On Kinetic Stability.

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1. The object of this paper is to illustrate the theory of kinetic stability, so far as such a theory can be said to exist, by a few simple examples. As the theory itself appears to be by no means widely known, some preliminary recapitulation seems advisable.

The difficulty of framing a definition of kinetic stability which shall be comprehensive and at the same time conform to natural prepossessions has long been recognised.* Thus, according to one definition which has been proposed, the vertical fall of a particle under gravity would be unstable; according to another the revolution of a particle in a circular orbit about a centre of force varying inversely as the cube of the distance would be reckoned as stable, although the slightest disturbance would cause the particle either to fall ultimately into the centre, or to recede to infinity, after describing in either case a spiral path with an infinite number of convolutions.

There are, however, certain restricted classes of cases where a natural definition of stability is possible and the corresponding criterion can be formulated. Suppose, in the first place, that we have a dynamical system which is the seat of cylic motions whose *momenta* (in the generalised sense) are constant.† Apart from the cyclic motions the configuration depends on a certain number of "palpable" co-ordinates $q_1, q_2, \ldots, q_n$, and an "equilibrium" configuration is one in which these co-ordinates can remain constant when the system is left to itself. Such an equilibrium configuration is said to be stable when the extreme variations of these co-ordinates, consequent on an arbitrary disturbance, are confined within limits which diminish indefinitely with the energy of the disturbance. Any arrangement of frictionless gyrostats gives a system of this kind; on a larger scale we have the problem of the free rotation of a liquid mass under its own gravitation.

In a second class of cases we have (again) certain co-ordinates whose values do not affect the kinetic or the potential energy, and the corre-

* *Cf.* F. Klein u. A. Sommerfeld, 'Ueber die Theorie des Kreisels,' Leipzig, 1898, ..., p. 342.

† *Cf.* Thomson and Tait, 'Natural Philosophy,' § 319, example (G); Lamb, 'Hydrodynamics,' 1906, §§ 140, 141.

sponding *velocities* are now supposed to be maintained constant by the application of suitable forces.* We have then to investigate the stability (in the same sense as before) of an "equilibrium" configuration in which the remaining co-ordinates $q_1, q_2, \ldots, q_n$ have constant values. As an example, the system may be attached to a rigid body which rotates with constant speed. The theory of the stability of an ocean covering a rotating globe also comes under this class.†

It has been customary, in treatises on dynamics, to discuss all such questions by the classical method of "small oscillations." If the variations of the co-ordinates $q_1, q_2, \ldots, q_n$ be regarded as infinitely small, the solution of the equations of disturbed motion is obtained in the form

$$\delta q_r = \Sigma C e^{\pm \lambda t}, \tag{1}$$

the values of λ^2 being determined by an algebraical or (in the case of an infinite number of degrees of freedom) a transcendental equation. If these values of λ^2 are found to be all real and negative, the undisturbed configuration is reckoned as stable, whilst if any of them are positive or complex, it is accounted as unstable. As familiar instances of problems discussed from this standpoint, we have the stability of the conical pendulum, of the steady precessional motion of a top, and so on. The general theory of the method, including the conditions of stability (in this sense), has been investigated by Routh.‡

M. Poincaré§ has, however, insisted on the fact that this method may, from a practical point of view, be altogether misleading as to the ultimate behaviour of the system. If deviations from the equilibrium configuration be resisted (as in practice they always are) by forces of a viscous character affecting the co-ordinates $q_1, q_2, \ldots, q_n$, then in the case of absolute (statical) equilibrium the usual criterion of stability, viz., that the potential energy must be a minimum, is not affected. But in such cases of kinetic equilibrium as have been referred to, it may happen that the effect of the viscous forces is gradually to *increase* the deviation, even although the equilibrium configuration is *primâ facie* (*i.e.*, from the "classical" standpoint) thoroughly stable. A distinction is accordingly drawn between "ordinary" or "temporary" stability, *i.e.*, stability as judged by the method of small oscillations, and "secular" or "permanent" stability, *i.e.*, stability when regard is had to possible viscous forces affecting the co-ordinates

* Thomson and Tait, § 319, example (F′).

† 'Hydrodynamics,' 1906, §§ 202, 203, 204.

‡ 'Stability of Motion,' 1877; 'Advanced Rigid Dynamics, 6th ed., 1905, chap. vi.

