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On Kähler fiber spaces over curves*

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Introduction.

(0.0) In the study of higher-dimensional varieties we often face such questions: Let $f: M \to S$ be a fiber space. Suppose that S and a general fiber of f enjoy such and such properties. Then how is M?

One of the most important problems of this type is the addition conjecture for Kodaira dimensions (See [10], [16]). Recent developments in the classification theory of algebraic varieties (see [17], [18], [2], [17a]) throw light upon the relation between the above conjecture and 'positivity' of $f_*\omega_{M/S}$. In this paper we prove the numerical semi-positivity of $f_*\omega_{M/S}$ in case S is a curve.

(0.1) To be precise we fix our notation and terminology. *Variety* means an irreducible reduced compact complex analytic space. *Manifold* means a smooth variety. *Fiber space* is a triple (f, M, S), where f is a surjective morphism $M \to S$ whose general fiber is connected. Moreover, M and S are assumed to be smooth unless otherwise stated explicitly. This fiber space is said to be *Kähler* (resp. *projective*) if so is M. For a locally Macaulay variety V, ω_V denotes the dualizing sheaf of it (for the duality theory, see [7], [14], [15]). For a fiber space $f: M \to S$ we denote by $\omega_{M/S}$ the relative dualizing sheaf $\omega_M \otimes f^*\omega_S^\vee \cong \mathcal{O}_M(K_M - f^*K_S)$, where K_X denotes the canonical bundle of a manifold X.

The following three facts are well-known.

(0.2) ω_V is torsion free for any locally Macaulay variety V.

(0.3) $g_*\mathcal{F}$ is torsion free for any surjective morphism $g: X \to Y$ and for any torsion free sheaf $\mathcal{F}$ on X.

(0.4) Any torsion free sheaf on a smooth curve is locally free.

(0.5) Combining the above facts we infer that $f_*\omega_{M/C}$ is locally free for any fiber space $f: M \to C$ over a curve C. Moreover $\operatorname{rank} f_*\omega_{M/C} = p_g(F) = h^{n,0}(F)$ where F is a general fiber of f and $n = \dim F$.

(0.6) MAIN THEOREM (see (2.7)). *$f_*\omega_{M/C}$ is numerically semi-positive for any Kähler fiber space over a curve C. Namely, the invertible sheaf $\mathcal{O}(1)$ on*

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$\boldsymbol{P}(f_*\omega_{M/C})$ *(see EGA, Chap. II,* (4.1.1)) *is numerically semi-positive (see* (2.2) *and* [8]). *In particular, any quotient invertible sheaf of $f_*\omega_{M/C}$ is of degree* $\geqq 0$.

(0.7) The key of our proof is the proposition (1.2). § 1 is devoted to the proof of (1.2), and § 2 for the main theorem. The method looks rather elementary and purely computational, but it depends deeply (often implicitly) on the theory on variation of Hodge structures (see [3], [4]). The most essential part of this paper is the elementary calculations in § 1.

(0.8) In § 3 we give a structure theorem for $f_*\omega_{M/C}$, which can be treated independently of the results in §1 and § 2. However, this theorem is related to a certain positivity of $f_*\omega_{M/C}$.

(0.9) In § 4 we give several applications including those for fiber spaces over higher dimensional bases.

(0.10) If M is projective, $f_*\omega_{M/C}$ enjoys a better property than the semi-positivity. This topic will be treated in a forthcoming paper of the author.

(0.11) Perhaps our result is closely related with the problem about the (quasi-)projectivity of moduli spaces. Of course, however, the relation will not be simple.

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§ 1. Pseudo-semipositivity.

(1.1) DEFINITION. A locally free sheaf $\mathcal{F}$ (or the vector bundle corresponding to it) on a curve C is said to be *pseudo-semipositive* if $\deg L \geqq 0$ for any invertible sheaf $\mathcal{L}$ which is a homomorphic image of $\mathcal{F}$.

The purpose of this section is to prove the following

(1.2) PROPOSITION. *Let $f: M \to C$ be a Kähler fiber space over a curve C. Then $f_*\omega_{M/C}$ is pseudo-semipositive.*

(1.3) Put $m = \dim M$, $n = \dim F = m-1$ where F is a general fiber of f, $r+1 = \operatorname{rank} f_*\omega_{M/C} = p_g(F) = h^{n,0}(F)$ and let E be the vector bundle with $\mathcal{O}_C[E] \cong f_*\omega_{M/C}$. We should show $\deg L \geqq 0$ for any quotient line bundle L of E. Set $\Sigma = \{p \in C \mid f^{-1}(p) = F_p \text{ is singular}\}$. For any subset X of C we denote $X - \Sigma$ by X°. Σ is clearly a finite set.

(1.4) We define a Hermitian C^∞-metric of $E|_{C^\circ}$ in the following way. Let $\mathfrak{e} \in \Gamma(U, f_*\omega_{M/C})$ where U is an open set in C°. For $x \in U$, let t be a local parameter in a neighbourhood U_x of x. Then $f^*(dt) \in (f^{-1}(U_x), f^*\omega_C)$ and $\mathfrak{e} \in \Gamma(U, f_*\omega_{M/C}) = \Gamma(f^{-1}(U), \omega_{M/C}) = \Gamma(f^{-1}(U), \mathcal{H}om_{\mathcal{O}_M}(f^*\omega_C, \omega_M))$, hence we have $\mathfrak{e}(f^*(dt)) \in \Gamma(f^{-1}(U_x), \omega_M)$. Take a sufficiently fine open covering $\{V_\alpha\}$ of $F_x = f^{-1}(x)$ in M. Then we find a holomorphic n-form ψ_α on V_α such that $dt \wedge \psi_\alpha = \mathfrak{e}(f^*dt)$ on V_α, where dt on the left hand side is considered to be a 1-form on V_α. Easily we see $\psi_\alpha|_{F_x} = \psi_\beta|_{F_x}$ as n-forms on $V_\alpha \cap V_\beta \cap F_x$. Patching them together we obtain a holomorphic n-form $\psi_{e,x}$ on F_x. It is easy to see that $\psi_{e,x}$ is defined independently of the choice of U_x, t, $\{V_\alpha\}$ and $\{\psi_\alpha\}$. Moreover, $\psi_{e,x}$ is differentiable in x.

For a manifold X with $\dim X=n$, we define a Hermitian form $(\ ,\ )_X$ on $H^n(X;\boldsymbol{C})$ by $(\varphi, \psi)_X=\sigma_n\int_X \bar{\varphi}\wedge\psi$, where $\sigma_n=(\sqrt{-1})^{n(n+2)}$. It is easy to see that the restriction of this form to $H^{n,0}(X)$ is positive definite.

Now, for $e_1, e_2\in\Gamma(U, f_*\omega_{M/C})$, we define a C^∞-function (e_1, e_2) on U° by $(e_1, e_2)(x)=(\psi_{e_1,x}, \psi_{e_2,x})_{F_x}$. Clearly this gives rise to a Hermitian metric of E on C°.

(1.5) We take a sufficiently fine open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of C such that any point on the singular locus Σ of f is contained in only one U_λ. Let $e_{(\lambda)0}, \cdots, e_{(\lambda)r}$ be a local base of E on U_λ such that $\{e_{(\lambda)j}\}_{j\geq 1}$ is a base of $T=\mathrm{Ker}(E\to L)$. Note that the image $\widehat{e_{(\lambda)0}}\in\Gamma(U_\lambda, L)$ of $e_{(\lambda)0}$ is a local base of L. $\widehat{e_{(\mu)0}}=l_{\lambda\mu}\widehat{e_{(\lambda)0}}$ on $U_{\lambda\mu}=U_\lambda\cap U_\mu=U_\lambda^\circ\cap U_\mu^\circ$ for $l_{\lambda\mu}\in\Gamma(U_{\lambda\mu}, \mathcal{O}_C^\times)$. This cocycle $\{l_{\lambda\mu}\}$ defines the line bundle L.

