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ON K-SEQUENCES

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1. We recall that a sequence $\{x_n\}$ in a topological group X is called a K-sequence if for every subsequence $\{y_n\}$ of $\{x_n\}$ there are a subsequence $\{t_n\}$ of $\{y_n\}$ and $t \in X$ such that

$$\sum_{n=1}^{\infty} t_n = t$$

(see [1]).

K-sequences converge to zero. Sequences converging to zero in a complete metric group are K-sequences.

In this note we prove

Theorem 1. *Assume that X is a topological group, $\{F_k\}$ is a nondecreasing sequence of closed subsets of X such that*

$$X = \bigcup_{k=1}^{\infty} F_k$$

and assume that $\{x_n\}$ is a K-sequence in X. Then there exists an index k_0 such that

$$x_n \in F_{k_0} + \Big\{ - \sum_{m \in A} x_m : A \subset \{1, \dots, k_0\} \Big\}$$

for every $n \in \mathbb{N}$.

As consequences of Theorem 1 we get the following theorems.

Theorem 2. *Assume that f_n for $n \in \mathbb{N}$ and f are sequentially continuous non-negative mappings defined on X such that the following conditions hold:*

(a) f_n *for* $n \in \mathbb{N}$ *are triangle mappings, i.e.*

$$f_n(x+y) \leqslant f_n(x) + f_n(y) \quad \text{for} \quad x, y \in \mathbb{N};$$

(b) $f(0) = 0$;
(c) $f_n(x) \to f(x)$ *for every* $x \in X$,
and assume that $\{x_n\}$ *is a* K*-sequence in* X.

Then $f_n(x_n) \to 0$ as $n \to \infty$.

Theorem 3. *If* X *is a Fréchet topological group such that every sequence converging to zero in* X *is a* K*-sequence, then* X *is of the second category.*

We recall that X is a Fréchet topological group if for every subset A of X and for every element x which belongs to the closure $\bar{A}$ of A there is a sequence $\{x_n\}$ of elements in A such that $x_n \to x$. In the case when X is a metric group, Theorem 3 was proved in [2]. Theorem 3 in the present form was proved in [3]. The proof of Theorem 3 produced in this paper is simpler than the proof in [3] and suggests a generalization of the theorem.

2. In this section we prove the theorems formulated in Section 1.

Proof of Theorem 1. Suppose that Theorem 1 does not hold. Then there are a topological group X, a nondecreasing sequence $\{F_k\}$ of closed subsets of X, a K-sequence $\{x_n\}$ in X and a subsequence $\{m_n\}$ of $\{n\}$ such that

$$x_{m_{n+1}} \notin F_{m_n} + \Big\{ - \sum_{m \in A} x_m : A \subset \{1, \ldots, m_n\} \Big\}.$$

Since $\{F_k\}$ is a nondecreasing sequence of subsets of X and subsequences of K-sequences are K-sequences, we may assume that $m_n = n$ for $n \in \mathbb{N}$ and

$$x_1 \notin G_1 = \{0\},$$
$$x_{n+1} \notin G_{n+1} = F_n + \Big\{ - \sum_{m \in A} x_m : A \subset \{1, \ldots, n\} \Big\}.$$

Since G_n for $n \in \mathbb{N}$ are closed subsets of X, there are continuous pseudonorms p_n on X and numbers $\varepsilon_n > 0$ such that

$$\inf \big\{ p_n(x_n - z) : z \in G_n \big\} > \varepsilon_n \tag{1}$$

for $n \in \mathbb{N}$. As $p_1(x_n) \to 0$, there is an index r_1 such that $p_1(x_{r_1}) < 2^{-2}\varepsilon_1$. As $p_2(x_n) \to 0$, there is an index r_2 such that

$$p_1(x_{r_2}) < 2^{-3}\varepsilon_1 \quad \text{and} \quad p_2(x_{r_2}) < 2^{-4}\varepsilon_2.$$

By induction, we select a subsequence $\{r_n\}$ or $\{n\}$ such that

$$(2) \qquad p_n(x_{r_m}) < 2^{-n-m}\varepsilon_n$$

for $n \leqslant m$ and $m,\ n \in \mathbb{N}$. Since $\{x_{r_n}\}$ is a subsequence of the K-sequence $\{x_n\}$, there are a subsequence $\{s_n\}$ of $\{r_n\}$ and $x \in X$ such that

$$\sum_{n=1}^{\infty} x_{s_n} = x.$$

Let n_0 be an index such that $x \in F_{s_{n_0-1}}$. We put

$$z = x - \sum_{n<n_0} x_{s_n}.$$

Then

$$z \in G_{s_{n_0}} \quad \text{and} \quad x_{s_{n_0}} - z = \sum_{n=n_0+1}^{\infty} x_{s_n}$$