§ "Sur l'Equilibre d'une Masse Fluide animée d'un Mouvement de Rotation," 'Acta Math.,' 1885, vol. 7, p. 259.

$q_1, q_2, \ldots, q_n$. The question of permanent stability is, of course, the important one in physical and cosmical applications.

Fortunately, the criteria of permanent stability are much simpler than the elaborate criteria of temporary stability investigated by Routh. In the former class of problems (that of constant cyclic momenta), the condition is that a certain function $V+K$ should be a minimum, where V is the potential energy, and K denotes the kinetic energy of the cyclic motions alone. In the second class of cases (where certain velocities are maintained constant), the condition is that the "kinetic potential" $V-T_0$ should be a minimum; here T_0 denotes the kinetic energy of the system when at "rest" in any prescribed configuration $(q_1, q_2, \ldots, q_n)$.*

These principles were clearly laid down by Poincaré in 1885, and applied to the problem of rotating fluid; but it is doubtful whether they have received adequate recognition beyond the necessarily somewhat narrow circle of writers who have been concerned with the special question.† It is for this reason only that I venture to call attention to a few practical exemplifications of the theory. These relate to the second class of cases above referred to, and in particular to the question of stability of equilibrium relative to a rigid body which is maintained in constant rotation about a fixed axis.

2. The trivial character of the first example may be excused on the ground that it shows almost intuitively the necessity for some qualification to the doctrine of "ordinary" stability. We consider a particle movable on the inner surface of a spherical bowl which rotates with constant angular velocity (ω) about the vertical diameter. If the bowl be smooth the equilibrium of the particle when in the lowest position is "ordinarily" stable, since the rotation of the bowl is quite irrelevant. But if we admit the existence of friction, however slight, between the particle and the bowl, the lowest position is "permanently" stable only so long as $\omega < \sqrt{(g/a)}$, where a is the radius. This results immediately from the consideration of the formula for the kinetic potential,

$$V - T_0 = -Mga\cos\theta - \tfrac{1}{2}M\omega^2a^2\sin^2\theta, \tag{2}$$

where M is the mass of the particle, and θ is its angular distance from the lowest point. When the above value of ω is exceeded, the only permanently stable position is that in which

$$\cos\theta = \frac{g}{\omega^2 a}, \tag{3}$$

* See Poincaré, *loc. cit.*, or the author's 'Hydrodynamics,' *ll. cit.*

† The latest edition (1905) of Routh's 'Advanced Rigid Dynamics' contains no reference to the matter.

when the particle rotates with the bowl like the bob of a conical pendulum. To examine in detail the initial stage when the particle is slightly disturbed from its lowest position we may (for mathematical convenience) adopt the hypothesis of a frictional force varying as the relative velocity. If we employ horizontal rectangular axes Ox, Oy passing through the lowest point, and rotating with the bowl, we have, when x, y are small,

$$\left.\begin{aligned} \ddot{x}-2\omega \dot{y}-\omega^2 x &= -k\dot{x}-\frac{gx}{a}, \\ \ddot{y}+2\omega \dot{x}-\omega^2 y &= -k\dot{y}-\frac{gy}{a}, \end{aligned}\right\} \tag{4}$$

where k is the frictional coefficient. These equations may be combined into

$$\ddot{\zeta}+(2i\omega+k)\,\dot{\zeta}+\left(\frac{g}{a}-\omega^2\right)\zeta = 0, \tag{5}$$

where $\zeta = x+iy$. If we assume

$$\zeta = Ce^{\lambda t}, \tag{6}$$

we find $$\lambda = -i\omega \pm i\sqrt{\frac{g}{a}}-\tfrac{1}{2}k\left(1 \mp \omega\sqrt{\frac{a}{g}}\right), \tag{7}$$

if the square of k be neglected. If ξ, η be Cartesian co-ordinates referred to *fixed* axes through O, the complete solution is

$$\xi+i\eta = \zeta e^{i\omega t} = C_1 e^{\rho_1 t+i\sigma t}+C_2 e^{\rho_2 t-i\sigma t}, \tag{8}$$

where $$\sigma = \sqrt{\frac{g}{a}}, \quad \left.\begin{matrix}\rho_1 \\ \rho_2\end{matrix}\right\} = -\tfrac{1}{2}k\left(1 \mp \omega\sqrt{\frac{a}{g}}\right). \tag{9}$$

If this be put in real form we perceive that the motion is made up of two superposed circular vibrations, in opposite directions, of period $2\pi/\sigma$; moreover that, if $\omega^2 > g/a$, ρ_1 is positive, so that that circular vibration whose sense agrees with ω continually increases in amplitude. The particle works its way outwards in an ever widening spiral path, approximating to the stable position of relative equilibrium indicated by (3).

