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ON KLEIN-MASKIT COMBINATION THEOREM IN SPACE II

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Abstract

In this paper, which is sequel to [10], we give a generalisation of the second Klein-Maskit combination theorem, the one dealing with HNN extensions, to higher dimension. We give some examples constructed as an application of the main theorem.

1. Introduction

The combination theorems for classical Kleinian groups, i.e. for those in $PSL_2\mathbf{C}$, are ways to generate new Kleinian groups as amalgamated free products or HNN extensions of given Kleinian groups. The first of such theorems was given by Klein [9] in the case of free products. Maskit in [12, 13, 14, 15, 16, 17] gave several generalisations of Klein's combination theorem, among which are the first combination theorem dealing with amalgamated free products and the second combination theorem dealing with HNN extensions.

In our previous paper [10], we considered a generalisation of the first combination theorem of Maskit to higher dimension. In the present paper, which is its sequel, we shall generalise his second combination theorem. Maskit's second combination theorem asserts that under some conditions, two Kleinian groups G_0, G_1, where $G_1 = \langle f \rangle$ is infinite cyclic and G_0 has two isomorphic geometrically finite subgroups J_1 and J_2 conjugated by f, generate a Kleinian group isomorphic to the HNN extension of G_0 by f, and also that under the same conditions the resulting group is geometrically finite if and only if G_0 is geometrically finite.

As in the case of amalgamated free products, a first attempt to generalise Maskit's second combination theorem to higher dimension was likewise made

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by Apanasov in his pioneering work [3, 4]. Ivascu [8] also dealt with such a generalisation with a bit different approach. In particular, they showed that under the same assumptions as Maskit combined with some extra conditions, one can get a discrete group which is an HNN extension of a discrete group of n-dimensional Möbius transformations and a Möbius transformation with infinite order. To be more precise, what they proved is the following.

THEOREM 1.1 [Apanasov, Ivascu]. *Let G_0 be an n-dimensional Kleinian group with subgroups H_1 and H_2. Suppose that closed domains $\overline{D}_1$ and $\overline{D}_2$ bounded by hypersurfaces S_1 and S_2 of n-sphere S^n are precisely invariant in G_0 with respect to H_1 and H_2, respectively. Suppose also that an n-dimensional Möbius transformation f maps D_1 onto $S^n \backslash D_2$, and that the following conditions hold*:

(1) *For fundamental domains Δ_1, Δ_2 and $F_0 \subset \Delta_1 \cap \Delta_2$ of the groups H_1, H_2 and G_0, there exist neighbourhoods V_1 and V_2 of the surfaces S_1 and S_2 such that $\Delta_i \cap V_i \subset F_0$, $i = 1, 2$.*
(2) $\Delta_i \cap \overline{D}_i = \overline{D}_i \cap F_0$.
(3) $F = (F_0 \cap (S^n \backslash (\overline{D}_1 \cup \overline{D}_2)))^\circ \neq \emptyset$, *where B° is the interior of B in S^n.*
(4) $fH_1f^{-1} = H_2$.
(5) $g(\overline{D}_1) \cap \overline{D}_2 = \emptyset$ *for all* $g \in G_0 \backslash \{id\}$.

Then we have the following.

(1) *The group $G = \langle G_0, f \rangle$ is a Kleinian group and isomorphic to $G_0 *_f$, the HNN extension of G_0 by f.*
(2) *F is a fundamental domain for the group G.*
(3) $m_n(\Lambda(G)) = 0$ *if and only if* $m_n(\Lambda(G_0)) = 0$.
(4) *Each elliptic or parabolic element of G is conjugate in G to an element of G_0.*

In this paper, we shall give a generalisation of the second Maskit combination theorem in higher dimension without any additional assumptions, imposing only natural ones corresponding to those in Maskit's. Our theorem also asserts that the group obtained as an HNN extension is geometrically finite if and only if the original group is, under some conditions. We note that in the present paper as in the previous one, we say that a Kleinian group is geometrically finite when the ε-neighbourhood of its convex core has finite volume for some $\varepsilon > 0$, and there is an upper bound for the orders of torsions in the group. We do not assume that the group has a finite-sided fundamental polyhedron. Our main result (Theorem 3.1) and its proof will appear in §3.

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2. Preliminaries

2.1. Basic notions. We follow the notations used in [10]. We use the symbol $\overline{\mathbf{R}}^n$ to denote the one-point compactification of the n-dimensional

Euclidean space $\mathbf{R}^n$ and $M(\overline{\mathbf{R}}^n)$ to denote the group of orientation-preserving Möbius transformations. We identify $\overline{\mathbf{R}}^n$ with the sphere at infinity of the hyperbolic $n+1$-space. We shall use both the ball model and the upper half-space model for the hyperbolic space. We use the symbol $\mathbf{B}^{n+1}$ for the ball model and $\mathbf{H}^{n+1}$ for the upper half-space model.

We denote the *limit set* of a discrete group $G \subset M(\overline{\mathbf{R}}^n)$ by $\Lambda(G)$. We call points of $\Lambda(G)$ *limit points*. The complement $\Omega(G) = \overline{\mathbf{R}}^n \backslash \Lambda(G)$ is called the *region of discontinuity* of G.

A discrete group $G \subset M(\overline{\mathbf{R}}^n)$ is said to act discontinuously at a point $x \in \overline{\mathbf{R}}^n$ if there is a neighbourhood U of x such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is a finite set. The group G acts discontinuously at every point of $\Omega(G)$, and at no point of $\Lambda(G)$.

The complement of the fixed points of elliptic elements in $\Omega(G)$ is called the *free regular set*, and is denoted by $^\circ\Omega(G)$. When $^\circ\Omega(G) \neq \emptyset$, a *fundamental set* of G is defined to be a set which contains one representative of each orbit $G(y)$ of $y \in {}^\circ\Omega(G)$. It is obvious that $^\circ\Omega(G) \neq \emptyset$ if and only if $\Omega(G) \neq \emptyset$. If $\Omega(G) \neq \emptyset$, then we call G a *Kleinian group*.

For the limit set $\Lambda(G)$, we have the following useful lemma ([10]).

Lemma 2.1. *Let $\{g_m\}$ be a sequence of distinct elements of the Kleinian group $G \subset M(\overline{\mathbf{R}}^n)$. Then there are a subsequence $\{g_{m_i}\}$ and limit points x and y of G such that $g_{m_i}(z) \to x$ uniformly on every compact subset of $\overline{\mathbf{R}}^{n+1} \backslash \{y\}$.*

We shall use the following terms in the same way as in [10].

Definition 2.1. Let H be a subgroup of a group G of $M(\overline{\mathbf{R}}^n)$. A subset V is said to be *precisely invariant* under H in G if $h(V) = V$ for all $h \in H$ and $g(V) \cap V = \emptyset$ for all $g \in G - H$.

Definition 2.2. Let $T_1, \dots, T_m$ be sets and $J_1, \dots, J_m$ be subgroups of the group $G \subset M(\overline{\mathbf{R}}^n)$. We say that $(T_1, \dots, T_m)$ is *precisely invariant* under $(J_1, \dots, J_m)$ in G, if each T_k is precisely invariant under J_k in G, and if for $i \neq j$, and for all $g \in G$, $g(T_i) \cap T_j = \emptyset$.