(1.6) Let $h_{(\lambda)\bar{i},j}$ $(0\leqq i, j\leqq r)$ denote the C^∞-function $(e_{(\lambda)i}, e_{(\lambda)j})$ on U_λ°. For any $x\in U_\lambda^\circ$, $(h_{(\lambda)\bar{i},j}(x))_{0\leq i,j\leq r}$ is a positive definite Hermitian matrix. Therefore the submatrix $(h_{(\lambda)\bar{i},j}(x))_{1\leq i,j\leq r}$ is also positive definite, and hence regular. Let $(h_{(\lambda)}{}^{i,j}(x))_{1\leq i,j\leq r}$ be the inverse matrix of it. Clearly $h_{(\lambda)}{}^{i,j}(x)$ is a C^∞-function on U_λ°. We put $l_\lambda=e_{(\lambda)0}-\sum_{i=1}^{r}\sum_{j=1}^{r}e_{(\lambda)i}h_{(\lambda)}{}^{i,j}h_{(\lambda)\bar{j},0}\in\Gamma(U_\lambda^\circ, C^\infty(E))$. Then we have $l_\lambda\equiv e_{(\lambda)0} \bmod C^\infty(T)$, and $(t, l_\lambda)=0$ for any $t\in\Gamma(U_\lambda^\circ, C^\infty(T))$. From this we infer that $l_\mu=l_{\lambda\mu}l_\lambda$ on $U_{\lambda\mu}$.

(1.7) We put $g_\lambda=(l_\lambda, l_\lambda)\in\Gamma(U_\lambda^\circ, C^\infty)$. Then $g_\mu=|l_{\lambda\mu}|^2 g_\lambda$ on $U_{\lambda\mu}$. Setting $\omega_\lambda=(2\pi i)^{-1}\partial\bar{\partial}\log g_\lambda$, we have $\omega_\lambda=\omega_\mu$ on $U_{\lambda\mu}$. Patching them together we obtain a global (1, 1)-form ω on C°.

(1.8) LEMMA. $\dfrac{\partial^2}{\partial t\partial\bar{t}}\log g_\lambda(x)\leqq 0$ *for any* $x\in U_\lambda^\circ$ *and for any parameter* t *at* x.

PROOF. The problem is local with respect to C, so we consider everything in a sufficiently small neighbourhood U of x. Especially the family $M|_U\to U$ is differentiably trivial. Consequently we have an isomorphism $\iota_t\colon H^n(F_t)\to H^n(F_x)$ for any $t\in U$. Let Ω be the Kähler class of M and let $\hat{H}^n(F_t)=\{\varphi\in H^n(F_t)\mid \varphi\wedge\Omega|_{F_t}=0\}$. Then $\iota_t(\hat{H}^n(F_t))=\hat{H}^n(F_x)$ since Ω is a global class on M. Moreover $\hat{H}^n(F_t)=\bigoplus_{p+q=n}\hat{H}^{p,q}(F_t)$ where $\hat{H}^{p,q}(F_t)=\hat{H}^n(F_t)\cap H^{p,q}(F_t)$. Note that $(\varphi, \psi)_{F_t}=(\iota_t(\varphi), \iota_t(\psi))_{F_x}$ for $\varphi, \psi\in H^n(F_t)$.

Following Griffiths [4], § 6, with the help of the classical Hodge theory on Kähler manifolds, we take a base $(a_0, \cdots, a_r, b_1, \cdots, b_q, c_1, \cdots, c_p)$ of $\hat{H}^n(F_x)$ such that $a_\zeta\in\hat{H}^{n,0}(F_x)$, $b_\xi\in\hat{H}^{n-1,1}(F_x)$, $c_\eta\in\bigoplus_{q\geq 2}\hat{H}^{n-q,q}(F_x)$ and $(a_\sigma, a_\tau)_{F_x}=\delta_{\sigma\tau}$, (b_σ, b_τ)

$=-\delta_{\sigma\tau}$.

Let $(\mathfrak{e}_0, \cdots, \mathfrak{e}_r)$ be a local base of E on U as in (1.5). We take $\psi_{j,t}=\psi_{\mathfrak{e}_j,t} \in H^{n,0}(F_t)$, $h_{\bar{i},j}=(\mathfrak{e}_i, \mathfrak{e}_j)$, $h^{i,j}$, l and $g=(l, l)$ as in (1.4), (1.6) and (1.7). Without loss of generality we can assume that $\psi_{j,x}=a_j$ for $0\leqq j\leqq r$, since a linear transformation of $(\mathfrak{e}_0, \cdots, \mathfrak{e}_r)$ does not change $\partial\bar{\partial}\log g$.

We write $\iota_t(\psi_{j,t})=\sum_{\zeta=0}^{r}\alpha_{j,\zeta}(t)a_\zeta+\sum_{\xi=1}^{q}\beta_{j,\xi}(t)b_\xi+\sum_{\eta=1}^{p}\gamma_{j,\eta}(t)c_\eta$. Clearly $\alpha_{j,\zeta}(x)=\delta_{j\zeta}$ and $\beta_{j,\xi}(x)=\gamma_{j,\eta}(x)=0$ for any j, ζ, ξ, η. In view of Lemma (1.6) in [3] p. 811, we infer that $\alpha_{j,\zeta}$, $\beta_{j,\xi}$ and $\gamma_{j,\eta}$ are holomorphic in t. Moreover, as in [3], we have $\gamma_{j,\eta}'(x)=0$.

Now we make following calculations:

$$h_{\bar{i},j}=\sum_{\zeta=0}^{r}\overline{\alpha_{i,\zeta}}\alpha_{j,\zeta}-\sum_{\xi=1}^{q}\overline{\beta_{i,\xi}}\beta_{j,\xi}+\sum\overline{\gamma_{i,\eta_1}}\gamma_{j,\eta_2}(c_{\eta_1}, c_{\eta_2})_{F_x}.$$

$$h_{\bar{i},j}(x)=\delta_{ij} \quad\text{for}\quad 0\leqq i, j\leqq r, \quad h^{i,j}(x)=\delta_{ij} \quad\text{for}\quad 1\leqq i, j\leqq r.$$

$$\frac{\partial h_{\bar{0},j}}{\partial t}(x)=\alpha_{j,0}'(x) \quad\text{and}\quad \frac{\partial h_{\bar{0},j}}{\partial \bar{t}}(x)=\overline{\alpha_{0,j}'(x)}.$$

$$\frac{\partial h_{\bar{0},0}}{\partial t\partial\bar{t}}(x)=\sum_{\zeta=0}^{r}|\alpha_{0,\zeta}'(x)|^2-\sum_{\xi=1}^{q}|\beta_{0,\xi}'(x)|^2.$$

$$g(t)=h_{\bar{0},0}-\sum_{i=1}^{r}\sum_{j=1}^{r}h_{\bar{0},i}h^{i,j}h_{\bar{j},0} \quad\text{and}\quad g(x)=1.$$

$$\frac{\partial g}{\partial t}(x)=\frac{\partial h_{\bar{0},0}}{\partial t}(x)=\alpha'_{0,0}(x),\ \frac{\partial g}{\partial\bar{t}}(x)=\overline{\alpha'_{0,0}(x)}.$$

$$\frac{\partial^2 g}{\partial t\partial\bar{t}}(x)=\frac{\partial^2 h_{\bar{0},0}}{\partial t\partial\bar{t}}-\sum_{i=1}^{r}\left|\frac{\partial h_{\bar{0},i}}{\partial t}\right|^2-\sum_{i=1}^{r}\left|\frac{\partial h_{\bar{0},i}}{\partial\bar{t}}\right|^2$$

$$=|\alpha_{0,0}'(x)|^2-\sum_{\xi=1}^{q}|\beta_{0,\xi}'(x)|^2-\sum_{i=1}^{r}|\alpha_{i,0}'(x)|^2.$$

$$\frac{\partial^2}{\partial t\partial\bar{t}}\log g=g^{-2}\Big(g\frac{\partial g}{\partial t\partial\bar{t}}-\frac{\partial g}{\partial t}\frac{\partial g}{\partial\bar{t}}\Big).$$

$$\frac{\partial^2}{\partial t\partial\bar{t}}\log g(x)=-\sum_{\xi=1}^{q}|\beta'_{0,\xi}(x)|^2-\sum_{i=1}^{r}|\alpha_{i,0}'(x)|^2\leqq 0.$$ Thus we prove the lemma.

