϕ,ξ,η,g) be a h-quasi conformally semisymmetric (k,μ)-paracontact metric manifold with $k \neq -1$. The condition (14) holds on M and, implies

$$(\widetilde{C}(X,Y)\cdot h)Z = \widetilde{C}(X,Y)hZ - h\widetilde{C}(X,Y)Z = 0, \tag{15}$$

for any vector fields X, Y and Z.

Using (13) in (15), we get

$$\begin{aligned} (\widetilde{C}(X,Y).h)Z &= a[R(X,Y)hZ - hR(X,Y)Z] \\ &+ b[S(Y,hZ)X - S(Y,Z)hX - S(X,hZ)Y + S(X,Z)hY \\ &+ g(Y,hZ)QX - g(Y,Z)hQX - g(X,hZ)QY \\ &+ g(X,Z)hQY] - \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y,hZ)X \\ &- g(Y,Z)hX - g(X,hZ)Y + g(X,Z)hY] = 0. \end{aligned} \tag{16}$$

In a paracontact metric manifold M^{2n+1} (ϕ, ξ, η, g) with $k \neq -1$, the following relation holds [10].

$$
\begin{aligned}
&R(X,Y)hZ - hR(X,Y)Z \qquad (17)\\
&= \{k[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] + \mu(1+k)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\}\xi\\
&+k\{g(X,\phi Z)\phi hY - g(Y,\phi Z)\phi hX + g(Z,\phi hX)\phi Y - g(Z,\phi hY)\phi X\\
&+\eta(Z)[\eta(X)hY - \eta(Y)hX)]\} - \mu\{(1+k)\eta(Z)[\eta(Y)X - \eta(X)Y]\\
&+2g(X,\phi Y)\phi hZ\},
\end{aligned}
$$

for all vector fields X, Y and Z.
By virtue of the relation (17), we obtain from (16) that

$$
\begin{aligned}
&a[\{k(g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)) \qquad (18)\\
+\ &\mu(1+k)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\}\xi\\
+\ &k\{g(X,\phi Z)\phi hY - g(Y,\phi Z)\phi hX + g(Z,\phi hX)\phi Y\\
-\ &g(Z,\phi hY)\phi X + \eta(Z)(\eta(X)hY - \eta(Y)hX)\}\\
-\ &\mu\{(1+k)\eta(Z)(\eta(Y)X - \eta(X)Y) + 2g(X,\phi Y)\phi hZ\}]\\
+\ &b[S(Y,hZ)X - S(Y,Z)hX - S(X,hZ)Y + S(X,Z)hY + g(Y,hZ)QX\\
-\ &g(Y,Z)hQX - g(X,hZ)QY + g(X,Z)hQY]\\
-\ &\frac{r}{(2n+1)}(\frac{a}{2n} + 2b)[g(Y,hZ)X - g(Y,Z)hX - g(X,hZ)Y + g(X,Z)hY] = 0.
\end{aligned}
$$

Substituting X by hX in (18) and using $h\xi = 0$, (7) and the symmetric property of h, we have

$$
\begin{aligned}
&a[(1+k)\{k(\eta(X)\eta(Z) - g(X,Z)) - \mu g(hX,Z)\}\eta(Y)\xi \qquad (19)\\
+\ &k\{g(hX,\phi Z)\phi hY - g(Z,\phi hY)\phi hX\\
-\ &(1+k)[g(Y,\phi Z)\phi X - g(Z,\phi X)\phi Y + \eta(Y)\eta(Z)(X - \eta(X)\xi)]\}\\
-\ &\mu\{(1+k)\eta(Z)\eta(Y)hX + 2g(hX,\phi hZ)\}]\\
+\ &b[S(Y,hZ)hX - (1+k)S(Y,Z)(X - \eta(X)\xi) - S(hX,hZ)Y + S(hX,Z)hY\\
+\ &g(Y,hZ)\phi hX - g(Y,Z)hQhX - g(hX,hZ)QY + g(hX,Z)hQY]\\
-\ &\frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y,hZ)hX - (1+k)(X - \eta(X)\xi)\\
-\ &g(hX,hZ)Y + g(hX,Z)hY] = 0.
\end{aligned}
$$

Taking inner product with ξ in (19) and making use of (2) and (3), we obtain

$$
\begin{aligned}
&a[(1+k)\{k(\eta(X)\eta(Z) - g(X,Z)) - \mu g(hX,Z)\}\eta(Y)] \qquad (20)\\
+\ &b[-S(hX,hZ)\eta(Y) - g(hX,hZ)g(QY,\xi)]\\
+\ &\frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)g(hX,hZ)\eta(Y) = 0.
\end{aligned}
$$