For the domain of discontinuity $\Omega(G)$, we have the following proposition. Refer to Proposition II.E.4 in Maskit [15] or Theorem 5.3.12 in Beardon [5].

Proposition 2.2. *Suppose that $\Omega(G)$ is not empty. Then a point $x \in \overline{\mathbf{R}}^n$ is contained in $\Omega(G)$ if and only if*

(1) *the stabiliser* $\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}$ *of x in G is finite, and*
(2) *there is a neighbourhood U of x in $\overline{\mathbf{R}}^n$ which is precisely invariant under* $\mathrm{Stab}_G(x)$ *in G.*

Definition 2.3. A fundamental domain for a discrete group G of $M(\overline{\mathbf{R}}^n)$ with non-empty region of discontinuity is an open subset D of $\Omega(G)$ satisfying the following.

(1) D is precisely invariant under the trivial subgroup in G.
(2) For every $z \in \Omega(G)$, there is an element $g \in G$ such that $g(z)$ is contained in $\overline{D}$, where $\overline{D}$ denotes the closure of D in $\overline{\mathbf{R}}^n$.
(3) Fr D, the frontier of D in $\overline{\mathbf{R}}^n$, consists of limit points of G, and a finite or countable collection of codimension-1 compact smooth submanifolds with boundary, whose boundary is contained in $\Omega(G)$ except for a subset with $(n-1)$-dimensional Lebesgue measure 0. The intersection of each submanifold with $\Omega(G)$ is called a side of D.
(4) For any side σ of D, there are another side σ' of D, which may coincide with σ, and a nontrivial element $g \in G$ such that $g(\sigma) = \sigma'$. Such an element g is called the side-pairing transformation from σ to σ'.
(5) If $\{\sigma_m\}$ is a sequence of distinct sides of D, then the diameter of σ_m with respect to the ordinary spherical metric on $\overline{\mathbf{R}}^n$ goes to 0.
(6) For any compact subset K of $\Omega(G)$, there are only finitely many translates of D that intersect K.

A fundamental set F for a discrete subgroup G whose interior is a fundamental domain is called *a constrained fundamental set*.

2.2. Normal forms. Let G_0 be a discrete subgroup of $M(\overline{\mathbf{R}}^n)$ with isomorphic subgroups J_1 and J_2, and f a transformation in $M(\overline{\mathbf{R}}^n)$ of infinite order satisfying $fJ_1f^{-1} = J_2$ and $G_0 \cap \langle f \rangle = \{id\}$. Following Maskit [15], we define normal forms as follows.

A *normal form* is a word of the form

$$f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1,$$

such that
(1) each g_k is contained in G_0,
(2) g_k is not the identity except possibly for the last one g_1,
(3) the exponents α_k are assumed to be non-zero except for the first one α_n,
(4) if $\alpha_k < 0$ and $g_{k+1} \in J_1 - \{id\}$, then $\alpha_{k+1} < 0$, and
(5) if $\alpha_k > 0$ and $g_{k+1} \in J_2 - \{id\}$, then $\alpha_{k+1} > 0$.

The length of a normal form $g = f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1$ is defined to be $|g| = \sum |\alpha_k|$. Two normal forms are defined to be *equivalent* if we can transform one to the other by repeating the following operations finitely many times: inserting a word of the form $fjf^{-1}(fjf^{-1})^{-1}$ for some $j \in J_1$ and deleting a word of the same form. The set of the equivalence classes of normal forms with concatenation as binary operation corresponds one-to-one to the HNN extension of G_0 by f, which we denote by $G_0 *_f$ preserving the group structures.

We call a normal form $g = f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1$ *positive* if $\alpha_n > 0$, *negative* if $\alpha_n < 0$, and *null* if $\alpha_n = 0$. More specifically, we call g a (j,k)-form, with j either $+$ or $-$, or 0 when g is positive or negative or null, respectively, and $k = +$ if $\alpha_1 > 0$, $k = -$ if $\alpha_1 < 0$.

Let $\langle G_0, f \rangle$ be the subgroup of $M(\overline{\mathbf{R}}^n)$ generated by G_0 and $\langle f \rangle$. Then, there is a natural homomorphism $\Phi : G_0 *_f \to \langle G_0, f \rangle$ which is defined by

$\Phi(f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1) = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1} \circ g_1$ for a normal form $f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1$ representing an element of $G_0 *_f$, and $\Phi(j) = j$ for $j \in G_0$. It is easy to see that this is independent of a choice of a representative of the equivalence class. The map is obviously an epimorphism. If Φ is an isomorphism, then we write $\langle G_0, f \rangle$ also as $G_0 *_f$ identifying elements of $G_0 *_f$ and their images by Φ.

Since G_0 is embedded in $\langle G_0, f \rangle$, each non-trivial element in the kernel of Φ can be written in a normal form. Therefore the following is obvious.

Lemma 2.3. *$\langle G_0, f \rangle = G_0 *_f$ if and only if Φ maps no non-trivial normal forms to the identity.*

2.3. Interactive triples. Following Maskit, we shall define interactive triples as follows.

We assume that G_0 is a discrete subgroup of $M(\overline{\mathbf{R}}^n)$ with isomorphic subgroups J_1 and J_2, and $f \in M(\overline{\mathbf{R}}^n)$ has infinite order, where $fJ_1 f^{-1} = J_2$ and $G_0 \cap \langle f \rangle = \{id\}$. Let Z, X_1, X_2 be disjoint nonempty subsets of $\overline{\mathbf{R}}^n$. The triple (Z, X_1, X_2) is said to be an *interactive triple* (for G_0, f, J_1 and J_2) when the following hold.

(1) (X_1, X_2) is precisely invariant under (J_1, J_2) in G_0.

(2) For every $g \in G_0$ and $m = 1, 2$, we have $g(X_m) \subset Z \cup X_m$.

(3) We have $f(Z \cup X_2) \subset X_2$ and $f^{-1}(Z \cup X_1) \subset X_1$.

If there exists a non-empty G_0-invariant subset of $Z \backslash G_0(X_1 \cup X_2)$, then the interactive triple is said to be *proper*. We can easily see that if (Z, X_1, X_2) is an interactive triple and $g \in G_0 - J_1$, then $g(X_1) \subset Z$, and also if $g \in G_0 - J_2$, then $g(X_2) \subset Z$.

Example 2.1. For $n \geq 2$, let $e_0, e_1, \ldots, e_{n-1}$ be the standard basis of $\mathbf{R}^n$, where $e_0 = (1, 0, \ldots, 0)$. Set $X_1 = \{x = \sum_{i=1}^n x_i e_{i-1} \in \mathbf{R}^n \mid x_n < 0\}$, $X_2 = \{x \in \mathbf{R}^n \mid x_n > 0\}$, and $Z = \{\mathbf{0}\}$. We define $G_0 = J_1 = J_2$ to be $\langle j_1, j_2, \ldots, j_{n-1} \rangle$, where $j_i(x) = x + e_{i-1}$ $(i = 1, 2, \ldots, n-1)$. Let $f(x) = x + e_{n-1}$. It is obvious that (Z, X_1, X_2) is an interactive triple for G_0, f, J_1 and J_2. Since $Z \backslash G_0(X_1 \cup X_2) = Z$ does not have a G_0-invariant subset however, (Z, X_1, X_2) is not proper.

