

# On $k$ -saturated graphs with restrictions on the degrees

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## Abstract

A graph  $G$  is called  $k$ -saturated, where  $k \geq 3$  is an integer, if  $G$  is  $K^k$ -free but the addition of any edge produces a  $K^k$  (we denote by  $K^k$  a complete graph on  $k$  vertices). We investigate  $k$ -saturated graphs, and in particular the function  $F_k(n, D)$  defined as the minimal number of edges in a  $k$ -saturated graph on  $n$  vertices having maximal degree at most  $D$ . This investigation was suggested by Hajnal, and the case  $k = 3$  was studied by Füredi and Seress. The following are some of our results. For  $k = 4$ , we prove that  $F_4(n, D) = 4n - 15$  for  $n > n_0$  and  $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n - 2$ . For arbitrary  $k$ , we show that the limit  $\lim_{n \rightarrow \infty} F_k(n, cn)/n$  exists for all  $0 < c \leq 1$ , except maybe for some values of  $c$  contained in a sequence  $c_i \rightarrow 0$ . We also determine the asymptotic behaviour of this limit for  $c \rightarrow 0$ . We construct, for all  $k$  and all sufficiently large  $n$ , a  $k$ -saturated graph on  $n$  vertices with maximal degree at most  $2k\sqrt{n}$ , significantly improving an upper bound due to Hanson and Seyffarth.

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# 1 Introduction

A graph  $G = (V, E)$  is called *k-saturated* for an integer  $k \geq 3$  if  $G$  does not contain a complete graph on  $k$  vertices  $K^k$ , but the addition of any edge to  $G$  yields a  $K^k$ . The theorem of Erdős, Hajnal and Moon ([2]) states that if  $G$  is a  $k$ -saturated graph on  $n \geq k - 2$  vertices, then  $|E(G)| \geq (k - 2)n - \binom{k-1}{2}$ . However, for every  $k$  the extremal example for this theorem contains a vertex of degree  $n - 1$  (we call such a vertex *conical*). Hajnal ([6]) asked what is the minimal number of edges in a  $k$ -saturated graph on  $n$  vertices with no conical vertex, or, more generally, what is the minimal number of edges in a  $k$ -saturated graph on  $n$  vertices with all vertex degrees at most  $D$ . The case  $k = 3$  was treated by Füredi and Seress in [5]. Some additional results were obtained in [3]. Both papers used a linear programming method introduced by Pach and Surányi ([11]) for the study of the problem of determining the minimal number of edges in a graph of diameter two and all degrees at most  $D$ . In this paper we study the case  $k \geq 4$ . Our methods are similar to those of Füredi and Seress, but contain several new ingredients.

The following related problem was considered by Duffus and Hanson in [1]: what is the minimal number of edges in a  $k$ -saturated graph on  $n$  vertices with minimum degree  $\delta$ ? Some results on this problem are presented as well.

We also address the problem of the lowest possible maximal degree in a  $k$ -saturated graph on  $n$  vertices. Clearly, every  $k$ -saturated graph has diameter two, therefore it can easily be deduced that the maximal degree in a  $k$ -saturated graph is at least  $(n - 1)^{1/2}$ . Hanson and Seyffarth ([7]) constructed  $k$ -saturated graphs on  $n$  vertices with maximal degree  $O(n^{\alpha_k})$ , where  $\alpha_k < 1$ , but  $\alpha_k \rightarrow 1$  as  $k \rightarrow \infty$ . They also conjectured that the correct value of the lowest possible maximal degree is asymptotically  $c_k n^{1/2}$  as  $n \rightarrow \infty$ , where  $c_k$  is a constant depending only on  $k$ . In this paper we build  $k$ -saturated graphs with maximal degree  $O(n^{1/2})$  for each  $k$ , thus matching the lower bound up to a constant (depending on  $k$ ) factor. The case  $k = 3$  has already been done by Hanson and Seyffarth, and a better value for the constant was obtained by Füredi and Seress.

We end this section with some notation. For a graph  $G$  we denote by  $\bar{G}$  the complement of  $G$ . For a subset  $U \subseteq V(G)$  we denote by  $G[U]$  the induced subgraph of  $G$  on  $U$ . We also write  $G \setminus U$  instead of  $G[V(G) \setminus U]$ . The degree of a vertex  $x$  is denoted by  $d(x)$ . We denote by  $\Delta(G)$  and  $\delta(G)$  the maximal and the minimal degree of  $G$ , respectively. Let

$$\begin{aligned} F_k(n, D) &= \min\{|E(G)| : G \text{ is } k\text{-saturated, } |V(G)| = n, \Delta(G) \leq D\} , \\ F_k^*(n, D) &= \min\{|E(G)| : G \text{ is } k\text{-saturated, } |V(G)| = n, \Delta(G) = D\} \end{aligned}$$

(for triples  $(k, n, D)$ , for which the corresponding graphs do not exist we set  $F_k(n, D) =$

$\infty$  or  $F_k^*(n, D) = \infty$ ). The above definitions clearly imply

$$F_k(n, D') \leq F_k(n, D) \leq F_k^*(n, D)$$

for every  $D' \geq D$ . Using this notation, Hajnal's question is to determine  $F_k(n, D)$  and in particular  $F_k(n, n-2)$ .

The rest of the paper is organized as follows. In Section 2 we treat the values of  $F_k(n, D)$  and  $F_k^*(n, D)$  for  $D = n-2$  and  $D = n-3$ . In Section 3 we obtain some structural results for  $k$ -saturated graphs which are used to treat the case  $D = cn, 0 < c < 1$ . In Section 4 we consider the case  $D = o(n)$ . Some additional results on 4-saturated graphs and  $k$ -saturated graphs,  $k > 4$ , are presented in Sections 5 and 6, respectively.

## 2 Graphs with maximal degree $n-2$ or $n-3$

In this section we study the values of  $F_k^*(n, n-2)$  and  $F_k^*(n, n-3)$ . The results obtained supply some information about  $F_k(n, n-2)$  and  $F_k(n, n-3)$  as well. Later we will obtain additional results about these functions.

The following two propositions appear essentially (in dual form) in [10] (p. 447).

**Proposition 1** *Let  $k \geq 4$  and  $G = (V, E)$  be a  $k$ -saturated graph on  $n$  vertices with  $\Delta(G) = n-2$ . If  $d(x) = n-2$  and  $(x, y) \notin E(G)$ , then  $(y, z) \in E(G)$  for every vertex  $z \in V \setminus \{x, y\}$  and the graph  $G' = G \setminus \{x, y\}$  is  $(k-1)$ -saturated with no conical vertex. Conversely, given a  $(k-1)$ -saturated graph  $G'$  on  $n-2$  vertices with no conical vertex, one can add two non-adjacent vertices  $x$  and  $y$  and join them to all other vertices, thus obtaining a  $k$ -saturated graph  $G$  on  $n$  vertices with  $\Delta(G) = n-2$ .*

**Proposition 2** *Let  $k \geq 4$  and  $G = (V, E)$  be a  $k$ -saturated graph on  $n$  vertices with  $\Delta(G) = n-3$ . If  $d(x) = n-3$  and  $(x, u), (x, v) \notin E(G)$  then either*

1.  *$(u, v) \notin E(G)$  and then  $(u, z), (v, z) \in E(G)$  for every vertex  $z \in V \setminus \{x, u, v\}$  and  $G' = G \setminus \{x, u, v\}$  is  $(k-1)$ -saturated with  $\Delta(G') \leq n-6$ . Conversely, given a  $(k-1)$ -saturated graph  $G'$  on  $n-3$  vertices with  $\Delta(G') \leq n-6$ , one can add three independent vertices  $x, u$  and  $v$  and join them to all other vertices, thus obtaining a  $k$ -saturated graph  $G$  on  $n$  vertices with  $\Delta(G) = n-3$ ,*

or

2.  $(u, v) \in E(G)$  and then for each  $z \in V \setminus \{x, u, v\}$  at least one of the edges  $(u, z), (v, z)$  belongs to  $E(G)$  and the graph  $G'$ , obtained from  $G \setminus \{x, u, v\}$  by adding a new vertex  $w$  and joining it to the vertices that are joined in  $G$  to both  $u$  and  $v$ , is  $(k-1)$ -saturated with  $\Delta(G') \leq n-5$ . Conversely, given a  $(k-1)$ -saturated graph  $G'$  on  $n-2$  vertices with  $\Delta(G') \leq n-5$ , one can replace any vertex  $w \in V(G')$  by two new vertices  $u$  and  $v$ , join  $u$  and  $v$ , join both  $u$  and  $v$  to all vertices of  $V(G')$  to which  $w$  was joined, also join one of  $u, v$  to every other vertex of  $V(G') \setminus \{w\}$  so that both  $u$  and  $v$  are chosen at least once, and add a new vertex  $x$  joined to all other vertices but  $u$  and  $v$ , thus obtaining a  $k$ -saturated graph  $G$  on  $n$  vertices with  $\Delta(G) = n-3$ .

Turning to our notation, we can easily see that the above propositions imply:

**Proposition 3** For  $k \geq 4$  one has

1.  $F_k^*(n, n-2) = F_{k-1}(n-2, n-4) + 2n-4$  ;
2.  $F_k^*(n, n-3) = F_{k-1}(n-2, n-5) + 2n-5$  .

