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On Kenmotsu Manifolds Satisfying Certain Pseudosymmetry Conditions

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Abstract: We study on pseudosymmetric, Ricci-pseudosymmetric, Ricci-generalized pseudosymmetric and Weyl pseudosymmetric Kenmotsu manifolds.

Key words: Kenmotsu manifold . pseudosymmetric manifold . Ricci-generalized pseudosymmetric manifold . Weyl-pseudosymmetric manifold . Einstein manifold

INTRODUCTION

Let (M, g) be an n-dimensional, $n \geq 3$, differentiable manifold of class C^{∞}. We denote by ∇ its Levi-Civita connection. We define endomorphisms R(X, Y) and $X \wedge Y$ by:

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (1)$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M. The Riemannian Christoffel curvature tensor R is defined by R(X, Y, Z, W) = g(R(X, Y) Z, W), $W \in \chi(M)$. Let S and κ denote the Ricci tensor and the scalar curvature of M, respectively. The Weyl conformal curvature tensor is defined by:

$$C(X, Y)Z = R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} + \frac{\kappa}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}, \quad (2)$$

where, Q is the Ricci operator defined by g(QX, Y) = S(X,Y) [1].

We define tensors $R\cdot R$, $R\cdot S$, Q(g, R) and Q(g, S) by:

$$(R(X, Y)\cdot R)(X_1, X_2, X_3) = R(X, Y)R(X_1, X_2) X_3 - R(R(X, Y)X_1, X_2) X_3 - R(X_1, R(X, Y) X_2) X_3 - R(X_1, X_2) R(X, Y)X_3, \quad (3)$$

$$(R(X, Y) \cdot S)(X_1, X_2) = -S(R(X, Y) X_1, X_2) - S(X_1, R(X, Y)X_2), \quad (4)$$

$$Q(g,R)(X_1, X_2, X_3; X, Y) = (X\wedge Y)R(X_1, X_2) X_3 - R((X\wedge Y)X_1, X_2) X_3 - R(X_1, (X\wedge Y)X_2) X_3 - R(X_1, X_2)(X\wedge Y) X_3 \quad (5)$$

and

$$Q(g, S)(X_1, X_2; X, Y) = -S((X\wedge Y)X_1, X_2) - S(X_1, (X\wedge Y)X_2), \quad (6)$$

respectively, where $X_1, X_2, X_3, X, Y \in \chi(M)$. The tensors $R\cdot C$ and Q(g, C) are defined in the same manner as the tensors $R\cdot R$ and Q(g, R) [2].

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If the tensors R·R and Q(g, R) are linearly dependent then M is called *pseudosymmetric*. This is equivalent to:

$$R \cdot R = L_R\, Q(g, R) \tag{7}$$

holding on the set $U_R = \{x \in M\colon Q(g, R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R [2]. If R·R = 0 then M is called *semisymmetric*. Every semisymmetric manifold is pseudosymmetric but the converse statement is not true. If $\nabla R = 0$ then M is called *locally symmetric*. It is trivial that if M is locally symmetric then it is semisymmetric [3].

If the tensors R·S and Q(g, S) are linearly dependent then M is called *Ricci-pseudosymmetric*. This is equivalent to:

$$R \cdot S = L_S\, Q(g, S) \tag{8}$$

holding on the set $U_S = \{x \in M\colon S \neq \frac{\kappa}{n} g \text{ at } x\}$, where L_S is some function on U_S [2].

Every pseudosymmetric manifold is Ricci-pseudosymmetric but the converse statement is not true. If R·S = 0 then M is called *Ricci-semisymmetric*. Every semisymmetric manifold is Ricci-semisymmetric but the converse statement is not true. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse statement is not true [2].

If the tensors R·R and Q(S, R) are linearly dependent then M is called *Ricci-generalized pseudosymmetric* [2]. This is equivalent to

$$R \cdot R = L\, Q(S, R) \tag{9}$$

holding on the set $U = \{x \in M\colon Q(S,R) \neq 0 \text{ at } x\}$, where L is some function on U. The tensors Q(S, R) and $X \wedge_S Y$ are defined by:

$$\begin{aligned} Q(S, R)(X_1, X_2, X_3; X, Y) = {} & (X \wedge_S Y)R(X_1, X_2)X_3 - R((X \wedge_S Y)X_1, X_2)X_3 \\ & - R(X_1, (X \wedge_S Y)X_2)X_3 - R(X_1, X_2)(X \wedge_S Y)X_3 \end{aligned} \tag{10}$$

and

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y, \tag{11}$$

respectively.