If we change Z above to $Z' = \{x = \sum_{i=1}^n x_i e_{i-1} \in \mathbf{R}^n \mid x_n = 0\}$ preserving X_1 and X_2 to be the same as above, then (Z', X_1, X_2) is also an interactive triple for G_0, f, J_1 and J_2, and (Z', X_1, X_2) is proper since $Z' \backslash G_0(X_1 \cup X_2) = Z'$ itself is G_0-invariant.

The following lemma due to Maskit holds also in higher dimension without any change.

Lemma 2.4 (Lemma VII.D.11 in [15]). *Suppose that (Z, X_1, X_2) is an interactive triple for G_0, f, J_1 and J_2, and that A_0 is a non-empty G_0-invariant subset of Z, which has trivial intersection with $G_0(X_1 \cup X_2)$. Let $g = f^{\alpha_n} g_n \cdots f^{\alpha_1} g_1$ be a non-trivial normal form in $G_0 *_f$.*

(1) *If g is a $(+,+)$-form, then* $\Phi(g)(A_0 \cup X_2) \subset X_2$.
(2) *If g is a $(+,-)$-form, then* $\Phi(g)(A_0 \cup X_1) \subset X_2$.
(3) *If g is a $(-,+)$-form, then* $\Phi(g)(A_0 \cup X_2) \subset X_1$.
(4) *If g is a $(-,-)$-form, then* $\Phi(g)(A_0 \cup X_1) \subset X_1$.
(5) *If g is a $(0,+)$-form, then there is an element $h \in G_0$ such that* $\Phi(g)(A_0 \cup X_2) \subset h(B) \subset Z$, *where* $B = X_1$ *if* $\alpha_{n-1} < 0$, *and* $B = X_2$ *if* $\alpha_{n-1} > 0$.
(6) *If g is a $(0,-)$-form, then there is an element $h \in G_0$ such that* $\Phi(g)(A_0 \cup X_1) \subset h(B) \subset Z$, *where* $B = X_1$ *if* $\alpha_{n-1} < 0$, *and* $B = X_2$ *if* $\alpha_{n-1} > 0$.

The existence of a proper interactive triple forces Φ to be isomorphic. (Theorem VII.D.12 in Maskit [15] in the case when $n = 2$. The proof is the same in higher dimension using Lemmata 2.3 and 2.4.)

THEOREM 2.5. *Let G_0, f, J_1 and J_2 be as above and suppose that there is a proper interactive triple for G_0, f, J_1 and J_2. Then* $\langle G_0, f\rangle = G_0 *_f$.

Using Theorem 2.5, we get the following straightforward generalisation of Theorem VII.D.13 in [15].

THEOREM 2.6. *Let G_0 be a discrete group. Suppose that (Z, X_1, X_2) is an interactive triple for G_0, f, J_1 and J_2 and that $A_0 \subset Z \backslash G_0(X_1 \cup X_2)$ is a non-empty G_0-invariant set. Then A_0 is precisely invariant under G_0 in* $\langle G_0, f\rangle = G_0 *_f$.

Let D_0 be a fundamental set for G_0 satisfying $J(D_0 \cap X_m) = X_m \cap {}^{\circ}\Omega(J_m)$ for $m = 1, 2$, and set $D = D_0 \cap A_0$. If D is non-empty, then D is precisely invariant under $\{id\}$ in $\langle G_0, f\rangle$.

2.4. Geometric finiteness. As in the previous paper [10], we use the following definition of geometric finiteness, not assuming the existence of finite-sided fundamental polyhedron.

DEFINITION 2.4. Let G be a discrete subgroup of $M(\overline{\mathbf{R}}^n)$. We denote by $\mathrm{Hull}(\Lambda(G))$, the minimal convex set of $\mathbf{H}^{n+1}$ containing all geodesics whose endpoints lie on $\Lambda(G)$. This set is evidently G-invariant, and its quotient $\mathrm{Hull}(G)/G$ is called the *convex core* of G, and is denoted by $\mathrm{Core}(G)$. The group G is said to be *geometrically finite* if the following two conditions are satisfied:

(1) there exists $\varepsilon > 0$ such that the ε-neighbourhood of $\mathrm{Core}(G)$ in $\mathbf{H}^{n+1}/G$ has finite volume, and
(2) there is an upper bound for the orders of torsions in G.

A point x of $\Lambda(G)$ of a discrete group G of Möbius transformations is called *a parabolic fixed point* if $\mathrm{Stab}_G(x)$ contains parabolic elements. For a parabolic fixed point z, a horoball in $\mathbf{B}^{n+1}$ touching $\overline{\mathbf{R}}^n$ at z is invariant under $\mathrm{Stab}_G(z)$. In the case when $\mathrm{Stab}_G(z)$ has rank less than n, it is useful to consider a domain larger than a horoball, which we call an extended horoball.

Definition 2.5. Let G be a discrete subgroup of $M(\overline{\mathbf{R}}^n)$, and z a parabolic fixed point of G. Let $\mathrm{Stab}^*_G(z)$ be the maximal free abelian subgroup of the stabiliser $\mathrm{Stab}_G(z)$ of z in G. Suppose that the rank k of $\mathrm{Stab}^*_G(z)$ is less than n. Then there is a closed subset $B_z \subset \mathbf{B}^{n+1}$ invariant under $\mathrm{Stab}_G(z)$ which is in the form

$$B_z = h^{-1}\left\{ x \in \mathbf{B}^{n+1} \;\middle|\; \sum_{i=k+1}^{n+1} x_i^2 \geq t \right\},$$

where t (> 0) is a constant and $h \in M(\overline{\mathbf{R}}^n)$ is a Möbius transformation such that $h(z) = \infty$. We call B_z an *extended horoball* of G around z.

Related to this, there is a set called a peak domain, which was introduced by Apanasov.

Definition 2.6. A *peak domain* of a discrete group G of $M(\overline{\mathbf{R}}^n)$ at the parabolic fixed point z of G is an open subset $U_z \subset \overline{\mathbf{R}}^n$ such that

(1) U_z is precisely invariant under $\mathrm{Stab}_G(z)$ in G, and
(2) there exist a $t > 0$, and a transformation $h \in M(\overline{\mathbf{R}}^n)$ with $h(z) = \infty$ such that

$$\left\{ x \in \mathbf{R}^n \;\middle|\; \sum_{i=k+1}^{n} x_i^2 > t \right\} = h(U_z),$$

where $k = \operatorname{rank} \mathrm{Stab}^*_G(z)$, $1 \leq k \leq n-1$.