**Proof.** 1. Follows immediately from Proposition 1.

2. Suppose  $G$  is a  $k$ -saturated graph on  $n$  vertices with  $\Delta(G) = n-3$ . Let  $d(x) = n-3$ ,  $(x, u), (x, v) \notin E(G)$ . If  $(u, v) \notin E(G)$ , then according to part 1 of Proposition 2 the graph  $G' = G \setminus \{x, u, v\}$  is  $(k-1)$ -saturated with  $\Delta(G') \leq n-6$  and  $|E(G')| = |E(G)| - d(x) - d(u) - d(v) = |E(G)| - 3(n-3)$ , therefore

$$|E(G)| \geq F_{k-1}(n-3, n-6) + 3n-9 . \quad (1)$$

In case  $(u, v) \in E(G)$ , consider the graph  $G'$  described in part 2 of Proposition 2. Let  $V_1 = \{y \in V(G) \setminus \{x, u, v\} : (y, u) \in E(G), (y, v) \notin E(G)\}$ ,  $V_2 = \{y \in V(G) \setminus \{x, u, v\} : (y, v) \in E(G), (y, u) \notin E(G)\}$ , and  $V_3 = \{y \in V(G) \setminus \{x, u, v\} : (y, u) \in E(G), (y, v) \in E(G)\}$ . Then  $V_1 \cup V_2 \cup V_3 = V(G) \setminus \{x, u, v\}$ , and

$$\begin{aligned} |E(G')| &= |E(G)| - d(x) - 1 - |V_1| - |V_3| - |V_2| - |V_3| + |V_3| \\ &= |E(G)| - d(x) - 1 - |V_1| - |V_2| - |V_3| \\ &= |E(G)| - (n-3) - 1 - (n-3) \\ &= |E(G)| - (2n-5) . \end{aligned}$$

Recalling that  $G'$  is  $(k-1)$ -saturated on  $n-2$  vertices we obtain

$$|E(G)| \geq F_{k-1}(n-2, n-5) + 2n-5 . \quad (2)$$

Proposition 2 and inequalities (1), (2) imply that

$$F_k^*(n, n-3) = \min\{F_{k-1}(n-3, n-6) + 3n-9, F_{k-1}(n-2, n-5) + 2n-5\} .$$

Since one can obtain from a  $(k-1)$ -saturated graph  $G_1$  on  $n-3$  vertices with  $\Delta(G_1) \leq n-6$  and  $|E(G_1)| = F_{k-1}(n-3, n-6)$  a  $(k-1)$ -saturated graph  $G_2$  on  $n-2$  vertices with  $\Delta(G_2) \leq n-5$  and  $|E(G_2)| \leq |E(G_1)| + \Delta(G_1)$  just by fixing any vertex  $v \in V(G_1)$ , adding a new vertex  $u$  and joining it to all vertices of  $V(G_1)$  to which  $v$  is joined, we have

$$F_{k-1}(n-2, n-5) \leq F_{k-1}(n-3, n-6) + n-6 .$$

Therefore

$$F_k^*(n, n-3) = F_{k-1}(n-2, n-5) + 2n-5 . \quad \square$$

It follows from the results of Duffus and Hanson ([1], see also [5]) that  $F_3(n, n-2) = F_3(n, n-3) = 2n-5$  for  $n \geq 5$ . Hence we derive:

**Corollary 1**    1.  $F_4^*(n, n-2) = 4n-13$  for  $n \geq 7$  ;

2.  $F_4^*(n, n-3) = 4n-14$  for  $n \geq 7$  .

An extremal graph for  $F_4^*(n, n-2)$  can be obtained from the cycle  $C^5$  (where  $C^r$  denotes a cycle on  $r$  vertices) by replicating one vertex, adding two new non-adjacent vertices  $x$  and  $y$  and joining them to all other vertices of the graph. As for  $F_4^*(n, n-3)$ , an extremal graph can be obtained by replicating any vertex of the following graph  $G$  on seven vertices:  $V(G) = \{0, 1, \dots, 6\}$ ,  $E(G) = \{(i, i+1) \bmod 7 : 0 \leq i \leq 6\} \cup \{(i, i+3) \bmod 7 : 0 \leq i \leq 6\}$ . In the subsequent sections we will show that  $F_4(n, n-2) \leq 4n-15$  for  $n \geq 9$  (and a construction exists with maximal degree  $n-4$ ), and  $F_4(n, n-2) = 4n-15$  for sufficiently large  $n$ .

**Proposition 4**    1.  $F_k^*(n, n-2) = F_k^*(n, n-3) + 1$  for  $n \geq 2k-1$  ;

2.  $F_k(n, n-2) = F_k(n, n-3)$  for  $n \geq 2k-1$ .

**Proof.** By induction on  $k \geq 3$ . For  $k=3$  it was proved by Duffus and Hanson, and Füredi and Seress that  $F_3^*(n, n-2) = 2n-4$  and  $F_3^*(n, n-3) = F_3(n, n-3) = 2n-5$  for  $n \geq 5$ . Assuming that the proposition holds true for  $k-1$ , we obtain from Proposition 3

1.  $F_k^*(n, n-2) = F_{k-1}(n-2, n-4) + 2n-4 = F_{k-1}(n-2, n-5) + 2n-4 = F_k^*(n, n-3) - (2n-5) + (2n-4) = F_k^*(n, n-3) + 1$  ;
2. If  $G$  is a  $k$ -saturated graph with  $\Delta(G) \leq n-2$ , then either  $\Delta(G) = n-2$ , and then  $|E(G)| \geq F_k^*(n, n-2) = F_k^*(n, n-3) + 1 \geq F_k(n, n-3) + 1$ , or  $\Delta(G) \leq n-3$ , and then  $|E(G)| \geq F_k(n, n-3)$ .  $\square$

An upper bound for  $F_k^*(n, n-2)$  can be obtained by considering a complete  $(k-1)$ -partite graph  $K^{n-2(k-2), 2, \dots, 2}$  ( $n \geq 2k-2$ ), yielding

$$F_k^*(n, n-2) \leq 2(k-2)n - (2k^2 - 6k + 4) .$$

In Section 6 we will improve this bound slightly.

### 3 The structure of $k$ -saturated graphs

This section extends the proofs of [5] for the case of general  $k$ . It contains some new ideas as well.

A *hypergraph* (set system) is a pair  $\mathcal{H} = (V, \mathcal{E})$ , where  $V$  is a finite ground set (the *vertex set*) and  $\mathcal{E}$  is a family of distinct subsets of  $V$  (the *edge set*). We will occasionally identify a hypergraph with its edge set.

Suppose  $G = (V, E)$  is a  $k$ -saturated graph and suppose  $V_0 \subseteq V$  is such that  $V \setminus V_0$  is independent in  $G$ . Then the number of edges in  $G$  can be computed using the following description of  $G$ :

- (i) a graph  $G_0 = G[V_0]$ ;
- (ii) a hypergraph  $\mathcal{H}$  on  $V_0$ , whose edges  $H_1, \dots, H_m$  are the neighbourhoods of the vertices in  $V \setminus V_0$ , listed without repetitions;
- (iii) an assignment of weights  $y_1, \dots, y_m$  where  $y_i$  is the fraction of vertices of  $V \setminus V_0$  with neighbourhood  $H_i$ , thus,  $y_i \geq 0$  and  $\sum_{i=1}^m y_i = 1$ .

Then

$$|E(G)| = |E(G_0)| + |V \setminus V_0| \sum_{i=1}^m y_i |H_i| .$$

Moreover, it can be easily checked, using the fact that  $G$  is  $k$ -saturated, that the pair  $(G_0, \mathcal{H})$  satisfies the following conditions.

1.  $G_0$  is  $K^k$ -free;

2.  $G_0[H_i]$  is  $K^{k-1}$ -free for every  $H_i \in \mathcal{H}$ ;
3.  $H_i \cap H_j$  contains a  $K^{k-2}$  for every pair  $H_i, H_j \in \mathcal{H}$ ;
4. for every edge  $H_i \in \mathcal{H}$  and every vertex  $x \in V_0 \setminus H_i$  the subset  $H_i \cup \{x\}$  contains a  $K^{k-1}$ ;
5. if  $(x, y) \notin E(G_0)$  then either there exists in  $G_0$  a copy of  $K^{k-2}$  completely joined to  $x$  and  $y$ , or there exists a copy of  $K^{k-3}$  on the vertices  $v_1, \dots, v_{k-3} \in V_0$ , completely joined to  $x$  and  $y$ , and an edge  $H_i \in \mathcal{H}$  such that  $\{x, y, v_1, \dots, v_{k-3}\} \subseteq H_i$ .

It turns out that such pairs  $(G_0, \mathcal{H})$  play a crucial role in determining the functions  $F_k(n, D)$ , and therefore the following definition is very useful.

**Definition 1** Let  $G_0 = (V_0, E)$  be a graph and  $\mathcal{H} = (V_0, \mathcal{E})$  be a hypergraph on some set  $V_0$ . The pair  $(G_0, \mathcal{H})$  is a  $k$ -core if it satisfies the above conditions (1)–(5).

**Definition 2**  $(G_0, \mathcal{H})$  is a  $k$ -pre-core if it satisfies conditions (1)–(4).

Note that the definition of a  $k$ -core generalizes that of a core ( $k = 3$ ) given by Füredi and Seress. Observe also that if  $(G_0, \mathcal{H})$  is a  $k$ -pre-core, then one can add, if necessary, edges to  $E(G_0)$ , obtaining a new graph  $G'_0$  such that  $(G'_0, \mathcal{H})$  is a  $k$ -core.

Given a  $k$ -core  $(G_0, \mathcal{H})$  and weights  $y_i \geq 0$ ,  $1 \leq i \leq m$ , such that  $\sum_{i=1}^m y_i = 1$ , one can construct for  $n$  large enough a  $k$ -saturated graph  $G$  on  $n$  vertices as follows. Choose sets  $V_1, \dots, V_m$  disjoint from each other and from  $V_0$  such that  $\lfloor y_i(n - |V_0|) \rfloor \leq |V_i| \leq \lceil y_i(n - |V_0|) \rceil$  and  $\sum_{i=1}^m |V_i| = n$ , and define  $V = \bigcup_{i=1}^m V_i$ . Two vertices  $x, y \in V_0$  are adjacent in  $G$  if and only if they are adjacent in  $G_0$ . The set  $\bigcup_{i=1}^m V_i$  is independent in  $G$ . Finally, two vertices  $x \in V_0$  and  $y \in V_i$ ,  $1 \leq i \leq m$ , are adjacent in  $G$  if and only if  $x \in H_i$ .

The degree of a vertex  $x \in V_0$  in  $G$  is  $n \sum_{i: x \in H_i} y_i + O(1)$ , the number of edges in  $G$  is  $n \sum_{i=1}^m y_i |H_i| + O(1)$ . These observations lead us to the following linear programming formulation.