3. The next illustration is of a more practical character, and admits of being realised with considerable exactness. A pendulum symmetrical about a longitudinal axis hangs by a Hooke's joint from a vertical spindle which is made to rotate with a constant angular velocity ω. The pendulum used by the writer was constructed originally without any reference to the

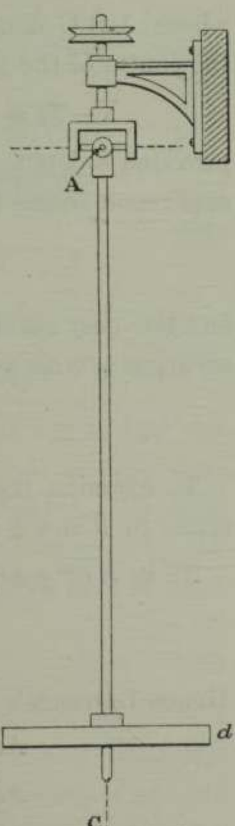

FIG. 1.

present question; no special pains were taken with the Hooke's joint, and the friction there was appreciable. Under these conditions, the instability of the vertical position when the rotation ω exceeds a certain critical value becomes rapidly apparent; the originally vertical axis of the pendulum describes an ever widening cone, tending towards the inclined position in which it can rotate as one body with the spindle. To examine the problem mathematically, let θ denote the inclination of one arm A of the joint, the other arm being, of course, horizontal, and let ϕ be the angle which a plane through the axis A and the axis of symmetry (C) makes with the vertical plane through A. The kinetic energy is then given by

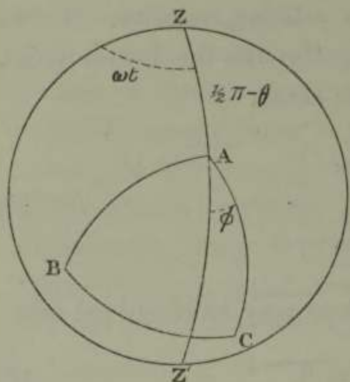

FIG. 2.

$$2T = A(\dot{\phi}+\omega\sin\theta)^2 + B(\dot{\theta}\cos\phi - \omega\cos\theta\sin\phi)^2 + C(\dot{\theta}\sin\phi + \omega\cos\theta\cos\phi)^2, \tag{10}$$

where A.B C denote the principal moments of inertia of the pendulum at the centre of the joint. Hence, if A = B, we have

$$V - T_0 = -Mgh\cos\theta\cos\phi - \tfrac{1}{2}\omega^2\{A-(A-C)\cos^2\theta\cos^2\phi\}, \tag{11}$$

provided h denote the distance of the centre of gravity from the joint. This expression ceases to be a minimum for $\theta = 0$, $\phi = 0$, if

$$\omega^2 > \frac{Mgh}{A-C}, \tag{12}$$

and the only stable positions are then those in which the pendulum makes an angle χ with the vertical, given by

$$\cos\chi = \cos\theta\cos\phi = \frac{Mgh}{(A-C)\omega^2}. \tag{13}$$

To examine the motion about the vertical position we neglect, in (10), terms in θ and ϕ of higher order than the second. Thus

$$2T = A(\dot{\theta}^2+\dot{\phi}^2) + 2A\omega(\theta\dot{\phi}-\dot{\theta}\phi) + 2C\omega\,\dot{\theta}\phi + (A-C)\,\omega^2(\theta^2+\phi^2) + \text{const.}, \tag{14}$$

$$2V = Mgh(\theta^2+\phi^2). \tag{15}$$

Hence Lagrange's equations give

$$\left.\begin{aligned} A\ddot{\theta} - (2A-C)\,\omega\dot{\phi} - \{(A-C)\,\omega^2 - Mgh\}\,\theta &= 0, \\ A\ddot{\phi} + (2A-C)\,\omega\dot{\theta} - \{(A-C)\,\omega^2 - Mgh\}\,\phi &= 0. \end{aligned}\right\} \tag{16}$$

As in the case of (4), we find that these are satisfied by

$$\theta + i\phi = Fe^{i\sigma t}, \tag{17}$$

provided
$$A\sigma^2+(2A-C)\,\omega\sigma+(A-C)\,\omega^2-Mgh = 0, \tag{18}$$
or
$$\sigma = -\left(1-\frac{C}{2A}\right)\omega \pm \frac{\sqrt{(C^2\omega^2+4AMgh)}}{2A}. \tag{19}$$

The vertical position is therefore "ordinarily" stable, whatever the value of ω.