Definition 2.7. Let z be a parabolic fixed point of the discrete group $G \subset M(\overline{\mathbf{R}}^n)$. If G has an extended horoball B around z, then the interior of its intersection with $\overline{\mathbf{R}}^n$ is a peak domain. Following Bowditch [6], we use the term *standard parabolic region* at z to mean an extended horoball when the rank of $\mathrm{Stab}_G(z)$ is less than n, and a horoball when the rank of $\mathrm{Stab}_G(z)$ is n.

We shall present definitions of terms which are commonly used in studying geometrically finite groups in $M(\overline{\mathbf{R}}^n)$.

Definition 2.8. A point $z \in \overline{\mathbf{R}}^n$ fixed by a parabolic element of a discrete group $G \subset M(\overline{\mathbf{R}}^n)$ is said to be a *parabolic vertex* of G if one of the following conditions is satisfied.

(1) The subgroup $\mathrm{Stab}^*_G(z)$ has rank n.
(2) There exists a peak domain U_z at the point z.

Definition 2.9. Let G be a discrete subgroup of $M(\overline{\mathbf{R}}^n)$. A point $x \in \overline{\mathbf{R}}^n$ is said to be a conical limit point (or a point of approximation in some literature) if there are $z \in \mathbf{H}^{n+1}$ and a geodesic ray l in $\mathbf{H}^{n+1}$ tending to x whose r-neighbourhood with some $r \in \mathbf{R}$ contains infinitely many translates of z.

As was shown in Theorem 12.2.5 in Ratcliffe [18], we have a characterisation of conical limit points as follows.

Proposition 2.7. *Let G be a discrete group of $M(\overline{\mathbf{R}}^n)$ regarded as acting on $\mathbf{B}^{n+1}$ by hyperbolic isometries. Then a point $z \in \partial\mathbf{B}^{n+1}$ is a conical limit point of G if and only if there exist $\delta > 0$, distinct elements g_m of G, and $x \in \partial\mathbf{B}^{n+1}\backslash\{z\}$ such that $g_m^{-1}(\mathbf{0})$ converges to z while $|g_m(x) - g_m(z)| > \delta$ for all m. Furthermore, if this condition holds, then for every $x \in \partial\mathbf{B}^{n+1}\backslash\{z\}$, there is $\delta > 0$ such that $|g_m(x) - g_m(z)| > \delta$ for all m.*

The following result due to Bowditch [6] or [7] will be essentially used in the proof of our main theorem.

Proposition 2.8. *Let $G \subset M(\overline{\mathbf{R}}^n)$ $(n \geq 2)$ be a discrete group. Then G is geometrically finite if and only if every point of $\Lambda(G)$ is either a parabolic vertex or a conical limit point.*

2.5. Dirichlet domains. Among fundamental domains of hyperbolic manifolds, what are called Dirichlet domains are most useful for us.

Definition 2.10. Let G be a discrete subgroup of $M(\overline{\mathbf{R}}^n)$, and x a point in $\mathbf{H}^{n+1}$, which is not fixed by any nontrivial element of G. Then, the set $\{y \in \mathbf{H}^{n+1} \mid d_h(y, x) \leq d_h(y, g(x))\ \forall g \in G\}$ is called the Dirichlet domain centred at x for G, where d_h denotes the hyperbolic distance.

We shall make use of the following result of Bowditch [6]. For a G-invariant set S on $\overline{\mathbf{R}}^n$, we say a collection of subsets $\{A_s\}_{s \in S}$ is *strongly invariant* if $gA_s = A_{gs}$ and for any $s \neq t \in S$, $A_s \cap A_t = \emptyset$. We should note that each A_s is in particular precisely invariant under $\mathrm{Stab}_G(s)$ in G.

Lemma 2.9. *Let Π be the set of all parabolic vertices of a discrete group $G \subset M(\overline{\mathbf{R}}^n)$. Then we can choose a standard parabolic region B_p for each $p \in \Pi$ in such a way that $\{B_p \mid p \in \Pi\}$ is strongly invariant.*

2.6. Blocks. Throughout this subsection, we assume that G is a discrete subgroup of $M(\overline{\mathbf{R}}^n)$, and J denotes a subgroup of G.

Definition 2.11. A closed J-invariant set B in $\overline{\mathbf{R}}^n$, containing at lease two points, is called a block, or more specifically (J, G)-block if it satisfies the following conditions.

(1) $B \cap \Omega(G) = B \cap \Omega(J)$, and $B \cap \Omega(G)$ is precisely invariant under J in G.
(2) If U is a peak domain for a parabolic fixed point z of J with the rank of $\mathrm{Stab}_J(z)$ being less than n, then there is a smaller peak domain $U' \subset U$ such that $U' \cap \mathrm{Fr}\, B = \emptyset$.

Let S be a topological $(n-1)$-dimensional sphere in $\overline{\mathbf{R}}^n$. Then S separates $\overline{\mathbf{R}}^n$ into two open sets. We say that S is *precisely embedded* in G if $g(S)$ is disjoint from one of the two open sets for any $g \in G$.

A (J, G)-block is said to be *strong* if every parabolic fixed point of J is a parabolic vertex of G.

We have the following in [10].

THEOREM 2.10. *Let J be a geometrically finite subgroup of G and $B \subset \overline{\mathbf{R}}^n$ be a (J, G)-block such that for every parabolic fixed point z of J with the rank of* $\mathrm{Stab}_J(z)$ *being less than n, there is a peak domain U_z for J with $U_z \cap B = \emptyset$. Let $G = \bigcup g_k J$ be a coset decomposition. If $\{g_k(B)\}$ is a sequence of distinct translates of B, then we have* $\mathrm{diam}(g_k(B)) \to 0$, *where* $\mathrm{diam}(M)$ *denotes the diameter of the set M with respect to the ordinary spherical metric on $\overline{\mathbf{R}}^n$.*

3. The second Klein-Maskit combination theorem

In this section, we shall show our main theorem (Theorem 3.1).

DEFINITION 3.1. Let J_1 and J_2 be subgroups of a discrete group $G_0 \subset M(\overline{\mathbf{R}}^n)$, and let $f \in M(\overline{\mathbf{R}}^n)$ be an element of infinite order. Following Maskit, we say that two closed topological n-dimensional balls B_1 and B_2 in $\overline{\mathbf{R}}^n$, are *jointly f-blocked* if the following conditions are satisfied.
(1) B_m is a (J_m, G_0)-block for $m = 1, 2$,
(2) $(B_1 \cap \Omega(G_0), B_2 \cap \Omega(G_0))$ is precisely invariant under (J_1, J_2) in G_0,
(3) f maps the exterior of B_1 in $\overline{\mathbf{R}}^n$ onto the interior of B_2 in $\overline{\mathbf{R}}^n$, and
(4) $fJ_1f^{-1} = J_2$.
If B_1 and B_2 are jointly f-blocked, then following Maskit, we say that a fundamental set D_0 for G_0 is *maximal* if $D_0 \cap B_m$ is a fundamental set for the action of J_m on B_m and $f(D_0 \cap \mathrm{Fr}\, B_1) = D_0 \cap \mathrm{Fr}\, B_2$ for $m = 1, 2$.