(1.9) COROLLARY. $\int_U\omega\geqq 0$ for any $U\subset C^\circ$.

PROOF. Combine (1.8) and (1.7).

(1.10) Now we want to study the behaviour of g in a sufficiently small neighbourhood U of $p\in\Sigma$. Let $(\mathfrak{e}_0, \cdots, \mathfrak{e}_r)$ be a local base of E on U as in (1.5) and define $h_{\bar{i},j}$, l and g as before. So g is a C^∞-function on $U^\circ=U-\{p\}$. We introduce the following notation: For $\mu=(\mu_0, \cdots, \mu_r)\in \boldsymbol{C}^{r+1}$ we put $\mathfrak{e}_\mu=\sum_{j=0}^{r}\mu_j\mathfrak{e}_j$ and $h_\mu=(\mathfrak{e}_\mu, \mathfrak{e}_\mu)$. Let $S=\{\mu\in \boldsymbol{C}^{r+1} \mid |\mu|=1\}$ where $|\mu|^2=\sum_{j=0}^{r}|\mu_j|^2$. It is clear that $h_{\alpha\mu}=|\alpha|^2 h_\mu$ for $\alpha\in \boldsymbol{C}$ and that S is homeomorphic to a topological sphere $\boldsymbol{S}^{2r+1}$.

(1.11) LEMMA. *For any $\mu\in S$, there is a neighbourhood W of μ in S and a neighbourhood U' of p in C and a positive number N such that $h_\nu(x)\geqq N$ for any $\nu\in W$, $x\in U'-\{p\}$.*

PROOF. Let $F_p=\Sigma\delta_i D_i$ be the prime decomposition of the divisor F_p. Since $(\mathfrak{e}_0, \cdots, \mathfrak{e}_r)$ is a local base of $f_*\omega_{M/C}$, we infer that the holomorphic m-form $\mathfrak{e}_\mu(f^*dt)$ does not vanish identically along F_p. Precisely speaking, we have a component D_j such that $\mathfrak{e}_\mu(f^*dt)$ does not vanish at order $\delta=\delta_j$ along D_j. Take a general point y of D_j and let $(z_0, \cdots, z_n)$ be a coordinate system in a neighbourhood V of y in M such that $f^*t=z_0^\delta$. Writing $\mathfrak{e}_\nu(f^*dt)=\varphi_\nu(z)dz_0\wedge\cdots\wedge dz_n$ on V and putting $\psi_\nu=\delta^{-1}z_0^{1-\delta}\varphi_\nu(z)dz_1\wedge\cdots\wedge dz_n$, we have $\mathfrak{e}_\nu(f^*dt)=dt\wedge\psi_\nu$ for $\nu\in \boldsymbol{C}^{r+1}$. Note that $\varphi_\nu(z)$ is holomorphic in ν. ψ_μ does not have a zero (possibly has a pole) at y since y is a general point on D_j. Therefore, if W is a sufficiently small neighbourhood of μ in S and if $V_\varepsilon=\{z=(z_0, \cdots, z_n)\in V \mid |z_j|<\varepsilon$ for any $j\}$ with ε being sufficiently small, we have $\mathrm{Min}_{\nu\in W, z\in V_\varepsilon}|z_0^{1-\delta}\varphi_\nu(z)|^2=k>0$. Then for any $x\in U'=f(V_\varepsilon)$ with $x\neq p$ we calculate $h_\nu(x)\geqq\sigma_n\int_{V_\varepsilon\cap F_x}\bar{\psi}_\nu\wedge\psi_\nu \geqq(2\pi\varepsilon^2)^n\delta^{-2}k>0$. This proves the lemma.

(1.12) LEMMA. *There is a neighbourhood U' of p and a positive number N such that $h_\mu(x)\geqq N$ for any $\mu\in S$, $x\in U'-\{p\}$.*

This follows from (1.11) since S is compact.

(1.13) LEMMA. *There is a neighbourhood U' of p and a positive number N such that $g(x)\geqq N$ for any $x\in U'-\{p\}$.*

PROOF. Let U' and N be as in (1.12). For $x\in U'-\{p\}$ we put $\mu_0=1$, $\mu_i=-\sum_{j=1}^{r}h^{i,j}(x)h_{j,0}(x)$ and $\mu=(\mu_0, \cdots, \mu_r)\in \boldsymbol{C}^{r+1}$. Then $g(x)=(l, l)(x)=(\mathfrak{e}_\mu, \mathfrak{e}_\mu)(x)$ $=h_\mu(x)=|\mu|^2 h_{\mu/|\mu|}(x)\geqq h_{\mu/|\mu|}(x)\geqq N$ since $\mu/|\mu|\in S$. This proves the lemma.

(1.14) Let t be a local parameter at p and let Γ_R be the circle $\{t \mid |t|=R\}$ around p. Put $I(R)=(2\pi i)^{-1}\int_{\Gamma_R}\bar{\partial}\log g$. Then we have the following

LEMMA. $\limsup_{R\to 0} I(R)\geqq 0$.

PROOF. Put $F(R)=\int_{\Gamma_R}\log g\, d\theta$, where $\log g$ is the real branch and (r, θ) is the real polar coordinate with $t=r(\cos\theta+i\sin\theta)$. Then, from an elementary calculation, follows $I(R)=-(4\pi)^{-1}RF'(R)$. Suppose that $\limsup_{R\to 0} I(R)=-k<0$. Then there exists $R_0>0$ such that $I(R)\leqq -k/4\pi$ for any $R\leqq R_0$. So $F'(R)\geqq kR^{-1}$. Consequently $F(R)=F(R_0)-\int_R^{R_0}F'(r)dr\leqq F(R_0)-\int kr^{-1}dr=F(R_0)-k\log R_0+k\log R$.

Hence $\lim_{R\to 0} F(R)=-\infty$. On the other hand, (1.12) implies $F(R)\geqq 2\pi \log N>-\infty$ for any small R. This contradiction proves the lemma.

(1.15) REMARK. $I(R)\geqq 0$ for any R.

PROOF. Using the theorem of Stokes we infer from (1.9) that $I(R_1)-I(R_2) =\int_{R_2\leq|t|\leq R_1}\omega\geqq 0$ for any $R_1\geqq R_2$. Combining this with (1.14) we prove the assertion.

(1.16) Now we prove the proposition (1.2). Take a covering $\{U_\lambda\}$ of C as in (1.5). Let g_λ and ω_λ be as in (1.7). For each $p\in\Sigma$, let U_p be the unique open set which contains p, and let t_p be the local parameter at p in U_p. Let $\Delta_p=\{t_p\,|\,|t_p|\leqq\varepsilon\}$ with ε being sufficiently small. So $\Delta_p\cap U_\lambda=\emptyset$ for $\lambda\neq p$. Take a positive C^∞-function $\tilde{g}_p$ on U_p such that $\tilde{g}_p(t_p)=g_p(t_p)$ if $|t_p|\geqq\varepsilon$. Put $\tilde{g}_\lambda=g_\lambda$ for $\lambda\notin\Sigma$. Then $\tilde{g}_\lambda=g_\lambda$ on $U_{\lambda\mu}$ for any $\lambda\neq\mu$, hence $\tilde{g}_\mu=|l_{\lambda\mu}|^2\tilde{g}_\lambda$ on $U_{\lambda\mu}$. Therefore we can patch $\tilde{\omega}_\lambda=(2\pi i)^{-1}\partial\bar{\partial}\log\tilde{g}_\lambda$ to obtain a global (1, 1)-form $\tilde{\omega}$ on C. Recall that the cocycle $\{l_{\lambda\mu}\}$ defines the line bundle L (see (1.5)). So the classical theory of Chern classes gives $\deg L=\int_C\tilde{\omega}$ (see, for example, [**11**] p. 127). $\int_{C-\cup\Delta_p}\tilde{\omega}=\int_{C-\cup\Delta_p}\omega\geqq 0$ follows from (1.9). On the other hand, using the theorem of Stokes, we infer $\int_{\Delta_p}\tilde{\omega}=(2\pi i)^{-1}\int_{\partial\Delta_p}\bar{\partial}\log\tilde{g}_p=(2\pi i)^{-1}\int_{\partial\Delta_p}\bar{\partial}\log g_p\geqq 0$ from (1.15). Combining things together we obtain $\deg L\geqq 0$.