If the tensors R·C and Q(g, C) are linearly dependent then M is called *Weyl-pseudosymmetric*. This is equivalent to

$$R \cdot C = L_C Q(g, C) \tag{12}$$

holding on the set $U_C = \{x \in M\colon C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. If R·C = 0 then M is called *Weyl-semisymmetric*. If M is Weyl-semisymmetric then it is trivially Weyl-pseudosymmetric. But the converse statement is not true [2].

Semisymmetric Kenmotsu manifolds were studied in [4], Ricci-semisymmetric and Weyl-semisymmetric Kenmotsu manifolds were studied in [5]. In this paper, we study on Kenmotsu manifolds satisfying several pseudosymmetry conditions. The paper is organized as follows: In Section 2, we give a brief account of Kenmotsu manifolds. In Section 3, we find the characterizations of Kenmotsu manifolds satisfying the pseudosymmetry conditions like $R \cdot R = L_R\, Q(g, R)$, $R \cdot R = LQ(S, R)$ and $R \cdot C = L_C Q(g, C)$.

KENMOTSU MANIFOLDS

Let M be an almost contact manifold [6] equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a (1, 1) tensor field φ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{13}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{14}$$

$$g(X,\varphi Y) = -g(\varphi X, Y), \qquad g(X,\xi) = \eta(X) \tag{15}$$

for all X, Y ∈ χ(M). An almost contact metric manifold M is called a *Kenmotsu manifold* if it satisfies [4]

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad X, Y \in \chi(M), \tag{16}$$

where, ∇ is Levi-Civita connection of the Riemannian metric g. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{17}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{18}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy [4]

$$S(X,\xi) = (1-n)\eta(X), \tag{19}$$

$$Q\xi = (1-n)\xi \tag{20}$$

and

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{21}$$

where n = 2m+1. Kenmotsu manifolds have been studied various authors. For example see [5, 7-12].

A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of φ equals -2dη⊗ξ) but not Sasakian. Moreover, it is also not compact since from equation (17) we get div ξ = n-1. In [4], Kenmotsu showed (i) that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant; and (ii) that a Kenmotsu manifold of constant φ-sectional curvature is a space of constant curvature -1 and so it is locally hyperbolic space.

RESULTS

In [4], K. Kenmotsu studied on semisymmetric Kenmotsu manifolds. In [5], it was considered Ricci-semisymmetric Kenmotsu manifolds. In this chapter, our aim is to find the characterizations of Kenmotsu manifolds which are pseudosymmetric, Ricci-generalized pseudosymmetric and Weyl-pseudosymmetric. Firstly we give:

Theorem 3.1: Let M be an n-dimensional, n ≥ 3, Kenmotsu manifold. If M is pseudosymmetric then either it is locally isometric to the hyperbolic space $H^n(-1)$ or $L_R = -1$ holds on M.

Proof: If M is semisymmetric then it is trivially pseudosymmetric. In [4], it was proved that a semisymmetric Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$. Now assume that M is not a semisymmetric, a pseudosymmetric Kenmotsu manifold. From (21) and (1), since

$$R(\xi, X)Y = (X \wedge \xi)Y, \tag{22}$$

it is easy to see that:

$$R(\xi, X)\cdot R = (X\wedge\xi)\cdot R,$$

which implies that the pseudosymmetry function $L_R = -1$. Thus the proof of our theorem is completed.
Therefore we have the following corollary:

Corollary 3.2: Every Kenmotsu manifold M^n, n ≥ 3, is a pseudosymmetric manifold of the form R·R = -Q(g, R).

Proof: If M is semisymmetric by the proof of the previous theorem $R \cdot R = Q(g, R) = 0$. If M is not semisymmetric then $L_R = -1$, hence $R \cdot R = -Q(g, R)$ holds on M. This proves the corollary.