DEFINITION 3.2. Let $\{S_j\}$ be a collection of topological $(n-1)$-spheres. We say that the sequence $\{S_j\}$ *nests about* a point x if the following are satisfied.
(1) The spheres S_j are pairwise disjoint.
(2) Each sphere S_j separates x from the precedent S_{j-1}.
(3) For any point $z_j \in S_j$, the sequence $\{z_j\}$ converges to x.

Now we state our main theorem.

THEOREM 3.1. *Let $G_0 \subset M(\overline{\mathbf{R}}^n)$ be a discrete group with geometrically finite subgroups J_1 and J_2, and $f \in M(\overline{\mathbf{R}}^n)$ an element of infinite order with*

$G_0 \cap \langle f \rangle = \{id\}$. Let B_1 and B_2 be closed topological balls in $\overline{\mathbf{R}}^n$. Suppose that B_1 and B_2 are jointly f-blocked and that $A_0 = \overline{\mathbf{R}}^n \backslash G_0(B_1 \cup B_2)$ is non-empty. Let D_0 be a maximal fundamental set for G_0. Set $A = \overline{\mathbf{R}}^n \backslash (B_1 \cup B_2)$, $G = \langle G_0, f \rangle$ and $D = D_0 \cap (A \cup \mathrm{Fr}\, B_1)$. Then the following hold.

(1) $G = G_0 *_f$.

(2) *G is discrete.*

(3) $\mathrm{Fr}\, B_m$ $(m = 1, 2)$ *is a precisely embedded (J_m, G)-block.*

(4) *If an element g of G is not loxodromic, then one of the following holds.*
   (a) *g is conjugate to an element of G_0.*
   (b) *g is parabolic and is conjugate to an element fixing a parabolic fixed point of either J_1 or J_2.*

(5) *If $\{W_k'\}$ is a sequence of distinct G-translates of* $\mathrm{Fr}\, B_m$, *then* $\mathrm{diam}(W_k') \to 0$ *as $k \to \infty$.*

(6) *There is a sequence of distinct translates of* $\mathrm{Fr}\, B_m$ *nesting about the point x if and only if $x \in \Lambda(G) \backslash G(\Lambda(G_0))$.*

(7) *D is a fundamental set for G. If D_0 is constrained,* $(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2) \cap \mathrm{Fr}\, D_0$ *consists of finitely many connected components, and the sum of their $(n-1)$-dimensional measures vanishes, then D is also constrained.*

(8) *Let $Q = (A_0 \cup G_0(\mathrm{Fr}\, B_1)) \cap \Omega(G_0)$. Then $\Omega(G)/G = Q/G_0$, and its boundary, which is possibly disconnected or empty, is equal to $(\mathrm{Fr}\, B_1 \cap \Omega(G_0))/J_1 = (\mathrm{Fr}\, B_2 \cap \Omega(G_0))/J_2$.*

Furthermore, under the assumption that each $\mathrm{Fr}\, B_m$ *is a strong (J_m, G)-block for $m = 1, 2$ if and only if each B_m is a strong (J_m, G_0)-block, two more statements hold.*

(9) *If each B_m is a strong (J_m, G_0)-block, then, except for G-translates of limit points of G_0, every limit point of G is a conical limit point of G.*

(10) *G is geometrically finite if and only if G_0 is geometrically finite.*

Let us explain what this theorem claims intuitively. We are given two geometrically finite subgroups J_1, J_2 of G_0 and a Möbius transformation f conjugating J_1 to J_2, none of whose non-zero powers is contained in G_0. The two topological balls B_1 and B_2 are invariant sets under J_1 and J_2 with some good conditions respectively, and f translates $\mathrm{Fr}\, B_1$ to $\mathrm{Fr}\, B_2$ inside out. In this situation, the theorem says that the group generated by G_0 and f is discrete and isomorphic to the HNN-extension of G_0 by f. The group G may contain a parabolic element which is not contained in G_0, but then it is conjugate to a parabolic element whose fixed point coincides with the fixed point of a parabolic element of J_1 (or J_2). Moreover, with further assumptions on parabolic fixed points, the theorem claims that the group G is also geometrically finite.

The following lemma constitutes the key step for the proof of our main theorem.

Lemma 3.2. *Let $m = 1, 2$. Under the assumptions of Theorem* 3.1, *the following naturally follow.*

(1) $\mathrm{Fr}\, B_m$ *is a* (J_m, G_0)*-block.*
(2) $\Lambda(G_0) \cap \mathrm{Fr}\, B_m = \Lambda(J_m) \cap \mathrm{Fr}\, B_m = \Lambda(J_m)$.
(3) $B_1^\circ \cup B_2^\circ \subset \Omega(G_0)$, *where* B_m° *is the interior of* B_m *in* $\overline{\mathbf{R}}^n$ *for each* m.
(4) B_m° *is precisely invariant under* J_m *in* G_0.
(5) *For any* $g \in G_0$, *we have* $g(B_m) \cap B_{3-m} = g(\mathrm{Fr}\, B_m) \cap \mathrm{Fr}\, B_{3-m} \subset \Lambda(G_0)$.
(6) *For any* $g \in G_0 - J_m$, *we have* $g(B_m) \cap B_m = g(\mathrm{Fr}\, B_m) \cap \mathrm{Fr}\, B_m \subset \Lambda(J_m)$.
(7) *Let* $G_0 = \bigcup_k g_{k,m} J_m$ *be a coset decomposition. If* $\{g_{k,m}(B_m)\}$ *is a sequence of distinct translates of* B_m, *then* $\mathrm{diam}(g_{k,m}(B_m)) \to 0$ *as* $k \to \infty$.
(8) $(A, B_1^\circ, B_2^\circ)$ *is an interactive triple, and* A_0 *is precisely invariant under* G_0 *in* G.
(9) $f(\mathrm{Fr}\, B_1 \cap \Omega(J_1)) = \mathrm{Fr}\, B_2 \cap \Omega(J_2)$.
(10) *We have* $D_0 \cap A = D_0 \cap A_0$.

Proof. We only need to prove (7) and (8).

(7) By (1), we know that $\mathrm{Fr}\, B_m$ is a (J_m, G_0)-block. Since J_m is geometrically finite, then by Theorem 2.10, we have $\mathrm{diam}(g_{k,m}(\mathrm{Fr}\, B_m)) \to 0$. The assumption that B_m is a (J_m, G_0)-block implies that $\mathrm{diam}(g_{k,m}(B_m)) = \mathrm{diam}(g_{k,m}(\mathrm{Fr}\, B_m)) \to 0$, which shows (7).

(8) By (4), B_m° is precisely invariant under J_m in G_0. If $g \in J_m$, then $g(B_m^\circ) = B_m^\circ$. If $g \in G_0 - J_m$, then $g(B_m^\circ) \cap B_m = \emptyset$ and $g(B_m^\circ) \cap B_{3-m} = \emptyset$. Therefore for any $g \in G_0$, we have $g(B_m^\circ) \subset B_m^\circ \cup A$.