§ 2. Semipositivity.

(2.1) PROPOSITION. *Let L be a line bundle on a projective variety V. Then the following conditions are equivalent to each other.*

a) $L^r\{W\}\geqq 0$ *for any subvariety W of V, where* $r=\dim W$.

b) $L\{C\}\geqq 0$ *for any curve C in V.*

c) $tL+A$ *is ample for any $t>0$ and for any ample line bundle A.*

PROOF. See Hartshorne [8], p. 34 and p. 30.

(2.2) DEFINITION. L is said to be *numerically semipositive* (or *semipositive*, as an abbreviated form) if the above conditions are satisfied.

(2.3) NOTATION. Let E be a vector bundle on X and let $E^\vee$ be the dual of it. By $\boldsymbol{P}(E)$ we denote the quotient of $E^\vee-\{\text{zero section}\}$ by the natural $\boldsymbol{C}^*$-action. The natural mapping $\pi:\boldsymbol{P}(E)\to X$ makes $\boldsymbol{P}(E)$ a fiber bundle over X with fiber $\boldsymbol{P}^r$, $r=\operatorname{rank} E-1$. Each point y on $\boldsymbol{P}(E)$ corresponds in a canonical way to a sub-vectorspace of $E^\vee_{\pi(y)}$ of dimension 1. Therefore we have a natural subline bundle L of $\pi^*E^\vee$. By $H(E)$ we denote the dual of L. It is well known that $\pi_*\mathcal{O}_{P(E)}[kH(E)]\cong\mathcal{O}_X(S^kE)$ for any $k\geqq 0$, where S^kE denotes the k-th symmetric product of E (see [5], Chap. II, § 4).

(2.4) PROPOSITION. *Let E be a vector bundle on a projective variety V. Then the following conditions are equivalent to each other.*

a) *$H(E)$ is numerically semipositive on $\boldsymbol{P}(E)$.*

b) *$kH(E)+\pi^*A$ is ample on $\boldsymbol{P}(E)$ for any $k>0$ and for any ample line bundle A on V.*

PROOF. Put $H=H(E)$. b) $\rightarrow$ a): $(kH+A)C>0$ for any curve C and for $k>0$. Letting $k\rightarrow\infty$ we obtain $HC\geqq 0$, the condition b) in (2.1). a) $\rightarrow$ b): $H+aA$ is ample for $a\gg 0$ since H is relatively ample. Hence $bH+aA$ is also ample for $b>0$ ((2.1)c). So $kH+A$ is ample as well as $a(kH+A)=akH+aA$.

(2.5) DEFINITION. A vector bundle (or the corresponding locally free sheaf) is said to be (*numerically*) *semipositive* if the above conditions are satisfied.

(2.6) REMARK. A locally free sheaf is semipositive if it is a homomorphic image of a semipositive locally free sheaf. Any pull back of a semipositive vector bundle is also semipositive. A semipositive locally free sheaf on a curve is pseudo-semipositive.

(2.7) THEOREM. *Let $f: M\rightarrow C$ be a Kähler fiber space over a curve C. Then $f_*\omega_{M/C}$ is locally free and numerically semipositive.*

We need several preparatory results to prove this theorem.

(2.8) PROPOSITION. *Let E be a vector bundle on a curve C. Suppose that f^*E is pseudo-semipositive for any finite morphism $f: C'\rightarrow C$. Then E is semipositive.*

PROOF. Let $P=\boldsymbol{P}(E)$ with $\pi: P\rightarrow C$ and put $H=H(E)$. It suffices to show $HC'\geqq 0$ for any curve C' in P. This is clearly valid if C' is contained in a fiber of π. So we may assume that the restriction $f: C'\rightarrow C$ of π is finite. The inclusion $C'\subset P$ gives rise to a section of $P'=P\underset{C}{\times}C'\cong\boldsymbol{P}(f^*E)$ over C'. Correspondingly we have a quotient line bundle L of f^*E. It is easy to see $\deg L=HC'$. On the other hand, $\deg L\geqq 0$ since f^*E is pseudo-semipositive. Thus we prove the assertion.

(2.9) PROPOSITION. *Let $f: \mathcal{F}\rightarrow\mathcal{G}$ be a homomorphism between locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on a curve C. Suppose that* Supp Coker f *is a finite set and that $\mathcal{F}$ is pseudo-semipositive. Then $\mathcal{G}$ is also pseudo-semipositive.*

PROOF. Let $\mathcal{L}$ be an invertible sheaf which is a homomorphic image of $\mathcal{G}$. Let $h: \mathcal{F}\rightarrow\mathcal{L}$ be the induced homomorphism and let $\mathcal{H}$ be the image of h. Since Supp $(\mathcal{L}/\mathcal{H})\subset$Supp Coker f, we infer that $\mathcal{H}$ is an invertible sheaf with $\deg\mathcal{H}\leqq\deg\mathcal{L}$. On the other hand, $\deg\mathcal{H}\geqq 0$ since $\mathcal{F}$ is pseudo-semipositive. This proves $\deg\mathcal{L}\geqq 0$, so the assertion.

(2.10) PROPOSITION. *Let $f: M\rightarrow V$ be a surjective morphism where M and V are locally Macaulay varieties of a same dimension n. Then there exists a non-trivial $\mathcal{O}_V$-homomorphism $f_*\omega_M\rightarrow\omega_V$.*

PROOF (for duality theory, see [7], [14], [15]). We consider the following spectral sequence of Leray: $E_2^{p,q}=H^p(V, R^qf_*\omega_M)\Rightarrow H^{p+q}(M, \omega_M)$. If $x\in$ Supp $R^qf_*\omega_M$, then $\dim f^{-1}(x)\geqq q$. Hence dim Supp $R^qf_*\omega_M<n-q$ for $q>0$ because

$f^{-1}(\operatorname{Supp} R^q f_*\omega_M)$ is a proper subvariety of M. Therefore $E_2^{p,q}=0$ if $p=n-q$, $q>0$. So $H^n(M, \omega_M)=E_\infty^{n,0}$, which is a homomorphic image of $E_2^{n,0}=H^n(V, f_*\omega_M)$ $=\operatorname{Ext}_V^0(f_*\omega_M, \omega_V)^\vee=\operatorname{Hom}_{\mathcal{O}_V}(f_*\omega_M, \omega_V)^\vee$. Since $H^n(M, \omega_M)=\operatorname{Ext}_M^n(\mathcal{O}_M, \omega_M)=H^0(\mathcal{O}_M)^\vee\neq 0$, this proves the assertion.

(2.11) REMARK. If M is smooth, then $R^q f_*\omega_M$ seems to vanish for $q>0$. In particular, $E_2^{p,q}=E_\infty^{p,q}$ and the above morphism is determined uniquely modulo scalar multiplication. However, the author can prove this assertion only when V is algebraic.