for $n \in \mathbb{N}$. Hence, by (2), we get

$$p_{s_{n_0}}(x_{s_{n_0}} - z) \leqslant \varepsilon_{s_{n_0}},$$

which contradicts (1). This contradiction completes the proof. □

Remark 1. Under the assumptions of Theorem 1 there is an index k_0 such that $x_n \in F_{k_0} - F_{k_0}$, and there are subsequence $\{y_n\}$ of $\{x_n\}$, an index k_0, a set $A \subset \{1, \ldots, k_0\}$ and a sequence $\{z_n\}$ in F_{k_0} such that

$$y_n = -\sum_{m\in A} x_m + z_n$$

for $n \in \mathbb{N}$. If, moreover, F_k for $k \in \mathbb{N}$ are subgroups of X, then there is an index k_0 such that $x_n \in F_{k_0}$ for $n \in \mathbb{N}$.

Proof of Theorem 2. Suppose that Theorem 2 does not hold. Then there are number $\varepsilon > 0$ and a subsequence $\{m_n\}$ of $\{n\}$ such that

$$(3) \qquad f_{m_n}(x_{m_n}) > \varepsilon$$

for $n \in \mathbb{N}$. Since f is continuous, $f(0) = 0$ and $x_n \to 0$, there is a subsequence $\{p_n\}$ of $\{m_n\}$ such that

$$(4) \qquad \sum_{n=1}^{\infty} \left[f(x_{p_n}) + f(-x_{p_n})\right] < \varepsilon/3.$$

We put

$$(5) \qquad F_k = \{x \in X : |f_{p_n}(x) - f(x)| \leqslant \varepsilon/4 \quad \text{for } n \geqslant k\}.$$

We note that F_k for $k \in \mathbb{N}$ are closed subsets of X,

$$X = \bigcup_{k=1}^{\infty} F_k$$

and $\{x_{p_n}\}$ is a K-sequence. Hence, by Theorem 1, there is an index k_0 such that

$$x_{p_n} \in F_{k_0} + \Big\{ -\sum_{k\in A} x_{p_n} : A \subset \{1, \dots, k_0\}\Big\}$$

for $n \in \mathbb{N}$. According to Remark 1, there is a subsequence $\{q_n\}$ of $\{p_n\}$, a set $A \subset \{1, \dots, k_0\}$ and a sequence $\{y_n\}$ in F_{k_0} such that

$$(6) \qquad x_{q_n} = -\sum_{m\in A} x_{p_m} + y_n$$

for $n \in \mathbb{N}$. It follows from (a) that

$$f_{q_n}(x_{q_n}) \leqslant f_{q_n}\Big(-\sum_{m\in A} x_{p_m}\Big) + |f_{q_n}(y_n) - f(y_n)| + f(y_n).$$

Since $y_n \in F_k$ and for sufficiently large n we have $q_n > k_0$, in view of (5) we get

$$|f_{q_n}(y_n) - f(y_n)| < \varepsilon/3$$

for sufficiently large n. Note that, by (6), (a), (c) and (4), we can write

$$f(y_n) \leqslant f(x_{q_n}) + \sum_{m\in A} f(x_{p_m}) < \varepsilon/3.$$

Since A is a finite set, we infer from (c) and (2) that

$$f_{q_n}\Big(-\sum_{m\in A} x_{p_m}\Big) < \varepsilon/3$$

for sufficiently large n. From the above estimates we get $f_{q_n}(x_{q_n}) < \varepsilon$ for sufficiently large n, which contradicts (3). This contradiction prove the theorem. □

We precede the proof of Theorem 3 with two lemmas.

Lemma 1. *If X is a Fréchet topological group, $x_{ij} \in X$ for $i, j \in \mathbb{N}$ and $x_{ij} \to 0$ as $j \to \infty$ for $i \in \mathbb{N}$, then there are two subsequences $\{p_i\}$, $\{q_i\}$ of $\{i\}$ such that $x_{p_i q_i} \to 0$.*

Proof. We may assume that, under the assumptions of Lemma 1, there is a sequence $\{x_n\}$ in X such that $x_n \neq 0$ for every $n \in \mathbb{N}$ and $x_n \to 0$. Otherwise the lemma is trivially true. We see that $x_{ij} + x_i \to x_i$ as $j \to \infty$ for $i \in \mathbb{N}$ and $x_i \neq 0$. Therefore, there is a subsequence $\{m_i\}$ of $\{i\}$ such that $x_{ij} \neq 0$ for $j \geqslant m_i$ and $i \in \mathbb{N}$. Assume that

$$A = \{x_{ij} : j \geqslant m_i, i, j \in \mathbb{N}\}.$$

Then $0 \notin A$ but $0 \in \operatorname{cl} A$. Since X is a Fréchet topological group, there are two sequences $\{r_i\}$ and $\{s_i\}$ of positive integers such that $m_i \leqslant s_i$ for $i \in \mathbb{N}$ and $x_{r_i s_i} \to 0$. We assert that $r_i \to \infty$. Otherwise there would exist a constant subsequence $\{v_i\}$ of $\{r_i\}$ such that $v_i = v$ for $i \in \mathbb{N}$ and $x_{v s_i} \to 0$ but $x_{v s_i} \to x_v$ and $x_v \neq 0$. Consequently, $r_i \to \infty$ and $s_i \to \infty$. Thus there is a subsequence $\{k_i\}$ of $\{i\}$ such that $\{r_{k_i}\}$ and $\{s_{k_i}\}$ are subsequences of $\{i\}$. Assuming $p_i = r_{k_i}$ and $q_i = s_{k_i}$ for $i \in \mathbb{N}$ we get the lemma. □