Since f maps the exterior of B_1 onto the interior of B_2, we have $f(A \cup B_2^\circ) \subset f(\overline{\mathbf{R}}^n \backslash B_1) = B_2^\circ$ and $f(B_1^\circ) = \overline{\mathbf{R}}^n \backslash B_2$. Hence $f^{-1}(A \cup B_1^\circ) \subset f^{-1}(\overline{\mathbf{R}}^n \backslash B_2) = B_1^\circ$. Thus we have shown that $(A, B_1^\circ, B_2^\circ)$ is an interactive triple.

It is easy to see that $A_0 = \overline{\mathbf{R}}^n \backslash G_0(B_1 \cup B_2) \subset A \backslash G_0(B_1^\circ \cup B_2^\circ)$. Therefore, A_0 is G_0-invariant. By Theorem 2.6, A_0 is precisely invariant under G_0 in G. □

Now we prove Theorem 3.1. Since the proofs of (1)–(8) of Theorem 3.1 are similar to those in [10, 15], we give proofs only for (9) and (10).

Proof of (9). Since we are assuming each B_m is a strong (J_m, G_0)-block for $m = 1, 2$, by our assumption mentioned above, $\mathrm{Fr}\, B_1$ is a strong (J_1, G)-block, and $\mathrm{Fr}\, B_2$ is a strong (J_2, G)-block. Let x be a limit point of G, which is not a translate of a limit point of G_0. By (6), there is a sequence $\{g_k(\mathrm{Fr}\, B_1)\}$ of distinct G-translates of $\mathrm{Fr}\, B_1$ with $|g_k| \to \infty$ such that $\{g_k(\mathrm{Fr}\, B_1)\}$ nests about x. We can assume that $g_1 = id$. Then $g_k^{-1}(x)$ and $g_k^{-1}(\mathrm{Fr}\, B_1)$ lie on opposite sides of $\mathrm{Fr}\, B_1$.

Since J_1 is geometrically finite, by Proposition 2.16 in [10], which is originally due to Bowditch, there are a Dirichlet domain P for J_1 and standard parabolic regions $B_{p_1}, \ldots, B_{p_k}$ such that $\bar{P} \backslash \bigcup_j (\mathrm{Int}\, B_{p_j} \cup \{p_j\})$ is compact and contains no limit point of J_1. Since P is a Dirichlet domain, the interior of $S = \bar{P} \cap \overline{\mathbf{R}}^n$ is a fundamental domain for J_1. Since $g_k^{-1}(x)$ is contained in $\Omega(J_1)$

for each k, there is an element $q_k \in J_1$ such that $q_k g_k^{-1}(x) \in S$. We denote $q_k g_k^{-1}$ by f_k.

We claim that $\{f_k(x)\}$ stays away from $\operatorname{Fr} B_1$. Suppose, seeking for a contradiction, that $f_k(x) \to w \in \operatorname{Fr} B_1 \cap \Lambda(G) = \operatorname{Fr} B_1 \cap \Lambda(J_1)$ passing to a subsequence. Then w is a parabolic fixed point of J_1, where the rank of $\operatorname{Stab}_{J_1}(w)$ is less than n, since S intersects $\Lambda(J_1)$ only at the p_j. This means that all $f_k(x)$ lie in some extended horoball B_{p_j} if we take a subsequence, where $p_j = w$. Let the rank of $\operatorname{Stab}_{J_1}(w)$ be s and the rank of $\operatorname{Stab}_G(w)$ be m.

If $s = m$, then we can assume that the interior of $B_w \cap \overline{\mathbf{R}}^n$, which is denoted by U_w, is also a peak domain for G. Hence we may assume that $\overline{U}_w \backslash \{w\}$ is contained in $\Omega(G)$. On the other hand, since x lies in $\Lambda(G)$, we have $f_k(x) \in \Lambda(G)$, which is a contradiction.

Therefore, there is $\delta > 0$ such that $d(f_k(x), z) > \delta$ for any $z \in \operatorname{Fr} B_1$, where d denotes the ordinary spherical metric on $\overline{\mathbf{R}}^n$. Since $\operatorname{Fr} B_1$ separates $g_k^{-1}(\operatorname{Fr} B_1)$ from $g_k^{-1}(x)$, we see that for all z on $\operatorname{Fr} B_1$ we have $\delta < d(f_k(x), z) < d(f_k(x), f_k(z))$. On the other hand, since $g_k(\operatorname{Fr} B_1)$ nest around x, we see that for any point y on $\operatorname{Fr} B_1$, the points $f_k^{-1}(y)$ converge to x. We now apply Proposition 2.7 to conclude that x is a conical limit point of G.

If $s < m$, by conjugation and Bieberbach's theorem (also refer to Theorem 2.10 in [10]), we may assume that $w = \infty$,

$$\operatorname{Stab}_G^*(w) = \langle j_1, \dots, j_m \rangle \quad \text{and} \quad \operatorname{Stab}_{J_1}^*(w) = \langle h_1, \dots, h_s \rangle,$$

where $j_i(y) = A_i(y) + e_{i-1}$ $(i = 1, \dots, m)$, $h_j(y) = U_j(y) + e_{j-1}$ $(j = 1, \dots, s)$, $y \in \mathbf{R}^n$, A_i and U_j are rotations, and A_i and U_j act on $\mathbf{R}^m$ trivially. It follows from $\{f_k(x)\} \subset B_w$ that $\sum_{i=1}^{s} |f_k(x)|_i^2$ are bounded away from ∞ for all k. Since $\operatorname{Fr} B_1$ is a strong (J_1, G)-block, there is $t > 0$ such that

$$U = \left\{ z \in \mathbf{R}^n : \sum_{i=m+1}^{n} |z_i|^2 > t \right\}$$

is a peak domain for G and $\overline{U} \backslash \{\infty\} \subset \Omega(G)$. We know that $\{f_k(x)\} \subset \Lambda(G)$. Hence $\sum_{i=m+1}^{n} |f_k(x)|_i^2 < t$. It follows from $f_k(x) \to \infty$ as $k \to \infty$ that

$$\sum_{i=s+1}^{m} |f_k(x)|_i^2 \to \infty.$$

For each $i = s+1, \dots, m$, if $|f_k(x)|_i^2 \to \infty$ $(k \to \infty)$, then we choose a sequence $\{i_k\}$ of integers such that for all k, $|j_i^{i_k} f_k(x)|_i^2 < M_1$, where $M_1 > 0$; if $|f_k(x)|_i^2 < M_2$ for some $M_2 > 0$, we let $i_k = 0$. Let $l_k = j_m^{m_k} \cdots j_{s+1}^{(s+1)_k}$. It follows that $|l_k(f_k(x))|^2 < M_3 (M_3 > 0)$, and for any $y \in \operatorname{Fr} B_1$

$$|l_k(y)|^2 = |j_{s+1}^{(s+1)_k}(y)|_{s+1}^2 + \cdots + |j_m^{m_k}(y)|_m^2 \to \infty.$$

Therefore, there is $\delta > 0$ such that $d(l_k f_k(x), l_k(z)) > \delta$ for all $z \in \operatorname{Fr} B_1$. Since $\operatorname{Fr} B_1$ separates $g_k^{-1}(x)$ from $g_k^{-1}(\operatorname{Fr} B_1)$ and hence $\operatorname{Fr} B_1$ separates $f_k(x)$

from $f_k(\mathrm{Fr}\, B_1)$, we see that for all z on $\mathrm{Fr}\, B_1$ we have $\delta < d(l_k f_k(x), l_k(z)) \leq d(l_k f_k(x), l_k f_k(z))$. By Lemma 2.1 and choosing a subsequence, we know that $l_k f_k(z) \to z'$ for all $z \in \overline{\mathbf{R}}^{n+1} \backslash \{x\}$ and $l_k f_k(x) \to x'$, where $z' \neq x'$. We now conclude that x is a conical limit point of G. □

Proof of (10). We first assume that G_0 is geometrically finite. Then B_1 and B_2 are both strong blocks of G_0, and hence each $\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block by our assumption.