It is evident that θ, ϕ are the rectangular co-ordinates, relative to rotating axes, of a point on the axis of the pendulum. For the corresponding co-ordinates relative to fixed axes we have

$$\xi+i\eta = (\theta+i\phi)\,e^{i\omega t} = Fe^{i(\sigma+\omega)t}, \tag{20}$$

where
$$\sigma+\omega = \frac{C\omega \pm \sqrt{(C^2\omega^2+4AMgh)}}{2A}. \tag{21}$$

The motion is therefore made up of two superposed circular vibrations of different periods $2\pi/(\sigma+\omega)$, the more rapid vibration being the one whose direction of revolution agrees with that of the spindle.

To investigate the question of permanent stability we introduce into the left-hand members of (16) terms $k\dot{\theta}$, $k\dot{\phi}$ to represent the viscous forces at the joint. The modified equations are satisfied by

$$\theta+i\phi = Fe^{\lambda t}, \tag{22}$$

provided
$$A\lambda^2+\{(2A-C)\,i\omega+k\}\,\lambda-\{(A-C)\,\omega^2-Mgh\} = 0. \tag{23}$$

If σ_1, σ_2 be the two values of σ given by (19), this may be written

$$A\,(\lambda-i\sigma_1)\,(\lambda-i\sigma_2)+k\lambda = 0, \tag{24}$$

the two roots of which are, if we neglect the square of k,

$$\lambda_1 = i\sigma_1-\frac{k\sigma_1}{A\,(\sigma_1-\sigma_2)}, \qquad \lambda_2 = i\sigma_2-\frac{k\sigma_2}{A\,(\sigma_2-\sigma_1)}. \tag{25}$$

When $\omega^2 < Mgh/(A-C)$ the two values of σ have opposite signs, and the real parts of λ_1, λ_2 are both negative. The vertical position is then permanently, as well as "ordinarily" stable. But if $\omega^2 > Mgh/(A-C)$ both values of σ are negative, and if σ_1 be the smaller in absolute magnitude, the real part of λ_1 will be positive, and that of λ_2 negative. If we pass to fixed axes, writing as before

$$\xi+i\eta = (\theta+i\phi)\,e^{i\omega t} = Fe^{(\lambda+i\omega)t}, \tag{26}$$

we find that the periods of the two circular vibrations are to a first approximation unaffected by a small degree of friction, but that the amplitude of one of these vibrations, viz., the one whose direction of revolution agrees with that of the spindle, increases exponentially with the time, whilst the amplitude of the other sinks asymptotically to zero. These points are illustrated in a striking manner by the apparatus referred to.* Substantially the same experiment can be made in a simpler form by means of a heavy metal ball hanging by a stout string from a hook at the lower end of the spindle. If due precautions be taken to check the violent evolutions which the ball is sometimes apt in the first instance to perform, the torsion of the string soon brings the latter into a state of steady rotation about a vertical diameter, with practically the angular velocity of the spindle. When the steady state has been attained the ball may be left to itself, with the string vertical. The friction of motion relative to the spindle is in this form of the experiment very slight, and although a close observation may soon detect the tendency to a circular vibration of continually increasing amplitude in the direction of revolution of the spindle, some time may elapse before this becomes really conspicuous. The final result is, however, unmistakable.†

4. The question is not seriously modified by a slight amount of deviation from the theoretical conditions, *e.g.*, in the problem of § 3, by a slight defect of alignment between the axis of rotation of the spindle and the centre of the joint. The configuration of relative equilibrium about which the observed oscillations take place is only slightly altered, except in the case of approximate agreement between the imposed period of rotation and what would be the natural period of vibration in the absence of rotation.