An important subclass of Ricci-generalized pseudosymmetric manifolds is formed by the manifolds realizing the condition:

$$R \cdot R = Q(S, R). \tag{23}$$

It is known from [13] that every 3-dimensional semi-Riemannian manifold satisfies (23).
Now our aim is to investigate Kenmotsu manifolds realizing the condition (23). So we have the following result:

Theorem 3.3: Let M be an n-dimensional, $n \geq 3$, Kenmotsu manifold. Then the condition (23) holds on M if and only if M is locally isometric to the hyperbolic space $H^n(-1)$.

Proof: Assume that M is locally isometric to the Hyperbolic space $H^n(-1)$. Then it is easy to see that the condition $R \cdot R = Q(S, R) = 0$ is satisfied on M. Now let X, Y, Z, W be vector fields on M. Then from (3) we have:

$$(R(\xi, X) \cdot R)(Y, Z, W) = R(\xi, X)R(Y, Z)W - R(R(\xi, X)Y, Z)W - R(Y, R(\xi, X)Z)W - R(Y, Z)R(\xi, X)W. \tag{24}$$

Using (21) and taking the inner product of (24) with ξ we get:

$$g((R(\xi, X) \cdot R)(Y, Z, W), \xi) = -R(Y, Z, W, X) - g(X, Y)g(Z, W) + g(X, Z)g(Y, W). \tag{25}$$

Similarly by the use of (10) we can write:

$$Q(S,R)(Y, Z, W; \xi, X) = (\xi \wedge_S X)R(Y, Z)W - R((\xi \wedge_S X)Y, Z)W - R(Y, (\xi \wedge_S X)Z)W - R(Y, Z)(\xi \wedge_S X)W. \tag{26}$$

So using (11, 19, 21) and taking the inner product of (26) with ξ we get:

$$\begin{aligned} g(Q(S, R)(Y, Z, W; \xi, X), \xi) = {} & S(X, R(Y, Z)W) - S(X, Y)\eta(W)\eta(Z) + S(X, Y)g(Z, W) \\ & - S(X, Z)g(Y, W) + S(X, Z)\eta(W)\eta(Y) + (1-n)g(X,Y)\eta(W)\eta(Z) \\ & - (1-n)g(X, Z)\eta(W)\eta(Y). \end{aligned} \tag{27}$$

Since the condition $R \cdot R = Q(S, R)$ holds on M, from (24) and (27) we have:

$$(R(\xi, X) \cdot R)(Y, Z, W) = Q(S, R)(Y, Z, W; \xi, X). \tag{28}$$

Taking the inner product of the equation (28) with ξ we also have:

$$g((R(\xi, X) \cdot R)(Y, Z, W), \xi) = g(Q(S, R)(Y, Z, W; \xi, X), \xi).$$

Using the equations (25) and (27) we obtain:

$$\begin{aligned} & -R(Y, Z, W, X) + g(X, Z)g(Y,W) - g(X,Y)g(Z,W) \\ & = S(X, R(Y, Z)W) - S(X, Y)\eta(W)\eta(Z) + S(X,Y)g(Z,W) - S(X, Z)g(Y,W) \\ & + S(X, Z)\eta(W)\eta(Y) + (1-n)g(X, Y)\eta(W)\eta(Z) - (1-n)g(X, Z)\eta(W)\eta(Y). \end{aligned} \tag{29}$$

Putting $Y = \xi$ in (29) we have:

$$\eta(W)[S(X, Z) - (1-n)g(X, Z)] = 0. \tag{30}$$

So we obtain:

$$S(X, Z) = (1\text{-}n)\, g(X, Z). \tag{31}$$

Then M is an Einstein manifold with the scalar curvature $\kappa = n(1\text{-}n)$. Hence putting (31) into (23) we find:

$$R\cdot R = (1\text{-}n)Q(g, R). \tag{32}$$

But from Corollary 3.2, we know that 1-n = -1. Since $n \geq 3$, this is impossible. So we get $R\cdot R = 0$. Then by [4], M is locally isometric to the hyperbolic space $H^n(-1)$. This proves the theorem.