Take any point $x \in \Lambda(G)$. Suppose first that x is a parabolic fixed point, where the rank of $H = \mathrm{Stab}_G(x)$ is $\kappa < n$. By (9), x is a translate of a limit point of G_0. Without loss of generality, we may assume that x lies on $\Lambda(G_0)$. Since G_0 is geometrically finite, x is a parabolic vertex or a conical limit point of G_0. If x is a conical limit point for G_0, then so is it for G. Since a parabolic fixed point cannot be a conical limit point, x is a parabolic vertex for G_0. If x lies in $G_0(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2)$, then, since each $\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block, x is a parabolic vertex of G. On the other hand, since $G_0(B_1^\circ \cup B_2^\circ) \subset \Omega(G_0)$, if x does not lie on any G_0-translate of either $\mathrm{Fr}\, B_1$ or $\mathrm{Fr}\, B_2$, then x is contained in A_0. Since A_0 is precisely invariant under G_0, we see that H is contained in G_0. Therefore we have $H = \mathrm{Stab}_{G_0}(x)$. There is a peak domain U centred at x for G_0. Since $U \cap \Lambda(G_0)$ is empty, by choosing U to be sufficiently small, we can assume that $\overline{U} \backslash \{x\} \subset \Omega(G_0)$. By conjugation, we may assume that $x = \infty$. By Bieberbach's theorem, we may further assume that for any $g \in H$, $g(z) = Az + \mathbf{a}$, where $\mathbf{a} \in \mathbf{R}^\kappa$ and A preserves the subspaces $\mathbf{R}^\kappa$ and $\mathbf{R}^{n-\kappa}$, respectively. Then U is in the form

$$U = \left\{ x \in \mathbf{R}^n : \sum_{i=\kappa+1}^{n} x_i^2 > t \right\},$$

for $t > 0$.

Claim 1. *We can choose U small enough so that $U \subset A_0$.*

Proof. Since B_1 and B_2 are bounded and for any $g \in H$, $\sum_{i=\kappa+1}^{n} |g(x)|_i^2 = \sum_{i=\kappa+1}^{n} |x|_i^2$, by taking sufficiently large t, we can make $g(B_1 \cup B_2) \cap U = \emptyset$ for any $g \in H$. Hence no H-translates of B_1 or B_2 intersect U if we choose U to be small enough.

Suppose that there is a sequence $\{g_k(B)\}$ of distinct G_0-translates of B_1 or B_2 such that the projections of $g_k(B)$ to the subspace $\mathbf{R}^{n-\kappa}$ converge to ∞ for $B = B_1$ or $B = B_2$. Without loss of generality, we may assume that $B = B_1$. Then taking a subsequence, we may assume that $g_k \in G_0 - (H \cup J_1)$ since J_1 fixes B_1. Lemma 3.2-(7) implies that $g_k(y) \to \infty$ for all $y \in B_1$. Since $g_k(U) \cap U = \emptyset$, the projections of $g_k(U)$ to the subspace $\mathbf{R}^{n-\kappa}$ are bounded. By Bieberbach's theorem, for each g_k, we can choose an element $j_k \in H$ so that all the $j_k \circ g_k(y_0)$ lie in a bounded set for a fixed $y_0 \in U$. Since the projections of $g_k(B_1)$ to the subspace $\mathbf{R}^{n-\kappa}$ converge to ∞, $\infty \notin g_k(B_1)$ and $\sum_{i=\kappa+1}^{n} |j_k(x)|_i^2 = \sum_{i=\kappa+1}^{n} |x|_i^2$, we may assume that all the $j_k \circ g_k(B_1)$ are distinct and that the projections of $j_k \circ g_k(B_1)$ to the subspace $\mathbf{R}^{n-\kappa}$ converge to ∞ by taking a subsequence. Lemma 3.2-(7) again implies that $j_k \circ g_k(y) \to \infty$ for all $y \in B_1$. By Lemma 2.1, we may assume that $j_k \circ g_k(y) \to \infty$ for all y except for a limit point of G_0. This leads to a contradiction since $y_0 \in \Omega(G_0)$ and $j_k \circ g_k(y_0) \not\to \infty$. $\square$

Claim 1 implies that $U \subset A_0$ is precisely invariant under H in G, which means that x is a parabolic vertex for G.

Next assume that x is a limit point of G, which is not a parabolic fixed point. If x is a translate of a limit point of G_0, then x is a conical limit point for G_0, and hence for G. If x is not a translate of a limit point of G_0, then x is a conical limit point for G by (9). This completes the proof of the "if" part.

To prove the "only if" part, we assume that G is geometrically finite. Then each $\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block, and hence each B_m is a strong (J_m, G_0)-block by our assumption. Let x be a point in $\Lambda(G_0)$. Since $G_0(B_1^\circ \cup B_2^\circ) \subset \Omega(G_0)$, we have either $x \in G_0(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2)$ or $x \in A_0$.

If $x \in G_0(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2)$, then for simplicity, we may assume that $x \in \mathrm{Fr}\, B_1$. So we have $x \in \mathrm{Fr}\, B_1 \cap \Lambda(J_1) = \mathrm{Fr}\, B_1 \cap \Lambda(G_0)$. Since J_1 is a geometrically finite subgroup of G_0, we see that x is either a conical limit point for J_1 or a parabolic fixed point for J_1. In the former case, x is a conical limit point for G_0. In the latter case, since B_1 is a strong (J_1, G_0)-block, x is a parabolic vertex for G_0.

Now let x be a point in A_0. If x is a parabolic fixed point of G, then since A_0 is precisely invariant under G_0 in G, $\mathrm{Stab}_G(x) = \mathrm{Stab}_{G_0}(x)$, which shows that x is a parabolic fixed point of G_0. We assume that the rank of $\mathrm{Stab}_G(x)$ is $\kappa < n$. Since G is geometrically finite, there is a peak domain U centred at x for G, which is also a peak domain for G_0. Therefore, x is a parabolic vertex for G_0. Suppose that x is not a parabolic fixed point of G, which means that it is a conical limit point for G. In this case, there is a sequence $\{h_k\}$ of distinct elements of G with $d(h_k(z), h_k(x))$ is bounded away from zero for all $z \in \overline{\mathbf{R}}^n \setminus \{x\}$ and $h_k^{-1}(z_0) \to x$ for some $z_0 \in \mathbf{H}^{n+1}$ by Proposition 2.7. Then there are points $x' \neq z' \in \overline{\mathbf{R}}^n$ such that $h_k(z) \to z'$ for any $z \in \overline{\mathbf{R}}^n \setminus \{x\}$ and $h_k(x) \to x'$ by passing to a subsequence if necessary.