(2.12) Now we prove the Theorem (2.7). Thanks to (2.8), it suffices to show that $\pi^*(f_*\omega_{M/C})$ is pseudo-semipositive for any finite morphism $\pi: C'\to C$. Clearly we may assume C' to be normal. Let $M'=M\underset{C}{\times}C'$ with $f': M'\to C'$ being the induced morphism. M' may not be smooth, but is always locally Macaulay (as a matter of fact, possible are only hypersurface singularities). Thanks to the theory of Hironaka ([**9**], [**9a**]), we can find a relatively projective birational morphism $\mu: M^\# \to M'$ such that $M^\#$ is a Kähler manifold. Applying (1.2) to $f^\#: M^\#\to C'$, we infer that $f^\#{}_*\omega_{M^\#/C'}$ is pseudo-semipositive. We use (2.10) to obtain a non-trivial homomorphism $\delta: \mu_*\omega_{M^\#}\to\omega_{M'}$. The support of $\operatorname{Coker}\delta$ is contained in the set of singular points of M'. δ induces $f'_*(\delta)$: $f^\#{}_*\omega_{M^\#/C'}\to f'_*\omega_{M'/C'}$. The support of $\operatorname{Coker}(f'_*(\delta))$ is contained in $f'(\operatorname{Supp}\operatorname{Coker}\delta)$ and hence is finite. So (2.9) implies that $f'_*\omega_{M'/C'}$ is pseudo-semipositive. We have $f'_*\omega_{M'/C'}=\pi^*(f_*\omega_{M/C})$ since f is flat. Combining things together, we prove the theorem.

(2.13) Using (2.10), we can generalize (2.7) into the following

THEOREM. *Let $f: M\to C$ be a surjective morphism onto a smooth curve C. Suppose that M is locally Macaulay and is dominated by a Kähler manifold, and that a general fiber of f is smooth and connected. Then $f_*\omega_{M/C}$ is locally free and numerically semipositive.*

§ 3. Decomposition.

In this section we prove the following

(3.1) THEOREM. *Let $f: M\to C$ be a Kähler fiber space over a curve C. Then $f_*\omega_{M/C}$ is a direct sum of $\mathcal{O}^{\oplus h}$ with $h=h^1(C, f_*\omega_M)$ and a locally free sheaf $\mathcal{E}$ with $H^1(C, \mathcal{E}[K_C])=0$, where $\mathcal{O}^{\oplus h}$ denotes the direct sum $\mathcal{O}\oplus\cdots\oplus\mathcal{O}$ of h pieces of $\mathcal{O}_C$.*

(3.2) Put $m=\dim M$ and $n=\dim F_t=m-1$ as before. Set $H=H^{n,0}(M)$ and let $N=\{\varphi\in H \mid \varphi_{F_t}=0 \text{ in } H^{n,0}(F_t) \text{ for any } t\in C^0\}$. Let $\wedge: H\to H^0(C, f_*\omega_{M/C})$ $=\operatorname{Hom}_{\mathcal{O}}(\omega_C, f_*\omega_M)$ be the natural homomorphism defined by the exterior product. It is clear that $\wedge(N)=0$. So $\wedge$ defines a natural mapping $\widetilde{\wedge}: H/N\to$ $\operatorname{Hom}_{\mathcal{O}}(\omega_C, f_*\omega_M)$.

(3.3) Let $\tau\in H^2(C;\boldsymbol{Z})\subset H^2(C)=H^1(C,\omega_C)$ be the Chern class of a divisor on C of degree 1. Then we have a homomorphism $v=H^1(\cdot)(\tau):\mathrm{Hom}_{\mathcal{O}}(\omega_C,f_*\omega_M)\to H^1(C,f_*\omega_M)$.

(3.4) Using the theory of Leray spectral sequence, we obtain a natural injective homomorphism $\iota: H^1(C,f_*\omega_M)\to H^1(M,\omega_M)=H^{m,1}(M)$.

(3.5) By definition of $\wedge$, v and ι we see easily $\iota\circ v\circ\wedge(\varphi)=f^*\tau\wedge\varphi$ for $\varphi\in H$.

(3.6) The duality theory gives a natural isomorphism $s: H^{m,1}(M)\to \bar{H}^\vee$ where $\bar{H}^\vee$ denotes the space of skew-linear functionals on H([**11**], p. 104).

(3.7) CLAIM. Image$(s\cdot\iota)\subset\overline{(H/N)}^\vee\subset\bar{H}^\vee$.

It suffices to show that $(\bar{\varphi}\wedge\iota(\psi))\{M\}=0$ for any $\varphi\in N$ and any $\psi\in H^1(C,f_*\omega_M)$. We take a sufficiently fine locally finite covering $\{U_\alpha\}$ of C and represent ψ by a 1-cocycle $\{\psi_{\alpha\beta}\}$ where $\psi_{\alpha\beta}\in\Gamma(U_{\alpha\beta},f_*\omega_M)=\Gamma(f^{-1}(U_{\alpha\beta}),\omega_M)$. Take a family $\{e_\alpha\}$ of C^∞-functions on C such that $0\leqq e_\alpha\leqq 1$, Supp$(e_\alpha)\subset U_\alpha$ and $1=\sum_\alpha e_\alpha$. Put $\psi_\alpha=\sum_\gamma e_\gamma\psi_{\gamma\alpha}$, which can be considered as a C^∞-$(m,0)$-form on $f^{-1}(U_\alpha)$. We see easily that $\psi_\beta-\psi_\alpha=\psi_{\alpha\beta}$ and $\bar{\partial}\psi_\beta=\bar{\partial}\psi_\alpha$ on $f^{-1}(U_{\alpha\beta})$. We patch $\{\bar{\partial}\psi_\alpha\}$ together to obtain a $(m,1)$-form Ψ on M. The cohomology class of this form is $\iota(\psi)$. $\bar{\varphi}\wedge\Psi=0$ for any $\varphi\in N$, because $\bar{\partial}\psi_\alpha=\bar{\partial}(\Sigma e_\gamma\psi_{\gamma\alpha})=\Sigma(\bar{\partial}e_\gamma)\wedge\psi_{\gamma\alpha}$. Thus the claim follows.

(3.8) CLAIM. $s\circ\iota\circ v\circ\widetilde{\wedge}: H/N\to\overline{H/N}^\vee$ defines a positive definite Hermitian form on H/N.

Let $\varphi\in H$. Then $\sigma_n(s\circ\iota\circ v\circ\wedge(\varphi),\varphi)=(\iota\circ v\circ\wedge(\varphi),\varphi)_M=(f^*\tau\wedge\varphi,\varphi)_M=(\varphi_{F_t},\varphi_{F_t})\geqq 0$ (see (1.4) and (3.5)). Moreover, the equality holds only when $\varphi_{F_t}=0$ for any $t\in C^\circ$. This proves the claim.

(3.9) $s\circ\iota$ is injective ((3.4) and (3.6)). This is surjective onto $\overline{H/N}^\vee$, because $s\circ\iota\circ v\circ\widetilde{\wedge}$ is bijective (see (3.8)). Hence $s\circ\iota$ is bijective. Therefore $v\circ\widetilde{\wedge}$ is also bijective and $v\circ\wedge: H\to H^1(C,f_*\omega_M)$ is surjective.

Take $\varphi_1,\cdots,\varphi_h\in H$ so that $\{v\circ\wedge(\varphi_j)\}$ is a base of $H^1(C,f_*\omega_M)$. $\wedge\varphi_1\oplus\cdots\oplus\wedge\varphi_h$ defines canonically $\Phi\in\mathrm{Hom}_{\mathcal{O}_C}(\omega_C{}^{\oplus h},f_*\omega_M)$.

(3.10) CLAIM. Φ corresponds to a sub-vector bundle. Or equivalently, Coker(Φ) is locally free.