**Definition 3** Given a hypergraph  $\mathcal{H} = \{H_1, \dots, H_m\}$  on a set  $V_0$  and a real number  $c > 0$ , let  $A(\mathcal{H}, c) = \min \sum_{i=1}^m |H_i| y_i$ , under the restrictions

$$\sum_{x \in H_i} y_i \leq c \quad \text{for all } x \in V_0, \quad (3)$$

$$y_i \geq 0 \quad \text{for all } 1 \leq i \leq m, \quad (4)$$

$$\sum_{i=1}^m y_i = 1. \quad (5)$$

Recall that the *fractional matching number*  $\nu^*(\mathcal{H})$  of a hypergraph  $\mathcal{H} = (V_0, \mathcal{E})$ , where  $\mathcal{E} = \{H_1, \dots, H_m\}$ , is defined as  $\nu^*(\mathcal{H}) = \max \sum_{i=1}^m f_i$  under the restrictions

$$\begin{aligned} \sum_{x \in H_i} f_i &\leq 1 && \text{for all } x \in V_0, \\ f_i &\geq 0 && \text{for all } 1 \leq i \leq m. \end{aligned}$$

Clearly,  $c \geq 1/\nu^*(\mathcal{H})$  is a necessary and sufficient condition for the feasibility of the conditions (3)-(5). Note also that if  $\mathcal{H}$  is  $r$ -uniform (that is,  $|H| = r$  for every  $H \in \mathcal{H}$ ) and  $c \geq 1/\nu^*(\mathcal{H})$  then  $A(\mathcal{H}, c) = r$ .

The following example shows that for every  $c > 0$  there exists a  $k$ -core  $(G_0, \mathcal{H})$  such that the conditions (3)–(5) are feasible for  $\mathcal{H}$ .

**Example 1.** Let  $q$  be a prime power. Define for every  $k \geq 3$  a graph  $G_0^k$  and a hypergraph  $\mathcal{H}^k$  on a set  $V_0^k$ . The set  $V_0^k$  consists of  $k - 2$  copies of two disjoint sets of size  $q^2 + q + 1$  each, denoted by  $A_1, B_1, \dots, A_{k-2}, B_{k-2}$ . Elements of  $A_i$  are identified with the points of a projective plane  $PG(2, q)$ , those of  $B_i$  with the lines of  $PG(2, q)$ . For each line  $l$ , there is an edge  $H \in \mathcal{H}^k$  consisting of all points of  $l$  in  $A_1, \dots, A_{k-2}$  and all singletons  $\{l\}$  in  $B_1, \dots, B_{k-2}$ . Thus,  $\mathcal{H}^k$  has  $q^2 + q + 1$  edges of size  $|H| = (k - 2)(q + 1) + k - 2 = (k - 2)(q + 2)$ . In the graph  $G_0^k$  the union  $\bigcup_{i=1}^{k-2} B_i$  is independent and each of the sets  $A_i$  is independent. Each  $A_i$  is completely joined to all  $A_j, B_j$ ,  $i \neq j$ , and within  $(A_i, B_i)$  a vertex in  $B_i$  corresponding to a line  $l$  is joined to all vertices in  $A_i$  corresponding to the points of  $PG(2, q)$  not lying on  $l$ . It can rather easily be checked that the pair  $(G_0^k, \mathcal{H}^k)$  is a  $k$ -core for all  $q \geq 2$ , while for  $q = 1$  (in this case  $PG(2, 1)$  denotes a triangle) it is a  $k$ -pre-core. Assigning  $y_i = 1/(q^2 + q + 1)$  we get a feasible solution of (3)–(5) for every  $c$  satisfying  $c \geq (q + 1)/(q^2 + q + 1)$ . This assignment implies also that  $A(\mathcal{H}^k, c) = (k - 2)(q + 2)$  for these values of  $c$ .

We note that the above example is in fact a generalization of Example 3.1 of [5].

**Definition 4** For every  $0 < c \leq 1$  define  $K_k(c) = \inf A(\mathcal{H}, c)$ , where  $\mathcal{H}$  ranges over all hypergraphs with  $\nu^*(\mathcal{H}) \geq 1/c$  such that, for a suitable graph  $G_0$ , the pair  $(G_0, \mathcal{H})$  is a  $k$ -core (or equivalently, a  $k$ -pre-core).

**Claim 1**  $K_k(c) \leq 2(k - 2)(1 + 1/c)$ .

**Proof.** Let  $q$  be a prime satisfying  $1/c \leq q \leq 2/c$ . Then Example 1 gives a  $k$ -core  $(G_0^k, \mathcal{H}^k)$  for which  $A(\mathcal{H}^k, c) = (k - 2)(q + 2) \leq (k - 2)(2/c + 2) = 2(k - 2)(1 + 1/c)$ .  $\square$



**Claim 2** *In the definition of  $K_k(c)$ , the infimum may be taken over hypergraphs with at most  $2(k-2)(1/c + 1/c^2) + 1$  edges.*

**Proof.** Let  $(G_0, \mathcal{H})$  be a  $k$ -core such that  $A(\mathcal{H}, c) \leq 2(k-2)(1 + 1/c)$ . By Claim 1, it suffices to consider such  $k$ -cores in determining  $K_k(c)$ . Let  $\mathcal{H} = \{H_1, \dots, H_m\}$ . The system of inequalities (3)–(5) defines a convex polytope  $P$  in  $R^m$ .  $P$  is bounded and non-empty, and therefore the function  $\sum_{i=1}^m |H_i|y_i$  attains its minimum at a vertex  $p$  of  $P$ . But every vertex of  $P$  is the intersection of at least  $m$  hyperplanes of the type:  $\sum_{x \in H_i} y_i = c$  for some  $x \in V_0$ , or  $y_i = 0$  for some  $1 \leq i \leq m$ , or  $\sum_{i=1}^m y_i = 1$ . Since

$$\sum_{x \in V_0} \sum_{x \in H_i} y_i = \sum_{i=1}^m |H_i|y_i = A(\mathcal{H}, c) \leq 2(k-2)\left(1 + \frac{1}{c}\right),$$

at most  $2(k-2)(1/c + 1/c^2)$  hyperplanes of the first type contain  $p$ . Therefore, for at least  $m - 2(k-2)(1/c + 1/c^2) - 1$  values of  $i$ , the equation  $y_i = 0$  occurs at  $p$ . Denote by  $\mathcal{H}_1$  the set of those edges  $H_i$  of  $\mathcal{H}$  for which  $y_i \neq 0$  at  $p$ . Clearly,  $A(\mathcal{H}_1, c) = A(\mathcal{H}, c)$  and  $|\mathcal{H}_1| \leq 2(k-2)(1/c + 1/c^2) + 1$ . Also, one can easily check that  $(G_0, \mathcal{H}_1)$  is a  $k$ -pre-core. Adding edges to  $E(G_0)$ , if necessary, we obtain a  $k$ -core  $(G_1, \mathcal{H}_1)$ , thus proving the assertion of the claim.  $\square$

The next step is to show that the number of vertices in the hypergraph for the definition of  $K_k(c)$  can be bounded from above by a function of  $c$  as well. It seems that the corresponding proof of Füredi and Seress cannot be extended for the case of general  $k$ , therefore we present a different proof. Let us call a hypergraph  $\mathcal{H} = (V, \mathcal{E})$  *separated* if for all  $x \neq y \in V$  there exists an edge  $H \in \mathcal{E}$  such that  $|H \cap \{x, y\}| = 1$ . This definition implies that for every pair  $x \neq y \in V$ , the sets of edges containing  $x$  and  $y$ , respectively, are different, and therefore the number of vertices in a separated hypergraph can be bounded from above by  $|V| \leq 2^{|\mathcal{E}|}$ . By identifying vertices, if necessary, we can obtain from every hypergraph  $\mathcal{H}$  a separated hypergraph  $\mathcal{H}_0$  with the same number of edges and the same fractional matching number:  $\nu^*(\mathcal{H}) = \nu^*(\mathcal{H}_0)$ . If  $V(\mathcal{H}_0) = \{x_1, \dots, x_p\}$  and  $x_i$  is obtained by identifying  $a_i$  vertices of  $\mathcal{H}$ , we say that  $\mathcal{H}$  is an  $(a_1, \dots, a_p)$  *blow-up* of  $\mathcal{H}_0$ . We let

$$B(\mathcal{H}_0) = \{(a_1, \dots, a_p) \in N^p : \text{the } (a_1, \dots, a_p) \text{ blow-up of } \mathcal{H}_0 \\ \text{forms a } k\text{-core with a suitable graph } G_0\}.$$

For a given  $c > 0$ , define a family of hypergraphs  $\mathbf{H}(c)$  by

$$\mathbf{H}(c) = \left\{ \mathcal{H}_0 : \mathcal{H}_0 \text{ is separated, } \nu^*(\mathcal{H}_0) \geq 1/c, V(\mathcal{H}_0) = \{x_1, \dots, x_p\}, \right. \\ \left. \mathcal{E}(\mathcal{H}_0) = \{H_1, \dots, H_m\}, m \leq 2(k-2)(1/c + 1/c^2) + 1, B(\mathcal{H}_0) \neq \emptyset \right\}.$$

The above observations imply that  $\mathbf{H}(c)$  is non-empty and finite. Using the above definitions and Claim 2, the problem of determining  $K_k(c)$  can be rewritten as

$$K_k(c) = \min_{\mathcal{H}_0 \in \mathbf{H}(c)} \inf_{(a_1, \dots, a_p) \in B(\mathcal{H}_0)} \min \sum_{i=1}^m \left( \sum_{x_j \in H_i} a_j \right) y_i$$

s.t.

$$\sum_{x_j \in H_i} y_i \leq c, \quad j = 1, \dots, p,$$

$$y_i \geq 0, \quad i = 1, \dots, m,$$

$$\sum_{i=1}^m y_i = 1.$$

In the infimum of the above expression for  $K_k(c)$  it suffices to consider only those  $(a_1, \dots, a_p)$  that are minimal elements of  $B(\mathcal{H}_0)$  in the natural partial order  $\prec$  of  $N^p$  ( $(a_1, \dots, a_p) \prec (a'_1, \dots, a'_p)$  iff  $a_i \leq a'_i$  for every  $1 \leq i \leq p$ ). Since the poset  $(N^p, \prec)$  has no infinite antichain, this enables us to restrict the choice of  $(a_1, \dots, a_p)$  to a finite set. Hence we obtain the following result.