Putting $Y=\xi$ and using (11), we obtain from the above equation,

$$\begin{aligned} S(hX,hZ) &= \frac{a}{b}\left[(1+k)\{k(\eta(X)\eta(Z)-g(X,Z))-\mu g(hX,Z)\}\right] \\ &+ \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-2nk\right]g(hX,hZ). \end{aligned} \tag{21}$$

Replacing X by hX and Z by hZ in (21) and using (1) and (7), we have

$$\begin{aligned} S(X,Z) &= \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-\frac{ak}{b}-2nk\right]g(X,Z) \\ &- \frac{a\mu}{b}g(hX,Z)+\left[4nk+\frac{ak}{b}-\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\eta(X)\eta(Z). \end{aligned} \tag{22}$$

If $\mu=0$, from (22) it follows that the manifold is η-Einstein. Conversely, if the manifold is η-Einstein, then we can write

$$S(X,Z)=a_1g(X,Z)+b_1\eta(X)\eta(Z), \tag{23}$$

where a_1 and b_1 are two scalars.
From the above equation and (20), we obtain

$$\begin{aligned} &a_1g(X,Z)+b_1\eta(X)\eta(Z) \\ =& \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)-\frac{ak}{b}-2nk\right]g(X,Z) \\ -& \frac{a\mu}{b}g(hX,Z)+\left[4nk+\frac{ak}{b}-\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\eta(X)\eta(Z). \end{aligned} \tag{24}$$

Putting $Z=\phi X$ then using (2) and $g(X,\phi X)=0$, we get from (24) that

$$\frac{a\mu}{b}g(hX,\phi X)=0,$$

for all X. Consequently, $\mu=0$.

Hence, we see that a $(2n+1)$-dimensional $(n>1)$ h-quasi conformally semi-symmetric (k,μ)-paracontact metric manifold is an η-Einstein manifold, if and only if $\mu=0$.

But from (9), it follows that a (k,μ)-paracontact metric manifold is an η-Einstein manifold, if and only if $2(n-1)+\mu=0$. If we consider a $(2n+1)$-dimensional $(n>1)$ h-quasi conformally semi-symmetric η-Einstein (k,μ)-paracontact metric manifold, then $n=1$, which contradicts the fact that $n>1$. Hence, M^{2n+1} cannot be an η-Einstein manifold. This leads to the following:

Theorem 3.2 *A $(2n+1)$-dimensional $(n>1)$ h-quasi-conformally semi-symmetric (k,μ)-paracontact metric manifold with $k\neq -1$ cannot be an η-Einstein manifold.*

4 ϕ-Quasi-conformally Semi-Symmetric (k,μ)-Paracontact Metric Manifolds with $k \neq -1$

Definition 4.1 *A (k,μ)-paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ $(n > 1)$ is said to be ϕ-quasi-conformally semi-symmetric if the quasi-conformal curvature tensor $\widetilde{C}$ satisfies the condition*

$$\widetilde{C}(X,Y)\cdot\phi = 0, \tag{25}$$

for all X and Y on M^{2n+1}.

Let M be a $(2n+1)$-dimensional $(n > 1)$ ϕ-quasi-conformal semi-symmetric (k,μ)-paracontact metric manifold with $k \neq -1$. The condition $\widetilde{C}(X,Y)\cdot\phi = 0$ implies that

$$(\widetilde{C}(X,Y)\cdot\phi)Z = \widetilde{C}(X,Y)\phi Z - \phi\widetilde{C}(X,Y)Z = 0, \tag{26}$$

for any vector fields X,Y and Z.
By virtue of (13), we obtain from (26) that

$$\begin{aligned}
(\widetilde{C}(X,Y)\cdot\phi)Z &= a[R(X,Y)\phi Z - \phi R(X,Y)Z] \\
&+ b[S(Y,\phi Z)X - S(X,\phi Z)Y + g(Y,\phi Z)QX - g(X,\phi Z)QY \\
&- S(Y,Z)\phi X + S(X,Z)\phi Y - g(Y,Z)\phi QX + g(X,Z)\phi QY] \\
&- \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y,\phi Z)X \\
&- g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0.
\end{aligned} \tag{27}$$

A paracontact metric manifold M^{2n+1} (ϕ,ξ,η,g) with $k \neq -1$, then for any vector fields X, Y and Z on M^{2n+1}, the following relation holds [10].