Theorem 3.4: Let M be an n-dimensional, $n \geq 3$, Ricci-generalized pseudosymmetric Kenmotsu manifold. If M is not semisymmetric then M is an Einstein manifold with scalar curvature $\kappa = n(1\text{-}n)$ and $L = \frac{1}{n-1}$ holds on M.

Proof: Suppose that M is a Ricci-generalized pseudosymmetric Kenmotsu manifold and X, Y, Z, $W \in \chi(M)$. Similar to the proof of the previous theorem, from (25) and (27) we can write:

$$\begin{aligned} &-R(Y, Z, W, X)+g(X, Z)g(Y, W)-g(X, Y)\, g(Z, W) \\ &= L\{S(X, R(Y, Z)W)-S(X, Y)\eta(W)\eta(Z)+S(X, Y)g(Z, W)-S(X, Z)g(Y,W) \\ &\quad + S(X, Z)\eta(W)\eta(Y)+(1\text{-}n)g(X, Y)\, \eta(W)\eta(Z)-(1\text{-}n)\, g(X, Z)\eta(W)\eta(Y)\}. \end{aligned} \tag{33}$$

Replacing Y with ξ in (33) we get:

$$\eta(W)L[S(X, Z)-(1\text{-}n)\, g(X, Z)] = 0. \tag{34}$$

Since M is not semisymmetric $L \neq 0$. So from (34):

$$S(X, Z) = (1\text{-}n)\, g(X, Z).$$

Then M is an Einstein manifold with the scalar curvature $\kappa = n(1\text{-}n)$. So putting $S = (1\text{-}n)g$ into (9) we obtain:

$$R\cdot R = (1\text{-}n)LQ(g, R).$$

But from Corollary 3.2, we know that $(1\text{-}n)L = -1$, which implies $L = \frac{1}{n-1}$. Hence we get the result, as required.

Corollary 3.5: Let M be an n-dimensional, $n \geq 4$, non-semisymmetric Ricci-generalized pseudosymmetric Kenmotsu manifold. Then $R\cdot R = R\cdot C$ holds on M.

Proof: Putting $S = (1\text{-}n)g$ and $\kappa = n(1\text{-}n)$ in (2) we get $C(X, Y)Z = R(X, Y)Z+\{g(Y, Z)X-g(X, Z)Y\}$ So using (3) we get the result.

Ricci pseudosymmetric Kenmotsu manifolds were studied in [14] and the following result was proved:

Theorem 3.6: [14]. Let M be an n-dimensional, $n \geq 3$, Kenmotsu manifold. If M is Ricci-pseudosymmetric then either M is an Einstein manifold with the scalar curvature $\kappa = n(1\text{-}n)$ or $L_S = -1$ holds on M.

So we have the following corollary:

Corollary 3.7: Every Kenmotsu manifold M, $n \geq 3$, is a Ricci-pseudosymmetric manifold of the form $R\cdot S = -Q(g, S)$.

Proof: If M is Ricci-semisymmetric by the proof of the previous theorem $R\cdot S = Q(g, S) = 0$. If M is not Ricci-semisymmetric, Ricci-pseudosymmetric then $R\cdot S = -Q(g, S)$ holds on M. Hence we get the result, as required.

Theorem 3.8: Let M be an n-dimensional, $n \geq 4$, Kenmotsu manifold. If M is Weyl-pseudosymmetric then either M is locally isometric to the hyperbolic space $H^n(-1)$ or $L_C = -1$ holds on M.

Proof: If M is Weyl-semisymmetric then by [5], it is conformally flat and hence it is locally isometric to the hyperbolic space $H^n(-1)$. Assume that M is not a Weyl-semisymmetric, a Weyl-pseudosymmetric Kenmotsu manifold. From (22), it is easy to see that $R(\xi, X)\cdot C = (X\wedge\xi)\cdot C$, which implies that the pseudosymmetry function L_C = -1. Therefore we have the following corollary:

Corollary 3.9: Every Kenmotsu manifold M^n, $n \geq 4$, is a Weyl-pseudosymmetric manifold of the form $R \cdot C = -Q(g, C)$.

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