PROOF. Note that $H^1(\Phi): H^1(\omega_C{}^{\oplus h})\to H^1(f_*\omega_M)$ is bijective. Each φ_j defines canonically a section $\hat{\varphi}_j$ of the vector bundle which corresponds to $f_*\omega_{M/C}$. Consider the values of $\hat{\varphi}_j$ at each point on C. If they are linearly independent at each point on C, then the assertion is true. If otherwise, we can find a linear combination φ of φ_j such that $\hat{\varphi}(x)=0$ at some $x\in C$. So $\wedge\varphi$ factors to $\omega_C\to\omega_C[D]\to f_*\omega_M$, where D is a positive divisor on C which corresponds to the zero of $\hat{\varphi}$. Since $H^1(\omega_C[D])=0$, we infer that $H^1(\wedge\varphi)=0$. This contradicts to the bijectivity of $H^1(\Phi)$. Thus we prove the assertion.

(3.11) CLAIM. Let $\mathcal{C}=$Coker(Φ). Then the exact sequence $0\to\omega_C{}^{\oplus h}\to f_*\omega_M$

$\to \mathcal{C} \to 0$ splits.

PROOF. Let $e \in H^1(C, \mathcal{H}om_{\mathcal{O}}(\mathcal{C}, \omega_C{}^{\oplus h}))$ be the obstruction class. By the natural isomorphisms $H^1(C, \mathcal{H}om_{\mathcal{O}}(\mathcal{C}, \omega_C{}^{\oplus h})) \cong \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{C}, \omega_C{}^{\oplus h}) \cong \mathrm{Hom}(H^0(\mathcal{C}), H^1(\omega_C{}^{\oplus h}))$, e maps to $\delta: H^0(\mathcal{C}) \to H^1(\omega_C{}^{\oplus h})$ which gives rise to the long cohomology exact sequence. The bijectivity of $H^1(\Phi)$ implies $\delta=0$. Hence $e=0$, which proves the assertion.

(3.12) Now, putting things together, we easily prove the theorem (3.1). Moreover, we see that each component of $\mathcal{O}_C{}^{\oplus h}$ comes from a holomorphic n-form on M such that the restriction of it to a general fiber does not vanish. It is clear that this decomposition is essentially unique.

§ 4. Applications.

(4.1) PROPOSITION. *Let $f: M \to C$ be a Kähler fiber space over a curve C of genus $g \geqq 2$. Then $f_*\omega_M$ is ample if it does not vanish.*

This follows from Theorem (2.7). (3.1) also implies this result, since every vector bundle E on a curve C of genus$\geqq 2$ with $H^1(E)=0$ is ample.

(4.2) COROLLARY. *Let $f: M \to C$ be a Kähler fiber space over a curve of genus $g \geqq 2$. Suppose that $p_g(F)>0$ for a general fiber F of f. Then $\kappa(M)=1+\kappa(F)$.*

PROOF. We have a natural non-trivial homomorphism $S^k(f_*\omega_M) \to f_*(\omega_M{}^{\otimes k})$. Consequently $0 \neq \Gamma(C, f_*(\omega_M{}^{\otimes k})[-K_C]) \cong \Gamma(M, kK_M - f^*K_C)$ for a sufficiently large k. Hence Proposition 1 of [2] applies.

REMARK. The above formula was originally proved by Ueno by a slightly different method ([17] and [**17a**]).

(4.3) LEMMA. *Let $f: M \to U$ be a smooth family of compact complex manifolds such that $K_M = f^*L$ for some line bundle L on U. Suppose that there is a holomorphic n-form Ψ on M ($n = \dim F$, F being a fiber of f) such that the restriction of it to each fiber does not vanish. Then this family is analytically locally trivial.*

PROOF. The problem is local with respect to U, so we consider everything in a small neighbourhood on U. In particular, we may assume that there is a covering $\{V_\alpha\}$ of M with coordinate system $(t_1, \cdots, t_s, z_\alpha^1, \cdots, z_\alpha^n)$ on V_α, where $(t_1, \cdots, t_s)$ is the coordinate on U. For the sake of simplicity we consider only the case in which $\dim U = s = 1$, because one can prove the general assertion by induction on s using a similar method. We write $\Psi_{V_\alpha} = \sum_{j=0}^{n} \psi_{\alpha,j}(z_\alpha) dz_\alpha^0 \wedge \cdots \wedge dz_\alpha^{j-1} \wedge dz_\alpha^{j+1} \wedge \cdots \wedge dz_\alpha^n$ where we set $z_\alpha^0 = t_1$. From the assumption we infer that $dt \wedge \Psi$ is a nowhere vanishing holomorphic m-form on M. This implies that $\psi_{\alpha,0}(z_\alpha)$ is an invertible function on V_α. We set

$$\gamma_\alpha=\sum_{j=0}^{n}(-1)^j\psi_{\alpha,j}(z_\alpha)\psi_{\alpha,0}^{-1}(z_\alpha)\frac{\partial}{\partial z_\alpha^j}.$$

We see easily that we can patch $\{\gamma_\alpha\}$ to obtain a global vector field γ on M. Moreover we have $f_*\gamma=\frac{\partial}{\partial t}$ at every point on M. Integrating this vector field γ, we obtain an analytic trivialization.

(4.4) LEMMA. *Let $f: M\to S$ be a fiber space with $K_M=f^*K_S$. Suppose that there exists a holomorphic n-form Ψ on M such that the restriction of Ψ to a general fiber does not vanish ($n=\dim F$). Then f is smooth and locally analytically trivial.*

PROOF. Let $\{U_\alpha\}$ be a sufficiently fine open covering of S and let ω_α be a local base of $K_S|_{U_\alpha}$. Then $\omega_\alpha\wedge\Psi$ is a non-trivial holomorphic m-form on $f^{-1}(U_\alpha)$. $K_M=f^*K_S$ implies that $\omega_\alpha\wedge\Psi$ vanishes nowhere on $f^{-1}(U_\alpha)$. This is impossible unless f is of maximal rank. Therefore f is smooth. Moreover, the restriction of Ψ to each fiber does not vanish, since otherwise $\omega_\alpha\wedge\Psi$ would have zero. So Lemma (4.3) applies.

(4.5) LEMMA. *Let $f: M\to U$ and $\tilde{f}:\tilde{M}\to\tilde{U}$ be smooth families of compact complex manifolds. Suppose ihat there exist finite unramified morphisms $\pi:\tilde{M}\to M$ and $\pi':\tilde{U}\to U$ such that $\pi'\circ\tilde{f}=f\circ\pi$. Then, if $\tilde{f}$ is locally trivial, so is f.*

PROOF. The problem is local with respect to U. Therefore, taking a sufficiently small neighbourhood on U if necessary, we may assume that each connected component of $\tilde{U}$ is isomorphic to U. Moreover, taking a connected component of $\tilde{M}$, we can assume that $\tilde{U}=U$. Let T_X denote the tangent bundle of a complex manifold X. The natural mapping $T_M\to f^*T_U$ is surjective since f is smooth. Let $T_{M/U}$ be the kernel of this mapping. Then we have a natural mapping $\tau_f: f_*(\mathcal{O}_M[f^*T_U])=\mathcal{O}_U[T_U]\to R^1f_*(\mathcal{O}_M[T_{M/U}])$. Infinitesimal version of this mapping is the well known one of Kodaira and Spencer ([**11**], p. 37). Similarly we have $\tau_{\tilde{f}}:\mathcal{O}_U[T_U]\to R^1f_*(\mathcal{O}_{\tilde{M}}[T_{\tilde{M}/U}])$. Note that $\tau_{\tilde{f}}=0$ since $\tilde{f}$ is locally trivial. On the other hand, using the trace mapping, we infer that the natural injection $\mathcal{O}_M\to\pi_*\mathcal{O}_{\tilde{M}}$ gives rise to a splitting exact sequence. Hence the induced mapping $\iota: R^1f_*(\mathcal{O}_M[T_{M/U}])\to R^1f_*(\pi_*(\mathcal{O}_{\tilde{M}}[\pi^*T_{M/U}]))=R^1\tilde{f}_*(\mathcal{O}_{\tilde{M}}[T_{\tilde{M}/U}])$ is injective. Consequently $\iota\circ\tau_f=\tau_{\tilde{f}}=0$ implies $\tau_f=0$. Therefore $f_*\mathcal{O}_M[T_M]\to f_*\mathcal{O}_M[f^*T_U]=\mathcal{O}_U[T_U]$ is surjective. Namely, any vector field on U can be lifted on M. Integrating this vector field, we obtain an analytic trivialization.