**Claim 3** *The infimum in the definition of  $K_k(c)$  is attained.* □

**Theorem 1** *The above defined function  $K_k(c)$  is monotone nonincreasing, piecewise linear and right-continuous. The points of discontinuity are all rational and contained in a sequence  $c_1 > c_2 > \dots \rightarrow 0$ .*

**Proof.** We use Lemma 3.6 of [5], which states that for an arbitrary hypergraph  $\mathcal{H}$  the function  $A(\mathcal{H}, c)$  is continuous, piecewise linear and monotone nonincreasing on the interval  $[1/\nu^*(\mathcal{H}), \infty)$ . It follows from the proof of Claim 3 that for every fixed  $\gamma > 0$  the value of  $K_k(c)$  is determined on  $[\gamma, 1]$  by a finite number of blow-ups of separated hypergraphs whose number in turn can be bounded from above by a function of  $\gamma$ . Therefore  $K_k(c)$  on  $[\gamma, 1]$  is the minimum of finitely many functions  $A(\mathcal{H}, c)$ , and hence  $K_k(c)$  is also monotone nonincreasing and piecewise linear. The only possible discontinuities are left-discontinuities at points of the form  $1/\nu^*(\mathcal{H})$  for some hypergraph  $\mathcal{H}$  from this finite collection; in particular, there are finitely many discontinuities in  $[\gamma, 1]$  and they are all rational. □

The following theorem, whose proof is shaped after Lemma 4.2 of [5], shows that every  $k$ -saturated graph with few edges is built on a  $k$ -core with a small number of vertices. (All logarithms are base two.)

**Theorem 2** *Let an integer  $k \geq 3$  and a real  $C$  be fixed. Then there exists an integer  $n_0$  such that for every  $n > n_0$ , if  $G = (V, E)$  is a  $k$ -saturated graph on  $n$  vertices with  $\leq Cn$  edges, then there exists a subset  $V_0 \subset V$  such that*

(a)  $|V_0| \leq (2C + 1)n / \log \log n$ ;

(b)  $V \setminus V_0$  is independent in  $G$ ;

(c) For every  $x \in V \setminus V_0$  let  $H(x) = \{y \in V_0 : (x, y) \in E(G)\}$ . Let  $\mathcal{H}$  be a hypergraph on  $V_0$  with edge set  $\{H(x) : x \in V \setminus V_0\}$  and let  $G_0 = G[V_0]$ . Then  $(G_0, \mathcal{H})$  is a  $k$ -core.

**Proof.** Let  $G = (V, E)$  be a  $k$ -saturated graph on  $n$  vertices with at most  $Cn$  edges. Let  $X = \{x \in V : d(x) \geq \log \log n\}$ . Then  $|X| \log \log n \leq \sum_{x \in V} d(x) \leq 2Cn$ , and therefore

$$|X| \leq \frac{2Cn}{\log \log n}. \quad (6)$$

For every  $y \in V \setminus X$  let  $H(y) = \{x \in X : (x, y) \in E(G)\}$ . Clearly,  $|H(y)| < \log \log n$  for all  $y \in V \setminus X$ . Define  $Y = \{y \in V \setminus X : \exists z \in V \setminus X \text{ such that } H(y) \cap H(z) \text{ does not contain a copy of } K^{k-2}\}$ . We claim that the set  $V_0 = X \cup Y$  satisfies the requirements of the theorem. Indeed, if  $u_1, u_2 \in V \setminus V_0$ , then  $K^{k-2} \subseteq H(u_1) \cap H(u_2)$ , but  $G$  is  $K^k$ -free, therefore  $(u_1, u_2) \notin E(G)$ , and hence  $V \setminus V_0$  is independent and (b) holds. As observed earlier in this section, in a  $k$ -saturated graph (b) implies (c). Therefore, in view of (6) it remains to prove that  $|Y| \leq n / \log \log n$ , provided that  $n$  is sufficiently large.

A hypergraph  $\mathcal{H} = (V, \mathcal{E})$  is a *sunflower* if  $H_i \cap H_j = \bigcap_{H \in \mathcal{E}} H$  for all  $H_i \neq H_j \in \mathcal{E}$ . The sets  $H_i \setminus \bigcap_{H \in \mathcal{E}} H$  are called *petals*. Let us prove now that the hypergraph  $\{H(y) : y \in Y\}$  does not contain a sunflower with more than  $\log \log^2 n + \log \log n$  petals. To show this, suppose that  $\{H(y_i) : 0 \leq i \leq \lfloor \log \log^2 n + \log \log n \rfloor\}$  is a sunflower for some  $y_i \in Y$ , and let  $U = \bigcap_i H(y_i)$ . By the definition of  $Y$ , there exists a vertex  $z \in Y$  such that  $K^{k-2} \not\subseteq H(y_0) \cap H(z)$ . Since all vertices in  $V \setminus X$  have degree  $< \log \log n$ , less than  $\log \log^2 n$  of the  $y_i$  are of distance at most two from  $z$  in  $G \setminus X$ . Therefore for more than  $\log \log n$  of the vertices  $y_i$  the distance between  $y_i$  and  $z$  in  $G \setminus X$  is more than two. Hence (recalling that  $G$  is  $k$ -saturated), there exists a copy of  $K^{k-2}$  (we denote it by  $T_i$ ), contained in  $X$  and completely joined to both  $y_i$  and  $z$ . Clearly,  $V(T_i) \not\subseteq U$  for every such  $y_i$  (otherwise  $K^{k-2} \subseteq H(y_0) \cap H(z)$ ), and hence there exists a point  $x_i \in V(T_i)$  such that  $x_i \notin H(y_j)$  for every  $j \neq i$ . All  $x_i$  are different and belong to  $H(z)$ , thus yielding  $|H(z)| > \log \log n$ , a contradiction.

Now, by a theorem of Erdős and Rado ([4]) if a hypergraph has more than  $r!m^r$  edges of size at most  $r$ , then some subhypergraph is a sunflower with  $m + 1$  petals. This implies that the set system  $\{H(y) : y \in Y\}$  has at most  $(\log \log n)!(\log \log^2 n + \log \log n)^{\log \log n} < n/\log \log^3 n$  members (here we use the assumption that  $n$  is sufficiently large). Finally, for each  $H \subseteq X$  we have  $|\{y \in Y : H(y) = H\}| \leq \log \log^2 n$ , because these  $y$  must be of distance at most two in  $G \setminus X$  from the vertex  $z \in Y$  for which  $H(z) \cap H$  does not contain a  $K^{k-2}$ .  $\square$

**Theorem 3** *If  $K_k(c)$  is continuous at  $c$ , then  $\lim_{n \rightarrow \infty} F_k(n, cn)/n = K_k(c)$ .*

**Proof.** Let us prove first that  $\limsup_{n \rightarrow \infty} F_k(n, cn)/n \leq K_k(c)$ . Suppose to the contrary that  $\limsup_{n \rightarrow \infty} F_k(n, cn)/n \geq K_k(c) + \epsilon$  for some positive constant  $\epsilon$ . Since  $K_k(c)$  is continuous at  $c$ , there exists a constant  $\delta > 0$  such that  $K_k(c - \delta) < K_k(c) + \epsilon$ . It follows from Claim 3, that there exists a  $k$ -core  $(G_0, \mathcal{H})$  on a set  $V_0$  and a weight function  $w$  on the edges of  $\mathcal{H}$  such that  $w$  is a feasible solution of (3)-(5) for  $c - \delta$  and  $A(\mathcal{H}, c - \delta) = \sum_{H \in \mathcal{H}} |H|w(H) = K_k(c - \delta)$ . As explained in the beginning of this section, we can use this  $k$ -core and weight function to construct, for sufficiently large  $n$ , a  $k$ -saturated graph  $G$  on  $n$  vertices with  $\Delta(G) \leq (c - \delta)n + O(1)$  and  $|E(G)| \leq K_k(c - \delta)n + O(1)$ . Therefore for sufficiently large  $n$  we have

$$\frac{F_k(n, cn)}{n} \leq \frac{F_k(n, (c - \delta)n + O(1))}{n} \leq K_k(c - \delta) + o(1) < K_k(c) + \epsilon,$$

a contradiction.