$$\begin{aligned}
& R(X,Y)\phi Z - \phi R(X,Y)Z \\
= \;& [(1+k)(g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)) + (\mu-1)(g(\phi hX,Z)\eta(Y) \\
- \;& g(\phi hY,Z)\eta(X))]\xi + g(Y-hY,Z)(\phi X - \phi hX) - g(X-hX,Z)(\phi Y - \phi hY) \\
- \;& g(\phi X - \phi hX,Z)(Y-hY) + g(\phi Y - \phi hY,Z)(X-hX) \\
+ \;& \eta(Z)[(1+k)(\eta(X)\phi Y - \eta(Y)\phi X) + (\mu-1)(\eta(X)\phi hY - \eta(Y)\phi hX)].
\end{aligned} \tag{28}$$

Using (28) in (27), we get

$$\begin{aligned}
& a[\{(1+k)(g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)) \\
+ \;& (\mu-1)(g(\phi hX,Z)\eta(Y) - g(\phi hY,Z)\eta(X))\}\xi \\
+ \;& g(Y-hY,Z)(\phi X - \phi hX) - g(X-hX,Z)(\phi Y - \phi hY) \\
- \;& g(\phi X - \phi hX,Z)(Y-hY) + g(\phi Y - \phi hY,Z)(X-hX)
\end{aligned} \tag{29}$$

$$
\begin{aligned}
&+ \quad \eta(Z)\{(1+k)(\eta(X)\phi Y - \eta(Y)\phi X) + (\mu-1)(\eta(X)\phi hY - \eta(Y)\phi hX)\}] \\
&+ \quad b[S(Y,\phi Z)X - S(X,\phi Z)Y + g(Y,\phi Z)QX - g(X,\phi Z)QY \\
&- \quad S(Y,Z)\phi X + S(X,Z)\phi Y - g(Y,Z)\phi QX + g(X,Z)\phi QY] \\
&- \quad \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0.
\end{aligned}
$$

Substituting X by ϕX, in (29) and using $\phi\xi = 0$, (1) and skew symmetric property of ϕ, we get

$$
\begin{aligned}
& \quad a[(1+k)\{g(X-\eta(X)\xi, Z)\eta(Y) - (\mu-1)g(hX,Z)\eta(Y)\}\xi \\
&+ \quad g(Y-hY,Z)(X-\eta(X)\xi+hX) - g(\phi X - h\phi X, Z)(\phi Y - \phi hY) \\
&- \quad g(X-\eta(X)\xi+hX, Z)(Y-hY) + g(\phi Y - \phi hY, Z)(\phi X - h\phi X) \\
&+ \quad \{(1+k)(\eta(X)\xi - X) + (\mu-1)hX\}\eta(Y)\eta(Z)] \\
&+ \quad b[S(Y,\phi Z)\phi X - S(\phi X,\phi Z)Y + g(Y,\phi Z)Q\phi X - g(\phi X,\phi Z)QY \\
&- \quad S(Y,Z)(X-\eta(X)\xi) + S(\phi X, Z)\phi Y - g(Y,Z)\phi Q\phi X + g(\phi X,Z)\phi QY] \\
&- \quad \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)[g(Y,\phi Z)\phi X - g(\phi X,\phi Z)Y \\
&- \quad g(Y,Z)(X-\eta(X)\xi) + g(\phi X,Z)\phi Y] = 0.
\end{aligned} \tag{30}
$$

Taking inner product with ξ in (30) and making use of (2) and (3), we obtain

$$
\begin{aligned}
& a[\{k(g(X,Z-\eta(X)\eta(Z)) - \mu g(hX,Z)\}\eta(Y)] - b[S(\phi X,\phi Z)\eta(Y) \\
&+ \quad \left[\frac{r}{(2n+1)}\left(\frac{a}{2n}+2b\right) - 2nkb\right] g(\phi X,\phi Z)\eta(Y) = 0.
\end{aligned} \tag{31}
$$

Putting $Y = \xi$ and using $\eta(\xi) = 1$ we have

$$
\begin{aligned}
S(\phi X,\phi Z) &= \frac{a}{b}[k(g(X,Z) - \eta(X)\eta(Z)) - \mu g(hX,Z)] \\
&+ \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right) - 2nk\right] g(\phi X,\phi Z).
\end{aligned} \tag{32}
$$

If $\mu = 0$, from (32) it follows that

$$
\begin{aligned}
S(\phi X,\phi Z) &= \frac{a}{b}[k(g(X,Z) - \eta(X)\eta(Z))] \\
&+ \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right) - 2nk\right] g(\phi X,\phi Z).
\end{aligned} \tag{33}
$$

Replacing X by ϕX and Z by ϕZ in (33) and using (1) and (2) we obtain

$$
\begin{aligned}
S(X,Z) &= \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right) - \frac{ak}{b} - 2nk\right] g(X,Z) \\
&+ \left[4nk + \frac{ak}{b} - \frac{r}{b(2n+1)}\left(\frac{a}{2n}+2b\right)\right]\eta(X)\eta(Z).
\end{aligned} \tag{34}
$$