(4.6) LEMMA. *Let M be a compact complex manifold and let F be a line bundle on M such that $kF=0$ for some $k>0$. Then there exists a finite unramified covering $\pi:\tilde{M}\to M$ such that $\pi^*F=0$.*

PROOF. Let $\{V_\alpha\}$ be a sufficiently fine open covering of M and let φ_α be a local base of F on V_α. $\varphi_\beta=f_{\alpha\beta}\varphi_\alpha$ on $V_{\alpha\beta}$ for $f_{\alpha\beta}\in\Gamma(V_{\alpha\beta},\mathcal{O}^\times)$. Let ζ_α be the fiber coordinate of $F|_{V_\alpha}$ such that $\zeta_\alpha=f_{\alpha\beta}\zeta_\beta$ on $V_{\alpha\beta}$. Since $kF=0$, we have $\{\psi_\alpha\}$

with $\psi_\alpha \in \Gamma(V_\alpha, \mathcal{O}^\times)$ such that $\psi_\alpha = f_{\alpha\beta}{}^k \psi_\beta$. Let D_α be the divisor in $F|_{V_\alpha}$ defined by $\zeta_\alpha{}^k = \psi_\alpha$. Then $D_\alpha = D_\beta$ over $V_{\alpha\beta}$. So $D = \bigcup_\alpha D_\alpha$ is a divisor in F. Let $\tilde{M}$ be an irreducible component of D. The restriction $\pi: \tilde{M} \to M$ of the projection $F \to M$ is unramified since $\{\psi_\alpha\}$ have no zero. Moreover, $\{\zeta_\alpha \pi^* \varphi_\alpha\}$ defines a non-vanishing section of $\pi^* F$. Thus we prove the lemma.

(4.7) PROPOSITION. *Let $f: M \to S$ be a Kähler fiber space and let Σ be its singular locus. Suppose that* codim $\Sigma \geqq 2$ *and that $k(K_M - f^* K_S) = 0$ for some $k > 0$. Then f is smooth and locally trivial.*

PROOF. First we consider the case in which $k=1$. Then $f_* \omega_{M/S} = \mathcal{O}_S$. Let $\mathfrak{e}$ be the non-trivial global section of $f_* \omega_{M/S}$. Similarly as in (1.4), we get a family $\{\psi_{\mathfrak{e},t}\}$ of n-forms along the fibers $\{F_t\}$, $t \in S^\circ = S - \Sigma$. Moreover, we have a C^∞-function $g = (\mathfrak{e}, \mathfrak{e})$ on S° as in (1.7). Note that $K_{F_t} = 0$ and $p_g(F_t) = 1$ for any $t \in S^\circ$. Let x be a point on S°. Taking a base of $H^n(F_x)$ and using the C^∞-trivialization, we obtain multi-valued holomorphic functions $\alpha, \beta_1, \cdots, \beta_q, \gamma_1, \cdots, \gamma_p$ on S° such that $g = |\alpha|^2 - \Sigma |\beta_\xi|^2 + \Sigma \bar{\gamma}_i \gamma_j c_{ij}$ quite similarly as in (1.8). By Φ we denote this $H^n(F_x)$-valued multi-valued function on S°. Let $\pi: \tilde{S} \to S$ be the universal covering of S and let $\tilde{\Sigma} = \pi^{-1}(\Sigma)$ and $\tilde{S}^\circ = \tilde{S} - \tilde{\Sigma}$. Then $\pi_1(\tilde{S}^\circ) = \pi_1(\tilde{S}) = \{1\}$ since codim $\tilde{\Sigma} \geqq 2$. Therefore $\pi^\circ: \tilde{S}^\circ \to S^\circ$ is the universal covering of S° and Φ is holomorphic on $\tilde{S}^\circ$. Using the extension theorem of Hartogs, we extend Φ to a holomorphic function on $\tilde{S}$. So we consider Φ to be a multi-valued function on S. Consequently g is extended to a C^∞-function on S. Similarly as in (1.8), we calculate $\frac{\partial^2}{\partial \bar{t}_i \partial t_j} \log g(x) = -\sum_{\xi=1}^{q} \overline{\left(\frac{\partial \beta_\xi}{\partial t_i}\right)(x)} \frac{\partial \beta_\xi}{\partial t_j}(x)$. Hence the Hessian matrix of $\log g$ at x is negative semidefinite. Making such calculations for each $y \in S^\circ$, we infer that $\log g$ is pluri-subharmonic on S°. Since S is compact and codim $\Sigma \geqq 2$, $\log g$ is constant on S. Therefore $\frac{\partial^2}{\partial \bar{t}_i \partial t_j} \log g(y) = 0$ for any $y \in S$. This implies $\frac{\partial \beta_\xi}{\partial t_j}(x) = 0$ for any ξ and j. Hence $\frac{\partial \Phi}{\partial t_j}(x)$ is a scalar multiple of $\Phi(x)$. Making similar arguments, we infer that $\frac{\partial \Phi}{\partial t_j}(y)$ is a scalar multiple of $\Phi(y)$ for any $y \in S$. This implies that $\Phi(y)$ is a scalar multiple of $\Phi(x)$ for any $y \in S$. Hence $\beta_\xi \equiv \gamma_\eta \equiv 0$. Moreover, $|\alpha(t)|^2 = g$ is constant on S. So α itself is constant. Thus we obtain a non-trivial section of the local system $\bigcup_{t \in S^\circ} H^n(F_t)$ on S°. In view of the theorem (4.1.1) in [**1**] (a Kähler version of this result, which we use here, can be proved by the same method as in [**1**]), we infer that this section comes from a holomorphic n-form on M. So Lemma (4.4) applies.

Second we consider the case in which $k > 1$. Using (4.6), we get a finite unramified covering $\pi: \tilde{M} \to M$ such that $K_{\tilde{M}} = (f \circ \pi)^* K_S$. Let $\pi_1(\tilde{M}) \to \pi_1(M) \to \pi_1(S)$ be the natural mapping. The image of this map is of finite index in

$\pi_1(S)$, since $\pi_1(f)$ is surjective. Correspondingly we take a finite unramified covering $\tilde{S}\to S$. It is easy to see that a general fiber of the induced mapping $\tilde{f}: \tilde{M}\to\tilde{S}$ is connected. Now, since $K_{\tilde{M}}=\tilde{f}^*K_{\tilde{S}}$, the preceding argument proves that $\tilde{f}$ is smooth and locally trivial. Hence f is also smooth. So (4.5) proves the assertion.

(4.8) THEOREM. *Let $f: M\to S$ be a surjective morphism from a Kähler manifold M onto a projective manifold S. Let $M\overset{g}{\to} W\overset{\nu}{\to} S$ be the Stein factorization of f (EGA, Chap. III, (4.3.1)). Suppose that $k(K_M-f^*K_S)=0$ for some $k>0$. Then ν is unramified and g is smooth and locally trivial. Moreover, if $k=1$, there is a holomorphic n-form on M such that the restriction of it to no fiber vanishes.*

PROOF. First we consider the case in which $k=1$. We use the induction on $s=\dim S$. Suppose $s=1$. Note that $W=\mathcal{S}pec(f_*\mathcal{O}_M)$ is normal. Let R be the ramification divisor of ν. Then $g_*\omega_{M/W}=g_*\omega_M[-(K_S+R)]=g_*\mathcal{O}_M[-R]=\mathcal{O}_W[-R]$. This is semipositive. So $R=0$ and ν is unramified. Moreover, $g_*\omega_{M/W}=\mathcal{O}_W$. Hence (3.1) and (3.12) give a holomorphic n-form on M with non-vanishing restriction to each fiber. So (4.4) applies.