Now we prove that  $\liminf_{n \rightarrow \infty} F_k(n, cn)/n \geq K_k(c)$ . Suppose to the contrary that  $\liminf_{n \rightarrow \infty} F_k(n, cn)/n \leq K_k(c) - \epsilon$  for some constant  $\epsilon > 0$ . This means that there exists an infinite increasing sequence  $\{n_i\}$  such that  $F_k(n_i, cn_i)/n_i \leq K_k(c) - \epsilon$ ; that is, there exists a sequence of graphs  $\{G^i\}$  such that every  $G^i$  is  $k$ -saturated with  $|V(G^i)| = n_i$ ,  $\Delta(G^i) \leq cn_i$ ,  $|E(G^i)| \leq (K_k(c) - \epsilon)n_i$ . The function  $K_k(c)$  is right-continuous (the argument in this direction does not depend on the assumption of continuity at  $c$ ), and therefore there exists a positive constant  $\delta$  such that  $K_k(c + \delta) > K_k(c) - \epsilon$ . For  $i$  sufficiently large, according to Theorem 2, there exists a subset  $V_0^i \subset V(G^i)$  and a  $k$ -core  $(G_0^i, \mathcal{H}^i)$  satisfying (a)-(c). Recalling the notation of Theorem 2, we define the weight function  $w$  on  $\mathcal{E}(\mathcal{H}^i)$  by

$$w(H) = \frac{|\{x \in V(G^i) \setminus V_0^i : H(x) = H\}|}{|V(G^i) \setminus V_0^i|}.$$

Clearly,  $\sum_{H \in \mathcal{E}(\mathcal{H}^i)} w(H) = 1$ . For  $z \in V_0^i$ , using the fact that  $d_{G^i}(z) \leq cn_i$ , we obtain

$$\sum_{z \in H} w(H) \leq \frac{cn_i}{|V(G^i) \setminus V_0^i|} \leq c + \delta$$

for sufficiently large  $i$  (the last inequality holds because  $|V_0^i| = o(n_i)$ ). This implies that  $w$  is a feasible solution of the problem (3)–(5) for  $c + \delta$ , and then according to the definition of  $K_k(c)$  we have  $\sum_{H \in \mathcal{E}(\mathcal{H}^i)} |H|w(H) \geq K_k(c + \delta)$ . But  $|E(G^i)| = |E(G_0^i)| + |V(G^i) \setminus V_0^i| \sum_{H \in \mathcal{E}(\mathcal{H}^i)} w(H)|H|$  and we obtain

$$|E(G^i)| \geq |V(G^i) \setminus V_0^i| K_k(c + \delta) > (K_k(c) - \epsilon)n_i$$

for  $i$  sufficiently large, thus obtaining a contradiction.  $\square$

The exact determination of  $K_k(c)$  seems to be hopeless in general. However, we can determine its asymptotic behaviour for  $c \rightarrow 0$ .

**Theorem 4** *Let  $G$  be a  $k$ -saturated graph on  $n$  vertices with  $\delta(G) = \delta$  and  $\Delta(G) = \Delta$ . Then*

$$\delta \geq \frac{(k-2)(n-1)}{\Delta + k - 3}.$$

**Proof.** Let  $x$  be a vertex with  $d(x) = \delta$ . Denote  $A = \{y : (x, y) \in E(G)\}$ ,  $B = V \setminus (A \cup \{x\})$ , then  $|A| = \delta$ ,  $|B| = n - \delta - 1$ . Since the addition of the edge  $(x, z)$  for  $z \in B$  yields a copy of  $K^k$  in  $G$ , every  $z \in B$  has at least  $k - 2$  neighbours in  $A$ , and therefore the number of edges between  $A$  and  $B$  is at least  $(k-2)|B| = (k-2)(n - \delta - 1)$ . On the other hand, this number of edges does not exceed  $|A|(\Delta - 1) = \delta(\Delta - 1)$ , and we conclude that  $(k-2)(n - \delta - 1) \leq \delta(\Delta - 1)$ , or

$$\delta \geq \frac{(k-2)(n-1)}{\Delta + k - 3}. \quad \square$$

**Theorem 5**  $\frac{k-2}{c} \leq K_k(c) \leq \frac{k-2+o(1)}{c}$  (here the  $o(1)$  term tends to 0 as  $c$  tends to 0).

**Proof.** The lower bound can be deduced from Theorem 4 (with the help of the previous theorems and some technicalities), but we give here a direct proof. Let  $\mathcal{H} = \{H_1, \dots, H_m\}$  be a hypergraph which forms a  $k$ -core with a suitable graph, and let  $y_1, \dots, y_m$  be a feasible solution of (3)–(5). Then for each  $H_i \in \mathcal{H}$  we have

$$|H_i|c \geq \sum_{x \in H_i} \sum_{x \in H_j} y_j = \sum_{j=1}^m |H_i \cap H_j| y_j \geq (k-2) \sum_{j=1}^m y_j = k-2$$

(using the third condition in the definition of a  $k$ -core). It follows that  $|H_i| \geq (k-2)/c$ , and therefore  $A(\mathcal{H}, c) \geq (k-2)/c$ , which proves the lower bound.

To prove the upper bound, we return to Example 1 and take a prime  $q$  satisfying  $1/c \leq q \leq 1/c + (1/c)^{7/12}$ . (Such a prime exists for all sufficiently small  $c$ , since by a theorem of Huxley [9], there always exists a prime between  $n$  and  $n + n^{7/12}$  for  $n$  sufficiently large.) Then we obtain a  $k$ -core  $(G_0^k, \mathcal{H}^k)$  for which  $A(\mathcal{H}^k, c) = (k-2)(q+2) \leq (k-2)(1/c + (1/c)^{7/12} + 2)$ . Therefore it follows from the definition of  $K_k(c)$  that

$$K_k(c) \leq A(\mathcal{H}^k, c) \leq \frac{k-2}{c}(1 + c^{5/12} + 2c) = \frac{k-2}{c}(1 + o(1)). \quad \square$$

## 4 Graphs with maximal degree $o(n)$

To construct  $k$ -saturated graphs with maximal degree  $o(n)$  we use the following  $k$ -core (which for  $k = 3$  coincides with Example 2.2 of [5]).

**Example 2.** Let  $q \geq k-1$  ( $q \geq 3$  for the case  $k = 3$ ) be a prime power. Enumerate the points  $p_0, \dots, p_{q^2+q}$  and the lines  $l_0, \dots, l_{q^2+q}$  of a projective plane  $PG(2, q)$  in such a way that  $p_{q^2+q} \in l_0, \dots, l_q$ , and  $p_{iq+j} \in l_i$  for every  $0 \leq i \leq q$ ,  $0 \leq j \leq q-1$ . For a point  $p = p_{iq+j}$  we call  $i$  the *level* of  $p$  and  $j$  the *place* of  $p$ . Deleting the point  $p_{q^2+q}$  and the lines  $l_0, \dots, l_q$  we obtain a truncated projective plane of order  $q$ . We describe now a set  $V_0^k$  and a  $k$ -core  $(G_0^k, \mathcal{H}^k)$  on it.  $V_0^k$  consists of  $k-1$  copies of  $T^k$ , where  $T^k$  is obtained from a truncated projective plane of order  $q$  by replacing each point  $p$  by  $k-2$  points  $x^0, \dots, x^{k-3}$ , where we refer to  $t$  as the *type* of  $x^t$ . Thus, each point of  $V_0^k$  has four coordinates: its level  $0 \leq i \leq q$ , its place  $0 \leq j \leq q-1$ , its type  $0 \leq t \leq k-3$  and the copy  $0 \leq s \leq k-2$  of  $T^k$  it belongs to. For each line  $l_r$  in the truncated plane, there is an edge  $H_{r-q} \in \mathcal{E}(\mathcal{H}^k)$ , consisting of all points of  $l_r$  (in all  $k-1$  copies, of all  $k-2$  types). The edges of  $G_0^k$  are as follows. Within each level, two vertices are joined if and only if they are in distinct copies and have either distinct places or distinct types. In the case  $k \geq 4$ , a point  $x$  in level  $i$  is joined to a point  $y$  in level  $i'$ , where  $i < i'$ , if and only if the type of  $y$  succeeds that of  $x$  (in  $Z_{k-2}$ ) and the place of  $y$  is one of the  $k-2$  successors of the place of  $x$  (in  $Z_q$ ). Then  $(G_0^k, \mathcal{H}^k)$  is a  $k$ -core. The verification of this assertion is technical and rather tedious. Let us prove, for example, that  $G_0^k$  is  $K^k$ -free. Suppose to the contrary that  $G_0^k[\{v_1, \dots, v_k\}] \cong K^k$ . It is easy to see that if  $x, y, z$  form a triangle in  $G_0^k$ , then the points  $x, y, z$  belong to at most two different levels. Therefore the points  $v_1, \dots, v_k$  belong to at most two different levels  $i_1$  and  $i_2$ . Suppose  $i_1 < i_2$ . Let  $v_1, \dots, v_r$  belong to level  $i_1$  and  $v_{r+1}, \dots, v_k$  belong to level  $i_2$ . Since there are  $k-1$  copies and two vertices from the same copy and the same level are non-adjacent, we obtain that  $1 \leq r \leq k-1$ . Therefore the type of each of the points  $v_{r+1}, \dots, v_k$  succeeds the type of each of the points  $v_1, \dots, v_r$ . Hence  $v_1, \dots, v_r$  have the

same type and also  $v_{r+1}, \dots, v_k$  have the same type. Thus the places of  $v_1, \dots, v_r$  are all distinct, and the same holds for  $v_{r+1}, \dots, v_k$ . The place of each  $v_h$ ,  $r+1 \leq h \leq k$ , is among the  $k-2$  successors of the place of each  $v_{h'}$ ,  $1 \leq h' \leq r$ . But now one can easily check that  $r$  distinct intervals of length  $k-2$  in  $Z_q$  (recall  $q \geq k-1$ ) have at most  $k-1-r$  points in common, and we obtain a contradiction.

Based on the above described  $k$ -core,  $(G_0^k, \mathcal{H}^k)$ , we can build a  $k$ -saturated graph  $G^k$  as follows. Let  $n \geq (k-1)(k-2)(q^2+q) + q^2$ . For  $1 \leq i \leq q^2$ , we choose sets  $V_i$  disjoint from each other and from  $V_0^k$  such that  $\lfloor (n - (k-1)(k-2)(q^2+q))/q^2 \rfloor \leq |V_i| \leq \lceil (n - (k-1)(k-2)(q^2+q))/q^2 \rceil$  and  $|V_0^k| + \sum_{i=1}^{q^2} |V_i| = n$ . Note that, by our assumption about  $n$ , all  $V_i$  are non-empty. Define  $V(G^k) = V_0^k \cup \bigcup_{i=1}^{q^2} V_i$ . Two vertices  $x, y \in V_0^k$  are adjacent in  $G^k$  if and only if they are adjacent in  $G_0^k$ . The set  $\bigcup_{i=1}^{q^2} V_i$  is independent in  $G^k$ . Finally,  $x \in V_0^k$  and  $y \in V_i$  are adjacent if and only if  $x \in H_i$ . Then  $G^k$  is  $k$ -saturated. If  $x \in \bigcup_{i=1}^{q^2} V_i$ , then  $d(x) = (k-1)(k-2)(q+1)$ . If  $x \in V_0^k$  then

$$\begin{aligned} d(x) &\leq q(\lfloor (n - (k-1)(k-2)(q^2+q))/q^2 \rfloor + 1) \\ &\quad + (k-2)((k-2)(q-1) + (k-3)) + (k-1)(k-2)q \\ &\leq n/q + ((k-2)^2 + 1)q . \end{aligned}$$

Finally,

$$\begin{aligned} |E(G^k)| &\leq (k-1)(k-2)(q+1)(n - (k-1)(k-2)(q^2+q)) \\ &\quad + (k-2)(2k-3)q(k-1)(k-2)(q^2+q)/2 \\ &< (k-1)(k-2)(q+1)n . \end{aligned}$$

**Theorem 6** For all  $1/2 < \epsilon < 1$  and all  $c > 0$

$$\left( \frac{k-2}{2c} - o(1) \right) n^{2-\epsilon} \leq F_k(n, cn^\epsilon) \leq \left( \frac{(k-1)(k-2)}{c} + o(1) \right) n^{2-\epsilon} .$$

(Here  $k$  is fixed and  $o(1)$  tends to zero as  $n$  tends to infinity.)