The effects of a want of perfect alignment in § 3 can be studied in their simplest form if we neglect the moment of inertia (C) about the axis of the pendulum. The case is then that of a particle suspended from the lower surface of a horizontal disc, which is made to rotate about a vertical axis. If l be the length of the string, and a the distance of the point of suspension from the axis of rotation, the inclination α of the string to the vertical in a position of relative equilibrium is given by

$$\frac{g}{\omega^2 l} = \cos\alpha + \sin^3\beta \cot\alpha, \tag{27}$$

where $\sin^3\beta = a/l$. If

$$\frac{g}{\omega^2 l} < \cos^3\beta, \tag{28}$$

this has *three* solutions, for two of which $\sin\alpha$ is negative; in one of these, moreover, $\sin\alpha$ is numerically greater, and in the other numerically less,

* It may be worth while to give roughly the dimensions. The steel rod shown in fig. 1 had a length of 36 in. and a thickness of $\frac{1}{2}$ in. The diameter of the iron disc d which could be fixed in various positions along the rod was 7 in. and its thickness $\frac{1}{2}$ in. The spindle was driven from a small electromotor, by means of the small pulley shown, at speeds ranging up to about 25 revolutions per second.

† In a typical experiment the ball was 3 in. in diameter, and was suspended by a string 33 in. long; and the speed was about 7 revolutions per second. The circular vibration took about 18 minutes to attain an amplitude of 1 inch.

than $\sin\beta$. These three positions are shown in fig. 3. The position is found to be both "ordinarily" and "permanently" stable, whilst the position III is on either reckoning unstable. Case II is "permanently" unstable, but the question of "ordinary" stability is less simple. For sufficiently great values of ω the equilibrium may become unstable from this point of view, but there is no difficulty in adjusting the conditions so that

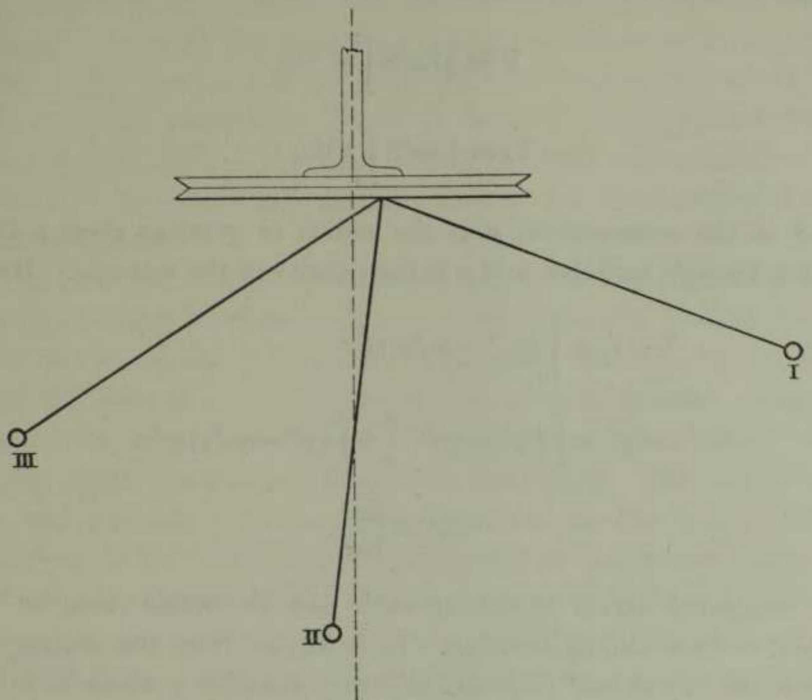

Fig. 3.

there may be "ordinary" stability with "permanent" instability. This was illustrated by an experiment in which the excentricity (a/l) was purposely made appreciable. The metal ball referred to was suspended by a stout string about 3 feet long from a point 1 inch out from the centre of the rotating disc. If the ball be carefully steadied in the central position before being left to itself, its subsequent demeanour differs in no essential way from what is observed when the suspension is made as nearly axial as possible.