Lemma 2. *If X is a Fréchet topological group and $\{A_n\}$ is a nonincreasing sequence of dense subsets A_n of X, then there is a sequence $\{x_n\}$ such that $x_n \in A_n$ for $n \in \mathbb{N}$ and $x_n \to 0$.*

Proof. Under the assumptions of Lemma 2, for every $i \in \mathbb{N}$ there is a sequence $\{x_{ij}\}$ such that $x_{ij} \in A_i$ for $j \in \mathbb{N}$ and $x_{ij} \to 0$ as $j \to \infty$ for $i \in \mathbb{N}$. By Lemma 1, there are two subsequences $\{p_i\}$ and $\{q_i\}$ of $\{i\}$ such that $x_{p_i q_i} \to 0$. Moreover, we have $x_{p_i q_i} \in A_{p_i} \subset A_i$ for $i \in \mathbb{N}$. Puting $x_i = x_{p_i q_i}$ for $i \in \mathbb{N}$ we get the assertion. □

Proof of Theorem 3. Suppose that X is a Fréchet topological group in which null sequences are K-sequences and X is not of the second category. Then there are closed subsets F_k of X such that $\operatorname{int} F_k = \emptyset$ for $k \in \mathbb{N}$ and

$$X = \bigcup_{k=1}^{\infty} F_k.$$

To get a contradiction we construct a matrix $\{x_{ij}\}$ such that

(i) $$x_{ij} \to 0 \text{ as } j \to \infty \quad \text{for} \quad i \in \mathbb{N}$$

and

$$\text{(ii)} \qquad x_{ij} \in \Big[F_j + \Big\{ - \sum_{(m,n)\in A} x_{mn} : A \subset \{(k,l)\colon 1 \leqslant k \leqslant i,\ 1 \leqslant l \leqslant j\}\Big\}\Big]'$$

for $i = 2, 3, \ldots$ and $j \in \mathbb{N}$. Let $\{x_{1j}\}$ be a sequence in X such that $x_{1j} \to 0$. Suppose that the first $(n-1)$ rows of the matrix have been constructed in such a way that (i) and (ii) hold. Assume that

$$F_{nj} = F_j + \Big\{ - \sum_{(m,n)\in A} x_{mn} : A \subset \{(k,l)\colon 1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant j\}\Big\}$$

for $j \in \mathbb{N}$. Then F_{nj} for $n, j \in \mathbb{N}$ are closed subsets of X, $\operatorname{int} F_{nj} = \emptyset$ and $F_{nj} \subset F_{n,j+1}$. Consequently, the components F'_{nj} are open dense subsets of X and $F'_{nj} \supset F'_{n,j+1}$ for $j \in \mathbb{N}$. Thus, by Lemma 2, there a sequence $\{x_{nj}\}$ such that

$$x_{nj} \in F'_{nj} \text{ for } j \in \mathbb{N} \quad \text{and} \quad x_{nj} \to 0 \text{ as } j \to \infty.$$

Consequently, (i) and (ii) hold for $i = n$. Hence, by induction, the existence of a matrix $\{x_{ij}\}$ such that (i) and (ii) hold follows. By Lemma 1, there are subsequences $\{p_i\}$ and $\{q_i\}$ of $\{i\}$ such that $x_{p_i q_i} \to 0$.

It follows from (ii) that

$$x_{p_i q_i} \notin F_{q_i} + \Big\{ - \sum_{k\in A} x_{p_k q_k} : A \subset \{1, \ldots, i\}\Big\}'$$

for $i \in \mathbb{N}$. On the other hand, $\{x_{p_i q_i}\}$ is a K-sequence. Hence, by Theorem 1, there exists an index i_0 such that

$$x_{p_i q_i} \in F_{i_0} + \Big\{ - \sum_{k\in A} x_{p_k q_k} : A \subset \{1, \ldots, i_0\}\Big\}$$

for $i \in \mathbb{N}$. This obvious contradiction completes the proof of Theorem 3. □

R e m a r k 2. Observe that we can modify the proof of Theorem 3 in such a way that the elements of $\{x_{ij}\}$ are in a given dense subset G of X. Therefore the assertion of Theorem 3 is valid whenever there exists a dense subset G of a Fréchet topological group X such that null sequences in G are K-sequences in X.

References

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