**Proof.** If  $G$  is  $k$ -saturated with  $\Delta(G) \leq cn^\epsilon$ , then according to Theorem 4

$$|E(G)| \geq n\delta(G)/2 \geq \frac{(k-2)(n-1)n}{2(\Delta + k - 3)} \geq \frac{(k-2)(n-1)n}{2(cn^\epsilon + k - 3)} = \frac{k-2}{2c} n^{2-\epsilon} (1 - o(1)) ,$$

thus proving the lower bound for  $F_k(n, cn^\epsilon)$ . To prove the upper bound, choose a constant  $b$  such that  $b > 2((k-2)^2 + 1)/c^3$  and let  $a = n^{1-\epsilon}/c + bn^{2-3\epsilon}$ . Let  $q$  be a

prime satisfying  $a \leq q \leq a + a^{7/12}$  (such a prime exists if  $n$  is large enough). Observe that  $q = (1/c)n^{1-\epsilon}(1 + o(1))$ . Now, Example 2 gives a  $k$ -saturated graph,  $G^k$ , on  $n$  vertices with

$$\begin{aligned}\Delta(G^k) &\leq \frac{n}{q} + ((k-2)^2 + 1)q \leq \frac{n}{a} + ((k-2)^2 + 1)q \\ &\leq cn^\epsilon - \frac{bc^2}{2}n^{1-\epsilon} + ((k-2)^2 + 1)q < cn^\epsilon\end{aligned}$$

for  $n$  sufficiently large. Also,

$$|E(G^k)| < (k-1)(k-2)(q+1)n = \frac{(k-1)(k-2)}{c} n^{2-\epsilon}(1 + o(1)). \quad \square$$

**Theorem 7** *For every  $k \geq 3$  there exists a  $k$ -saturated graph  $G^k$  on  $n$  vertices with*

$$\Delta(G^k) \leq \left( \frac{(k-2)(2k-3) + 1}{\sqrt{(k-1)(k-2) + 1}} + o(1) \right) \sqrt{n}.$$

(Here  $k$  is fixed and  $o(1)$  tends to zero as  $n$  tends to infinity.)

**Proof.** Turning again to Example 2, we denote  $a = (n/((k-1)(k-2) + 1))^{1/2} - n^{1/3}$  and choose a prime  $q$  satisfying  $a - n^{1/3} \leq a - a^{7/12} \leq q \leq a$ . Then

$$\begin{aligned}& ((k-1)(k-2) + 1)q^2 + (k-1)(k-2)q \\ &\leq ((k-1)(k-2) + 1) \left( \left( \frac{n}{(k-1)(k-2) + 1} \right)^{1/2} - n^{1/3} \right)^2 + (k-1)(k-2)n^{1/2} \\ &\leq n - \frac{2n^{5/6}}{((k-1)(k-2) + 1)^{1/2}} + n^{2/3} + (k-1)(k-2)n^{1/2} \\ &\leq n,\end{aligned}$$

and therefore we can substitute  $q$  in Example 2. Also,

$$\begin{aligned}r &:= n - ((k-1)(k-2) + 1)q^2 - (k-1)(k-2)q \\ &\leq n - ((k-1)(k-2) + 1) \left( \frac{n^{1/2}}{((k-1)(k-2) + 1)^{1/2}} - 2n^{1/3} \right)^2 \\ &\quad - (k-1)(k-2) \left( \frac{n^{1/2}}{((k-1)(k-2) + 1)^{1/2}} - 2n^{1/3} \right) \\ &= O(n^{5/6}).\end{aligned}$$



Now, as in [5] we use the fact (see, e.g., [8]) that the sizes of the sets  $V_i$  can be chosen in such a way that  $1 + \lfloor r/q^2 \rfloor \leq |V_i| \leq 1 + \lceil r/q^2 \rceil$ , and each vertex from  $V_0^k$  has degree at most  $((k-2)(2k-3) + 1)q + 2\lceil r/q \rceil$ . Then

$$\begin{aligned} \Delta(G^k) &\leq ((k-2)(2k-3) + 1)q + 2 \left\lceil \frac{r}{q} \right\rceil \\ &= \left( \frac{(k-2)(2k-3) + 1}{\sqrt{(k-1)(k-2) + 1}} + o(1) \right) \sqrt{n}. \quad \square \end{aligned}$$

This result improves significantly an upper bound, given by Hanson and Seyffarth ([7]). Our coefficient is asymptotically  $2k$  as  $k \rightarrow \infty$ . Hanson and Seyffarth proved a lower bound of  $\sqrt{(k-2)n} - O(1)$  for the lowest possible maximal degree (this can be deduced immediately from our Theorem 4). The existence of a constant  $c_k$  such that the lowest possible maximal degree in a  $k$ -saturated graph on  $n$  vertices is asymptotically  $c_k\sqrt{n}$  as  $n \rightarrow \infty$ , conjectured by Hanson and Seyffarth, remains open (but we know that such  $c_k$ , if it exists, must satisfy  $\sqrt{k-2} \leq c_k \leq 2k$ ).

## 5 More on 4-saturated graphs

We begin by noting the following construction of 4-saturated graphs.

**Example 3.** Let  $n \geq 9$  and let  $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n-4$ . Let  $G_0$  be the graph  $\overline{C^6}$ . Let  $\mathcal{H}$  be the hypergraph with edges  $H_1, H_2, H_3$  of size four, each obtained by deleting a pair of antipodal vertices of the cycle. Then  $(G_0, \mathcal{H})$  is a 4-core. We add  $n-6$  vertices, split into non-empty blocks  $V_1, V_2, V_3$ , and join every vertex in  $V_i$  to each vertex in  $H_i$ ,  $i = 1, 2, 3$ . We obtain a 4-saturated graph  $G$  on  $n$  vertices with  $4n-15$  edges. This graph has  $\delta(G) = 4$ , and the sizes of the blocks  $V_i$  can be chosen so as to have  $\Delta(G) = D$ , for any  $D$  in the indicated range.

The main result of this section is the optimality of this construction. The fact that every 4-saturated graph on  $n$  vertices with no conical vertex has at least  $4n - o(n)$  edges can be shown as follows. Hajnal [6] proved that if  $G$  is  $k$ -saturated and has no conical vertex then  $\delta(G) \geq 2(k-2)$ . (The case  $k=4$  of this is easy to prove.) Thus, every vertex in our graph has degree at least four. By Theorem 2 we may assume that the graph contains an independent set of vertices of size  $n - o(n)$ . These vertices are incident to at least  $4n - o(n)$  edges.

However, in order to replace  $o(n)$  by a sharp estimate we have to work harder. The following definition and lemma will be required. A graph  $G = (V, E)$  is *4-partite*

4-saturated with respect to the partition  $V_1, V_2, V_3, V_4$  of  $V$ , if each  $V_i$  is independent in  $G$ , no copy of  $K^4$  is contained in  $G$ , but adding any legal edge (with endpoints in distinct  $V_i$ 's) will create a  $K^4$ .

**Lemma 1** *If  $G$  is 4-partite 4-saturated with respect to the partition  $V_1, V_2, V_3, V_4$  of  $V(G)$ , where  $|V(G)| = n$ , and at most one of the  $V_i$ 's is empty, then  $|E(G)| \geq 2n - 3$ .*

**Proof.** We proceed by induction on  $n$ . If one of the  $V_i$ 's, say  $V_4$ , is empty, then  $G$  must be a complete tripartite graph with three non-empty parts  $V_1, V_2, V_3$ . The number of edges is minimum when two parts consist of one vertex each, in which case  $|E(G)| = 2n - 3$ . Thus, we may assume that all parts are non-empty.

We may also assume that  $\delta(G)$  is 2 or 3. Indeed, it is easy to check that there cannot be vertices of degree zero or one. If  $\delta(G) \geq 4$  then  $|E(G)| \geq 2n$ .

Let  $x$  be a vertex with  $d(x) = \delta(G)$ . Then the graph  $G \setminus \{x\}$  satisfies the assumptions of the lemma, except that it might be possible to add a legal edge to  $G \setminus \{x\}$  without creating a  $K^4$ . This may happen only if adding the same edge to  $G$  creates a  $K^4$  containing  $x$ . We distinguish two cases.

*Case 1.*  $d(x) = 2$ .

In this case,  $x$  does not participate in a  $K^4$  after adding an edge not containing  $x$ . Hence we may apply the induction hypothesis to  $G \setminus \{x\}$ . This yields  $|E(G \setminus \{x\})| \geq 2n - 5$ , and therefore  $|E(G)| \geq 2n - 3$ .

*Case 2.*  $d(x) = 3$ .