Now we consider the case in which $s\geqq 2$. Take a general hyperplane section H of S and let H_M and H_W be the pull-backs of H on M and W respectively. It is easy to see $K_{H_M}=f_H{}^*K_H$ and that $H_M\to H_W\to H$ is the Stein factorization of $f_H: H_M\to H$. We apply the induction hypothesis to infer that H_W is unramified over H. So W is unramified over S in codimension 1 since H is ample. This implies that ν is unramified because S is smooth ("purity of branch locus", [**6**], X, S. 8). Hence W is smooth. Let Σ be the singular locus of g. $H_W\cap\Sigma=\emptyset$ since $g_H: H_M\to H_W$ is smooth. Therefore $\operatorname{codim}\Sigma\geqq 2$ since H_W is ample. So (4.7) proves the assertion.

Second we consider the case in which $k\geqq 2$. Using (4.6), we take a finite unramified covering $\pi: \tilde{M}\to M$ such that $K_{\tilde{M}}=(f\circ\pi)^*K_S$. Let $\tilde{g}: \tilde{M}\to\tilde{W}=\mathcal{S}pec((f\circ\pi)_*\mathcal{O}_{\tilde{M}})$ be the natural morphism. We have a natural S-scheme morphism $\pi': \tilde{W}\to W$. The preceding argument proves that $\tilde{W}$ is unramified over S. Therefore π' is unramified and W is unramified over S. The preceding argument proves also that $\tilde{g}$ is smooth and locally trivial. Now we can apply (4.5) to prove that g is locally trivial.

(4.9) PROPOSITION. *Let $f: M\to V$ be a surjective morphism onto a projective variety V from a Kähler manifold M with $kK_M=0$ for $k>0$. Then $\kappa(V^*)\leqq 0$ for any smooth model $\mu: V^*\to V$. Moreover, if $\kappa(V^*)=0$, then $\nu: W=\mathcal{S}pec(f_*\mathcal{O}_M)\to V$ is unramified in codimension 1.*

PROOF. Take a finite unramified covering $\pi: \tilde{M}\to M$ with $K_{\tilde{M}}=0$ as in (4.6). We have a natural morphism $\tilde{W}=\mathcal{S}pec((f\circ\pi)_*\mathcal{O}_{\tilde{M}})\to W$. If $\tilde{W}\to V$ is unramified in codimension 1, then so is ν. Therefore, taking $f\circ\pi$ instead of f, we may assume $k=1$ and $\tilde{M}=M$. Clearly we can assume that V and W are normal. Let H be a very ample line bundle on V. Taking a general member of $|H|$ successively, we obtain a sequence $V=V_s\supset V_{s-1}\supset\cdots\supset V_1$ of subvarieties of V with $\dim V_j=j$ such that $V_j\in|H|_{V_{j+1}}$. Since V_j are chosen generally, we may assume that $M_j=f^{-1}(V_j)$ and $V_j^{\#}=\mu^{-1}(V_j)$ are smooth and that V_j and $W_j=\nu^{-1}(V_j)$ are normal. In particular, $V_1^{\#}\cong V_1$. So we have the following morphisms $M_1\to W_1\to V_1\to V_1^{\#}\subset V^{\#}$. Denoting by H also pull-backs of H by abuse of notation, we have $K_{M_1}=K_M+(s-1)H=(s-1)H$ and $K_{W_1}=K_{V_1}+R=K_{V^{\#}}+(s-1)H+R$, where R denotes the ramification divisor of $\nu_1: W_1\to V_1$. Hence $g_{1*}\omega_{M_1/W_1}=\mathcal{O}_{W_1}[-R-K_{V^{\#}}]$. Applying the semipositivity theorem to $g_1: M_1\to W_1$, we obtain $\deg_{W_1}K_{V^{\#}}\leqq-\deg R\leqq 0$. Therefore $H^{s-1}K_{V^{\#}}\{V^{\#}\}\leqq 0$. Since H is very ample on V, this implies that $\dim\mu(D)<s-1$ for any prime component D of a member of $|mK_{V^{\#}}|$ for any $m>0$. Hence $P_m(V^{\#})\leqq 1$ and $\kappa(V^{\#})\leqq 0$. If $\kappa(V^{\#})=0$, then $H^{s-1}K_{V^{\#}}\geqq 0$. Combining this inequality with $\deg(R+K_{V^{\#}})\leqq 0$, we infer that $\deg R=0$. Therefore $\nu_1: W_1\to V_1$ is unramified. Since H is ample, this proves the second assertion.

(4.10) LEMMA. *Let B be a subvariety of a complex torus T. Suppose that B is not contained in any proper subtorus of T. Then there exists a subtorus T_0 of T which satisfies the following conditions.*

a) *The quotient $T'=T/T_0$ is an abelian variety.*

b) *Let B' be the image of B in T'. Then* $\dim B'=\kappa(B')=\kappa(B)$.

c) *The natural mapping $B\to B'$ is a fiber bundle with fiber T_0.*

One finds a proof in Ueno [**16**], p. 120 Theorem 10.9 and p. 117, Theorem 10.3.

REMARK. For an effective divisor D on a complex torus T, we can find a subtorus T_0 of T and an ample divisor D' on $T'=T/T_0$ such that D is the pull-back of D' (see, for example, [**13**], p. 25).

(4.11) Now we can give a new proof of the following result of Matsushima ([**12**], p. 25, Compare also [**0**]).

THEOREM. *Let M be a Kähler manifold with $kK_M=0$ for some $k>0$. Then the Albanese mapping $a=a_M: M\to A(M)$ makes M a fiber bundle over $A(M)$.*

PROOF. Let $B=a(M)\subset A(M)=A$. We apply (4.10) to B. Let A_0 be the subtorus of A and let B' be the image of B in $A'=A/A_0$ as in (4.10). Since $B'\subset A'$ is projective, (4.9) implies $\kappa(B')\leqq 0$. So, from the condition b) of (4.10) follows that B' is a point. This is impossible unless $A_0=A$. Hence $B=A$, namely, a_M is surjective.

Let $W=\mathcal{S}pec_{an}(a_*\mathcal{O}_M)$ and let D be the branch locus of $\nu: W\to A$. If $D\neq 0$, then we find a subtorus A_0 of A and an ample divisor D' on $A'=A/A_0$ such that D is the pull-back of D'. Let $a': M\to A'$ be the induced natural mapping and let $W'=\mathcal{S}pec\,(a'_*\mathcal{O}_M)$. Then $W'\to A'$ is ramified along D'. This contradicts the second assertion of (4.9). Thus we infer that $D=0$. Since A is smooth, this implies that W is unramified over A. Consequently W is a complex torus. From the universality of the Albanese mapping we infer that ν is an isomorphism. Hence $a_*\mathcal{O}_M=\mathcal{O}_A$ and a general fiber of a_M is connected.

Let Σ be the singular locus of a_M. Suppose that Σ has a component Δ of codimension 1. Then, we find a subtorus A_0 of A and an ample divisor Δ' on $A'=A/A_0$ such that Δ is the pull-back of Δ'. Applying (4.8) to the induced morphism $a': M\to A'$, we infer that every fiber of a' is smooth. In particular, $a'^{-1}(x)$ is smooth for any $x\in\Delta'$. Therefore $a^{-1}(y)$ is smooth for a general point y on $q^{-1}(x)\cong A_0$, where q is the mapping $A\to A'$. This contradicts $y\in q^{-1}(\Delta')=\Delta\subset\Sigma$. Thus we infer that codim $\Sigma\geqq 2$. Now (4.7) proves the theorem.

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