Thus M^{2n+1} is an η-Einstein manifold.
Conversely, if the manifold is an η-Einstein manifold, then we can write

$$S(X,Z) \quad = \quad a_2 g(X,Z) + b_2 \eta(X)\eta(Z), \tag{35}$$

where a_2 and b_2 are two scalars.
Replacing X by ϕX and Z by ϕZ (35), we obtain

$$S(\phi X, \phi Z) = a_2 g(\phi X, \phi Z). \tag{36}$$

From the equation (32) and (36), we obtain

$$\begin{aligned} a_2 g(\phi X, \phi Z) \quad &= \quad \frac{a}{b}\left[k(g(X,Z) - \eta(X)\eta(Z)) - \mu g(hX,Z)\right] \\ &+ \quad \left[\frac{r}{b(2n+1)}\left(\frac{a}{2n} + 2b\right) - 2nk\right] g(\phi X, \phi Z). \end{aligned} \tag{37}$$

Putting $Z = \phi X$ in (37), then using (1) and $g(X, \phi X) = 0$, we obtain

$$\frac{a}{b}\mu g(hX, \phi X) = 0,$$

for all X. Consequently $\mu = 0$.

Hence, we see that a $(2n+1)$-dimensional $(n > 1)$ ϕ-quasi conformally semi-symmetric (k,μ)-paracontact metric manifold is an η-Einstein manifold, if and only if $\mu = 0$.

Again, from (9), we shall get the same result as in previous section. Hence, M^{2n+1} cannot be an η-Einstein manifold. Thus, we are able to state the following:

Theorem 4.2 *A $(2n+1)$-dimensional $(n > 1)$ ϕ-quasi-conformally semi-symmetric (k,μ)-paracontact metric manifold cannot be an η-Einstein manifold.*

Acknowledgement: The first author is thankful to UGC, New Delhi for financial support in the form of Major Research Project and also the second author is thankful to Karnatak University, Dharwad for financial support in the form of UGC-UPE fellowship.

References

[1] D.V. Alekseevski, C. Medori and A. Tomassini, Maximally homogeneous para-CR manifolds, *Ann. Glob. Anal. Geom.*, 30(2006), 1-27.

[2] D.V. Alekseevski, V. Cortes, A.S. Galaev and T. Leistner, Cones over pseudo-Riemannian manifolds and their holonomy, *J. Reine Angew. Math.*, 635(2009), 23-69.

[3] G. Calvaruso and D. Perrone, Geometry of H-paracontact metric manifolds, arXiv:1307.7662v1 [*math.DG*], 29 Jul (2013).

[4] G. Calvaruso, Homogeneous paracontact metric three-manifolds, *Illinois J. Math.*, 55(2011), 697-718.

[5] S. Erdem, On almost (para) contact (hyperbolic) metric manifolds and harmonicity of (ϕ, ϕ')-holomorphic maps between them, *Houston J. Math.*, 28(2002), 21-45.

[6] S. Ivanov, D. Vassilev and S. Zamkovoy, Conformal paracontact curvature and the local flatness theorem, *Geom. Dedicata*, 144(2012), 115-129.

[7] S. Kaneyuki and F.L. Williams, Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.*, 99(1985), 173-187.

[8] S. Kaneyuki and M. Konzai, Paracomplex structures and affine symmetric spaces, *Tokiyo J. Math.*, 8(1985), 301-318.

[9] D. Kowalczyk, On some subclass of semisymmetric manifolds, *Soochow J. Math.*, 27(2001), 445-461.

[10] B.C. Montano, I.K. Erken and C. Murathan, Nullity conditions in paracontact geometry, *Different. Geom. Appl.*, 30(2010), 79-100.

[11] C. Murathan and I.K. Erken, The harmonicity of the Reeb vector dield on paracontact metric 3-manifolds, arXiv:1305.1511v3 [*math.DG*], 27 Dec (2013).

[12] K. Srivastava and S.K. Srivastava, On a class of paracontact metric 3-manifolds, *Different. Geom. Appl.*, arXiv:1404.1569v2, 9 April (2014).

[13] Z.I. Szabo, Structure theorems on Riemannian sp-aces satisfying $R(X,Y)\cdot R=0$, I: The local version, *J. Differential Geom.*, 17(4) (1982), 531-582.

[14] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Differential Geom.*, 2(1968), 161-184.

[15] A. Yildiz and U.C. De, A classification of (k,μ)-contact metric manifolds, *Commun. Korean Math. Soc.*, 27(2012), 327-339.

[16] S. Zamkovoy, Canonical connections on paracontact manifolds, *Ann Glob Anal Geom.*, 36(2009), 37-60.