The only way to add an edge  $e$  to  $G \setminus \{x\}$ , which creates a  $K^4$  in  $G$  containing  $x$ , is for  $e$  to join two neighbours of  $x$ , say  $y$  and  $z$ . Moreover,  $y$  and  $z$  must both be joined in  $G$  to the remaining neighbour of  $x$ , and hence  $e$  is unique. Thus, either  $G \setminus \{x\}$  or  $(G \setminus \{x\}) + e$  satisfies the assumptions of the lemma. In either case, induction yields  $|E(G)| \geq 2n - 3$ .  $\square$

**Theorem 8** *If  $G$  is a 4-saturated graph with no conical vertex,  $|V(G)| = n$  and  $\delta(G) = 4$ , then  $|E(G)| \geq 4n - 15$ .*

**Proof.** We proceed by induction on  $n$ . We may assume that  $n \geq 8$ , since  $\delta(G) = 4$  and so for  $n \leq 7$  we have  $|E(G)| \geq 2n > 4n - 15$ . Furthermore, by Corollary 1 we may assume that  $\Delta(G) \leq n - 4$ .

The following observation will be useful. Suppose that the vertices  $x$  and  $y$  of  $G$  are *twins*, i.e., they have the same neighbours. It is easy to see that in this case  $G \setminus \{x\}$  is also 4-saturated and has no conical vertex. By Hajnal's result,  $\delta(G \setminus \{x\}) \geq 4$ . Since  $d(x) = d(y)$ , it follows that  $\delta(G \setminus \{x\}) = 4$ . Hence we can apply the induction

hypothesis to get  $|E(G \setminus \{x\})| \geq 4n - 19$  and therefore  $|E(G)| \geq 4n - 15$ . Thus we may assume that  $G$  has no pairs of twins.

For a vertex  $x \in V(G)$ , we denote by  $N(x)$  the open neighbourhood of  $x$  and by  $N[x]$  the closed neighbourhood of  $x$  (i.e.,  $N[x] = N(x) \cup \{x\}$ ). We shall make repeated use of the fact that in a 4-saturated graph two vertices  $x$  and  $y$  are adjacent if and only if  $N(x) \cap N(y)$  contains no edge.

Let  $x$  be a vertex of degree four, fixed for the rest of the proof. Let  $N(x) = \{x_1, x_2, x_3, x_4\}$ . For every vertex  $y \in V(G) \setminus N[x]$ , since  $y$  is not adjacent to  $x$ , there must be an edge in  $N(y) \cap N(x)$ . It follows that we can write  $V(G) \setminus N[x]$  as the disjoint union

$$V(G) \setminus N[x] = \bigcup_S V_S ,$$

where  $S$  varies over the subsets of  $N(x)$  which contain an edge, and

$$V_S = \{y \in V(G) \setminus N[x] : N(y) \cap N(x) = S\} .$$

Each  $V_S$  is an independent set, because the neighbourhoods of any two vertices in  $V_S$  have an edge in common. Moreover, if  $S \cap T$  contains an edge then, for the same reason,  $V_S \cup V_T$  is independent. In particular, if  $y \in V_{N(x)}$  then  $N(y) = N(x)$ , contradicting the absence of twins. Hence  $V_{N(x)} = \emptyset$ . To simplify notation, we write, for example,  $V_{12}$  for  $V_{\{x_1, x_2\}}$ . We also write  $V_S \sim V_T$ , meaning that every vertex in  $V_S$  is adjacent to every vertex in  $V_T$ , and  $V_S \not\sim V_T$ , meaning  $V_S \cup V_T$  is independent.

The graph  $G[N(x)]$  has the following property: for every vertex  $x_i$  there is an edge which does not contain  $x_i$ . Indeed, if all edges of  $G[N(x)]$  contained  $x_i$ , the degree of  $x_i$  in  $G$  would be at least  $n - 3$ , contradicting  $\Delta(G) \leq n - 4$ . The graph  $G[N(x)]$  is also triangle-free, because  $G$  is  $K^4$ -free. It follows that  $G[N(x)]$  can be, up to isomorphism, one of three graphs:  $2K^2$  (two disjoint edges),  $P^4$  (a path on 4 vertices) or  $C^4$  (a 4-cycle).

*Case 1.*  $G[N(x)] = 2K^2$ .

Without loss of generality, we assume that  $(x_1, x_2)$  and  $(x_3, x_4)$  are the two edges. By the above remarks, the graph  $G \setminus N[x]$  is bipartite, with parts

$$\begin{aligned} A &= V_{12} \cup V_{123} \cup V_{124} , \\ B &= V_{34} \cup V_{134} \cup V_{234} . \end{aligned}$$

If  $y \in V_{12}$  then, since  $y$  is not adjacent to  $x_4$ ,  $N(y) \cap N(x_4)$  must contain an edge. But  $N(y) \cap N(x_4) \subseteq B$ , so this is impossible. Thus  $V_{12} = \emptyset$ , and similarly  $V_{34} = \emptyset$ . Next, we claim that  $G \setminus N[x]$  is a complete bipartite graph on  $A, B$ . Indeed if, for example,  $y \in V_{123}$  were not adjacent to  $z \in V_{134}$ , then  $N(y) \cap N(z)$  would have to contain an

edge, which is not the case, since  $N(y) \cap N(z) = \{x_1, x_3\}$ . It follows that each of the sets  $V_{ijk}$  is a set of twins, and hence  $|V_{ijk}| \leq 1$ . Since  $\Delta(G) \leq n - 4$ , each  $V_{ijk}$  is non-empty. By now, the graph  $G$  is fully determined. It has 9 vertices and 22 edges, so  $|E(G)| > 4n - 15$ .

*Case 2.*  $G[N(x)] = P^4$ .

We assume that  $(x_1, x_2), (x_2, x_3), (x_3, x_4)$  are the edges. Now, in addition to the sets of Case 1, we also have  $V_{23}$ . Arguments similar to those given in Case 1 show that  $V_{12} = V_{34} = \emptyset$ , and the following relations hold between  $V_{134}$  and the remaining sets  $V_S$ :

$$V_{134} \not\sim V_{234}, V_{134} \sim V_{123}, V_{134} \sim V_{124}, V_{134} \sim V_{23} .$$

Hence  $V_{134}$  is a set of twins, and therefore  $|V_{134}| \leq 1$ . But this means that  $d(x_2) \geq n - 3$ , contradicting  $\Delta(G) \leq n - 4$ .

*Case 3.*  $G[N(x)] = C^4$ .

We assume that  $(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)$  are the edges. The sets  $V_S$  involved in this case are  $V_{12}, V_{23}, V_{34}, V_{41}, V_{234}, V_{134}, V_{124}, V_{123}$ . Arguments as above show that the following relations hold: if  $S$  is an edge and  $T$  is a triple, then  $V_S \not\sim V_T$  or  $V_S \sim V_T$  according as  $S \subseteq T$  or  $S \not\subseteq T$ ; if  $T$  and  $T'$  are triples, then  $V_T \not\sim V_{T'}$  or  $V_T \sim V_{T'}$  according as  $T \cap T'$  is an edge or not. It follows that for each triple  $T$ , the set  $V_T$  consists of twins, and therefore  $|V_T| \leq 1$ .

Let  $U = V_{12} \cup V_{23} \cup V_{34} \cup V_{41}$ . Then  $G[U]$  is 4-partite 4-saturated with respect to this partition. To see this, suppose for example that  $y \in V_{12}, z \in V_{23}$  and  $(y, z) \notin E(G)$ . Then  $N(y) \cap N(z)$  must contain an edge. But  $N(y) \cap N(z) \subseteq \{x_2\} \cup V_{134} \cup V_{34} \cup V_{41}$ , and since  $x_2$  and  $V_{134}$  are isolated in the latter, the edge must be found in  $V_{34} \cup V_{41}$ . The argument is similar if  $y$  and  $z$  come from other pairs of sets.

Suppose that at most one of the sets  $V_S$  in the partition of  $U$  is empty. Then it follows from Lemma 1 that  $|E(G[U])| \geq 2|U| - 3$ . Letting  $W = U \cup N[x]$  we obtain  $|E(G[W])| \geq 4|W| - 15$ . Each of the singletons  $V_T, |T| = 3$ , if present, adds at least four new edges (three joining it to the vertices in  $T$  and at least one to  $U$ ). Thus, regardless of how many  $V_T$  are non-empty, we have  $|E(G)| \geq 4n - 15$ .

If, on the other hand, two or more of the sets  $V_S$  in the partition of  $U$  are empty, then those which are non-empty must be singletons and joined to each other. From the fact that  $d(x_i) \leq n - 4$ , it can be seen that we must have two singletons  $V_S$  and  $V_{S'}$ , where  $S$  and  $S'$  are disjoint, as well as all singletons  $V_T, |T| = 3$ . The graph  $G$  is fully determined, it has 11 vertices and 31 edges, so  $|E(G)| > 4n - 15$ .  $\square$

**Corollary 2** *There exists an integer  $n_0$  such that if  $G$  is a 4-saturated graph with no conical vertex and  $|V(G)| = n > n_0$ , then  $|E(G)| \geq 4n - 15$ .*

**Proof.** By Hajnal's result,  $\delta(G) \geq 4$ . If  $\delta(G) \geq 5$ , then Theorem 2 gives at least  $5n - o(n)$  edges, which exceeds  $4n - 15$  for large  $n$ . If  $\delta(G) = 4$ , we apply Theorem 8.  $\square$

Together with Example 3, the corollary establishes that  $F_4(n, D) = 4n - 15$  for  $n > n_0$  and  $\lfloor \frac{2n-1}{3} \rfloor \leq D \leq n - 2$ .