5. The next example is one in which the number of degrees of freedom is infinite. For a reason to be given it is hardly a practical one, but it may serve to illustrate the limitations to which the application of the theory is subject. We consider a cylindrical shaft rotating in fixed bearings placed at isolated points, and the question is at what speed the straight form becomes unstable. If the circumferential velocity of the shaft be

small compared with the elastic wave-velocities of the material, the angular momentum about the axis may be ignored. Under this condition it is obvious that the straight form is "ordinarily" stable, the fact of the rotation being irrelevant. To investigate the "permanent" stability, consider, for definiteness, a length l between two bearings A, B. If the axis of x be taken along the length of the shaft, and if y denote the lateral deviation, we have, by the usual theory of flexure,

$$V = \tfrac{1}{2}E\kappa^2 S \int_0^l y''^2 dx, \tag{29}$$

$$T_0 = \tfrac{1}{2}\rho\omega^2 S \int_0^l y^2 dx, \tag{30}$$

where S is the cross-section, κ is the radius of gyration about a diameter, E denotes Young's modulus, and ρ is the density of the material. Hence,

$$\begin{aligned} V - T_0 &\propto \int_0^l (y''^2 - m_0^4 y^2)\, dx \\ &\propto \Big[y'y'' - yy''' \Big]_0^l + \int_0^l (y^{\text{iv}} - m_0^4 y)\, y dx, \end{aligned} \tag{31}$$

where
$$m_0^4 = \frac{\rho\omega^2}{E\kappa^2}. \tag{32}$$

The integrated terms vanish if each end be either free, or merely supported, or fixed also in direction. It is known from the ordinary theory of transversal vibrations* that any arbitrary function y which is subject to the given terminal conditions can be expanded, for $0 < x < l$, in a series of normal functions,

$$y = C_1 u_1 + C_2 u_2 + \dots. \tag{33}$$

Here $u_1, u_2, \dots$ satisfy the differential equations

$$u_1^{\text{iv}} = m_1^4 u_1, \quad u_2^{\text{iv}} = m_2^4 u_2, \dots, \tag{34}$$

and the proper terminal conditions, $m_1, m_2, \dots$, being the roots of a certain transcendental equation,† arranged in ascending order of magnitude. If we substitute from (33) in (31), and omit terms which vanish in consequence of the orthogonal property of different conjugate functions, we find

$$V - T_0 \propto (m_1^4 - m_0^4)\, C_1^2 \int_0^l u_1^2 dx + (m_2^4 - m_0^4)\, C_2^2 \int_0^l u_2^2 dx + \dots. \tag{35}$$

* See Rayleigh, 'Theory of Sound,' chap. viii.

† Thus, if the shaft be merely supported at the ends, the equation is $\sin ml = 0$; if it be fixed in direction at one end and free at the other, we have $\cos ml \cosh ml + 1 = 0$.

The frequencies $(\sigma/2\pi)$ of the various modes of natural vibration of the shaft are determined by the relation

$$\sigma_r^2 = \frac{E\kappa^2}{\rho} \cdot m_r^4. \tag{36}$$

Hence $V - T_0$ is a minimum, in the straight condition, or the equilibrium is permanently stable, only so long as $m_0^4 < m_1^4$, *i.e.*, so long as the period of rotation is greater than that of the gravest mode of transverse vibration. The incipient stages of the instability might be studied as in the previous problems. The motion can be analysed into circular vibrations, and it appears that the amplitude of one at least of these, having the same direction of revolution as the shaft, should increase exponentially with the time, provided m_0 exceed the smallest root of the transcendental equation which determines m.

We conclude that a truly symmetrical shaft, rotating accurately about its axis, in rigidly fixed bearings, with any speed exceeding that of the gravest mode of transverse vibration, would be rendered unstable by viscous forces affecting the *relative* motion, such as are, in fact, present owing to the internal friction of the substance. The instability might, indeed, take time to develop itself, but the result would be inevitable. The fact that shafts can be, and are, safely driven at speeds exceeding the critical limit thus indicated* must be ascribed to the operation of dissipative forces (so far ignored) affecting the absolute as well as the relative vibrations. The seat of such forces is probably to be found in a yielding of the bearings. For a similar reason the "permanent" instability illustrated by the experiments of §§ 2, 3 above might be wholly masked if the resistance of the air were very much greater than it actually is, or if the whole apparatus were immersed in a viscous liquid.

* The observed "whirling" of shafts at a series of critical speeds is due to a want of absolute symmetry, and is to be regarded as a forced oscillation of exaggerated amplitude, due to approximate synchronism. (See Dunkerley, 'Phil. Trans.,' A, vol. 185, 1894; Stodola, 'Die Dampfturbinen,' Berlin, 1904, p. 157.)