**Corollary 3** *If  $G$  is a 4-saturated graph,  $|V(G)| = n$  and  $\delta(G) = 4$ , then  $|E(G)| \geq 4n - 19$ . The lower bound is sharp for  $n \geq 11$ .*

**Proof.** If  $G$  has no conical vertex, we apply Theorem 8. Assume then that  $x$  is a conical vertex. Then  $G$  has the properties stated if and only if  $G \setminus \{x\}$  is a 3-saturated graph,  $|V(G \setminus \{x\})| = n - 1$  and  $\delta(G \setminus \{x\}) = 3$ . By a result of Duffus and Hanson [1], the graph  $G \setminus \{x\}$  must have at least  $3(n - 1) - 15$  edges, and the lower bound is sharp for  $n - 1 \geq 10$ . Adding the  $n - 1$  edges containing  $x$ , we get the desired result.  $\square$

We recall that Duffus and Hanson ([1]) investigated the function  $E(n, k, \delta)$ , defined as the minimal number of edges in a  $k$ -saturated graph on  $n$  vertices having minimal degree  $\delta$ . For the case  $k = \delta = 4$ , they showed that  $E(n, 4, 4) \leq 4n - 14$  for  $n \geq 7$ , with equality for  $n = 7$ . Our Corollary 3 establishes that  $E(n, 4, 4) = 4n - 19$  for  $n \geq 11$ . Example 3 shows that  $E(n, 4, 4) \leq 4n - 15$  for  $n \geq 9$ . The proof of Corollary 3 and the fact that  $E(8, 3, 3) = 12$ , shown by Duffus and Hanson, imply that  $E(9, 4, 4) = 20$ .

When  $D$  goes below  $\lfloor \frac{2n-1}{3} \rfloor$ , we do not know the exact behaviour of  $F_4(n, D)$ , but we do have the following construction.

**Example 4.** Let  $V_0$  consist of 12 vertices, denoted  $x_{ij}$ ,  $0 \leq i \leq 3$ ,  $1 \leq j \leq 3$ . Let  $V^i = \{x_{ij} : 1 \leq j \leq 3\}$  for  $0 \leq i \leq 3$ . Let  $G_0$  be the 4-partite graph on  $V_0$  with partition  $V^0, V^1, V^2, V^3$  obtained by joining each  $x_{ij}$  to all vertices of  $V^{i+j \pmod{4}}$ . Let  $\mathcal{H}$  be the hypergraph on  $V_0$  with edges  $V^l \cup \{x_{ij} : i + j \equiv l \pmod{4}\}$  for  $0 \leq l \leq 3$ . Then  $(G_0, \mathcal{H})$  is a 4-pre-core. Assigning a weight of  $1/4$  to each edge of  $\mathcal{H}$ , we get  $A(\mathcal{H}, 1/2) = 6$ .

This construction gives the estimate  $K_4(c) \leq 6$  for  $c \geq 1/2$ . Below that, we have the estimate  $K_4(c) \leq 8$  for  $c \geq 3/7$ , derived from the case  $k = 4, q = 2$  of Example 1.

## 6 More on $k$ -saturated graphs, $k > 4$

If  $G$  is a  $k$ -saturated graph on  $n$  vertices with no conical vertex, combining Hajnal's bound  $\delta(G) \geq 2(k - 2)$  and Theorem 2, we see that  $|E(G)| \geq 2(k - 2)n - o(n)$ . In

Section 2 we showed that there exist such graphs with  $2(k-2)n - O(1)$  edges and maximal degree  $n-2$  and  $n-3$ . But does there exist a  $k$ -saturated graph  $G$  on  $n$  vertices with  $|E(G)| = 2(k-2)n(1+o(1))$  and  $\Delta(G) \leq cn$  for some constant  $0 < c < 1$ ? We conjecture that the answer to this question is positive for every  $k \geq 4$ , that is,

**Conjecture 1** *For every  $k \geq 4$  there exists a constant  $0 < c_k < 1$  such that  $K_k(c) = 2(k-2)$  for every  $c_k \leq c < 1$ .*

Note that the above conjecture fails to be true for  $k = 3$ , as shown by Füredi and Seress. For  $k \geq 4$ , the case  $q = 1$  of Example 1 yields  $K_k(c) \leq 3(k-2)$  for  $c \geq 2/3$ , but we have better examples for infinitely many values of  $k$ , as shown by the following theorem.

**Theorem 9** *Conjecture 1 holds true in the following cases (with the indicated values of  $c_k$ ):*

- (i)  $k \equiv 0 \pmod{2}$ ,  $c_k = \frac{k-2}{k-1}$ ;
- (ii)  $k \equiv 2 \pmod{3}$ ,  $c_k = \frac{2k-4}{2k-1}$ ;
- (iii)  $k = 5$ ,  $c_5 = \frac{3}{5}$ ;
- (iv)  $k = 7$ ,  $c_7 = \frac{2}{3}$ ;
- (v)  $k = 17$ ,  $c_{17} = \frac{6}{7}$ .

**Proof.** In each of the above cases we describe a  $k$ -core  $(G_0, \mathcal{H})$ , yielding the cited result for  $K_k(c)$  with a uniform weight assignment. The verification of the required properties is left to the reader.

(i)  $G_0 = \overline{C^{2(k-1)}}$ ,  $H \in \mathcal{H}$  are obtained by omitting from  $V(G_0)$  a pair of antipodal vertices (this generalizes the 4-core of Example 3);

(ii)  $G_0 = \overline{C^{2k-1}}$ ,  $H \in \mathcal{H}$  are obtained by omitting from  $V(G_0)$  three equally spaced vertices;

(iii)  $G_0 = \overline{P}$  (where  $P$  denotes the Petersen graph),  $H \in \mathcal{H}$  are the complements of the 4-element independent sets of  $P$ ;

(iv)  $G_0$  is obtained from the graph  $\overline{C^{15}}$  on the vertices  $\{0, 1, \dots, 14\}$  by deleting the edges  $(0, 7), (1, 8), (5, 12), (6, 13), (10, 2), (11, 3)$ ;  $H \in \mathcal{H}$  are obtained by omitting from  $V(G_0)$  five equally spaced vertices;

(v)  $G_0$  is obtained from  $\overline{C^{35}}$  on the vertices  $\{0, 1, \dots, 34\}$  by deleting the edges  $(0, 13), (5, 18), (10, 23), (15, 28), (20, 33), (25, 3), (30, 8)$ ;  $H \in \mathcal{H}$  are obtained by omitting from  $V(G_0)$  five equally spaced vertices.  $\square$

Note that the conjecture remains open for  $k \equiv 1, 3 \pmod{6}$ ,  $k \neq 7$ . The values  $k = 5, 17$  are covered already by (ii), but the corresponding values of  $c_k$  are improved in (iii), (v).

Finally, we return to the investigation of  $F_k^*(n, n-2)$  and  $F_k^*(n, n-3)$ . We can now state sharper bounds, using the results of Sections 5 and 6. In view of Proposition 4 it suffices to state them for  $F_k^*(n, n-2)$ .

**Theorem 10** (a)  $F_5^*(n, n-2) \leq 6n - 27$  for  $n \geq 11$  and we have equality for  $n > n_0$ ;  
(b)  $F_k^*(n, n-2) \leq 2(k-2)n - (2k^2 - 5k + 4)$  for  $k \geq 6$ ,  $n \geq 2k + 5$ .

**Proof.** (a) Proposition 3 states that  $F_5^*(n, n-2) = F_4(n-2, n-4) + 2n - 4$ , while the results of Section 5 assert that  $F_4(n, n-2) \leq 4n - 15$  for  $n \geq 9$  and  $F_4(n, n-2) = 4n - 15$  for sufficiently large  $n$ . Combining these two facts we obtain

$$F_5^*(n, n-2) = F_4(n-2, n-4) + 2n - 4 \leq 4(n-2) - 15 + 2n - 4 = 6n - 27$$

for  $n \geq 11$ , with equality for  $n > n_0$ .

(b) By induction on  $k \geq 6$ . For the case  $k = 6$ , we use the 5-core  $(G_0, \mathcal{H})$  from the proof of case (iii) of the previous theorem to build a 5-saturated graph  $G$  on  $n \geq 15$  vertices with no conical vertex and with  $|E(G)| = 6n - 30$ , thus obtaining  $F_5(n, n-2) \leq 6n - 30$ . Then Proposition 3 gives

$$F_6^*(n, n-2) = F_5(n-2, n-4) + 2n - 4 \leq 6(n-2) - 30 + 2n - 4 = 8n - 46$$

for  $n \geq 17$ . For  $k > 6$ , using induction and Proposition 4, we obtain

$$\begin{aligned} F_k^*(n, n-2) &= F_{k-1}(n-2, n-4) + 2n - 4 \leq F_{k-1}^*(n-2, n-5) + 2n - 4 \\ &= F_{k-1}^*(n-2, n-4) - 1 + 2n - 4 \\ &\leq 2(k-3)(n-2) - (2(k-1)^2 - 5(k-1) + 4) + 2n - 5 \\ &= 2(k-2)n - (2k^2 - 5k + 4). \quad \square \end{aligned}$$

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## References

- [1] D.A. Duffus and D. Hanson, Minimal  $k$ -saturated and color critical graphs of prescribed minimum degree. *J. Graph Theory* 10 (1986) 55–67.
- [2] P. Erdős, A. Hajnal and J. W. Moon, A problem in graph theory, *Amer. Math. Monthly* 71 (1964) 1107–1110.
- [3] P. Erdős and R. Holzman, On maximal triangle-free graphs, *J. Graph Theory* 18 (1994) 585–594.
- [4] P. Erdős and R. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.* 35 (1960) 85–90.
- [5] Z. Füredi and Á. Seress, Maximal triangle-free graphs with restrictions on the degrees, *J. Graph Theory* 18 (1994) 11–24.
- [6] A. Hajnal, A theorem on  $k$ -saturated graphs, *Canad. J. Math.* 17 (1965) 720–724.
- [7] D. Hanson and K. Seyffarth,  $k$ -saturated graphs of prescribed maximum degree, *Congres. Numer.* 42 (1984) 169–182.
- [8] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*. Oxford Mathematical Monographs. Clarendon Press, Oxford (1979).
- [9] M. Huxley, The difference between consecutive primes. Proceedings of the Symposium in Pure Mathematics, Vol. 24, American Mathematical Society (1973) 141–146.
- [10] L. Lovász and M. D. Plummer, *Matching Theory*, North Holland, Amsterdam (1986).
- [11] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming. *Algebraic Methods in Graph Theory*, Colloquia Mathematica Societatis János Bolyai Vol. 25, North Holland (1981) 599–629.