Claim 2. *By taking a subsequence, we can assume that all the* $h_k(\mathrm{Fr}\, B_m)$ *are distinct for* $m = 1, 2$.

Proof. If this is not the case, by taking a subsequence, we can assume that all the $h_k(\mathrm{Fr}\, B_m)$ are the same for all k. Then $h_1^{-1} \circ h_k(\mathrm{Fr}\, B_m) = \mathrm{Fr}\, B_m$. Hence, for each k, there is an element $j_k \in J_m$ such that $h_k = h_1 \circ j_k$, where $j_1 = id$. Since the h_k are distinct elements of G, the j_k are distinct elements of J_m. Then $h_k^{-1}(z_0) = j_k^{-1}(h_1^{-1}(z_0)) \to x$ for $h_1^{-1}(z_0) \in \mathbf{H}^{n+1}$. This shows that x is a limit point of J_m, which is a contradiction since $x \in A_0$ and $\Lambda(J_m) \subset \mathrm{Fr}\, B_m$. $\square$

Now we shall prove that x is a conical limit point of G_0.

If $|h_k| = |f^{\alpha_{k_n}} \circ g_{k_n} \circ \cdots \circ f^{\alpha_{k_1}} \circ g_{k_1}| \geq 2$, then by taking a subsequence, we may assume that $\alpha_{k_1} > 0$ for all k; for the case $\alpha_{k_1} < 0$ can be dealt with in the same way. For each k, let h_k' be $h_k \circ g_{k_1}^{-1} \circ f^{-1}$. Then we have

$$h_k(\bar{A}_0) = h_k' \circ f \circ g_{k_1}(\bar{A}_0) \subset h_k'(B_2)$$

since $\bar{A}_0$ is G_0-invariant and $f(\bar{A}_0) \subset f(\overline{\mathbf{R}}^n \backslash B_1^\circ) = B_2$. If all the $h_k'(\mathrm{Fr}\, B_2)$ are distinct, then $\mathrm{diam}(h_k'(\mathrm{Fr}\, B_2)) \to 0$ for $\mathrm{Fr}\, B_2$ is a (J_2, G)-block satisfying the conditions in Theorem 2.10. It follows that $\mathrm{diam}(h_k'(B_2)) = \mathrm{diam}(h_k'(\mathrm{Fr}\, B_2)) \to 0$ and $d(h_k(z), h_k(x)) \to 0$ for all $z \in \mathrm{Fr}\, B_1 \subset \bar{A}_0$, which is a contradiction. Therefore, we may assume that $h_k'(\mathrm{Fr}\, B_2) = h_1'(\mathrm{Fr}\, B_2)$ for all k by taking a subsequence. For each k, there is an element $j_k \in J_2$ with $h_k' = h_1' \circ j_k$, where $j_1 = id$. Since j_k is contained in J_2, there is an element $i_k \in J_1$ such that $f \circ i_k = j_k \circ f$. These imply that $h_k = h_1' \circ f \circ i_k \circ g_{k_1}$. Since all the $h_k(\mathrm{Fr}\, B_2)$ are distinct, $\{i_k \circ g_{k_1}\}$ is a sequence of distinct elements of G_0. This implies that $g_{k_1}^{-1} \circ i_k^{-1}((h_1' \circ f)^{-1}(z_0)) \to x$ and that there is $\varepsilon > 0$ such that $d(i_k \circ g_{k_1}(z), i_k \circ g_{k_1}(x)) > \varepsilon$ for all k and any $z \in \overline{\mathbf{R}}^n \backslash \{x\}$. This implies that x is a conical limit point of G_0 by Proposition 2.7.

If $|h_k| = 1$ for all k, then set h_k to be $g_{k_2} \circ f^{\varepsilon_k} \circ g_{k_1}$, where $\varepsilon_k = \pm 1$. By taking a subsequence, we may assume that $\varepsilon_k = 1$ for all k. Then $g_{k_2} = id$ or $g_{k_2} \notin J_2$. If $g_{k_2} = id$ for all k, then $\{g_{k_1}\}$ is a sequence of distinct elements of G_0 since all the $h_k(\mathrm{Fr}\, B_1)$ are distinct. Thus, $g_{k_1}(x) \to f^{-1}(x')$ and $g_{k_1}(z) \to f^{-1}(z')$ for all $z \neq x$. Therefore, for all $z \in \overline{\mathbf{R}}^n \backslash \{x\}$ there is $\varepsilon > 0$ such that $d(g_{k_1}(z), g_{k_1}(x)) > \varepsilon$. Since $g_{k_1}^{-1}(f^{-1}(z_0)) \to x$ for $f^{-1}(z_0) \in \mathbf{H}^{n+1}$, x is a conical limit point of G_0 by Proposition 2.7.

If $g_{k_2} \notin J_2$ for all k, then $h_k(\bar{A}_0) \subset g_{k_2}(B_2)$. If all the $g_{k_2}(B_2)$ are distinct, then $\mathrm{diam}(g_{k_2}(B_2)) = \mathrm{diam}(g_{k_2}(\mathrm{Fr}\, B_2)) \to 0$ by Lemma 3.2-(7), which violates the fact that $d(h_k(z), h_k(x))$ is bounded away from zero for all $z \in \overline{\mathbf{R}}^n \backslash \{x\}$. Therefore we can assume that all the $g_{k_2}(B_2)$ are the same by taking a subsequence. For each k, there is an element $j_k \in J_2$ with $g_{k_2} = g_{1_2} \circ j_k$, with $j_1 = id$, where g_{1_2} denotes g_{k_2} with $k = 1$. Since $j_k \in J_2$, there is an element $i_k \in J_1$ such that $f \circ i_k = j_k \circ f$. These imply that $h_k = g_{1_2} \circ f \circ i_k \circ g_{k_1}$. Since all $h_k(\mathrm{Fr}\, B_2)$ are distinct, $\{i_k \circ g_{k_1}\}$ is a sequence of distinct elements of G_0. It follows that $(i_k \circ g_{k_1})^{-1}((g_{1_2} \circ f)^{-1}(z_0)) \to x$, where $(g_{1_2} \circ f)^{-1}(z_0) \in \mathbf{H}^{n+1}$, and for all $z \in \overline{\mathbf{R}}^n \backslash \{x\}$

$$i_k \circ g_{k_1}(z) \to f^{-1} \circ g_{1_2}^{-1}(z')$$

and $i_k \circ g_{k_1}(x) \to f^{-1} \circ g_{1_2}^{-1}(x')$. Therefore, for all $z \in \overline{\mathbf{R}}^n \backslash \{x\}$, there is a $\varepsilon > 0$ such that $d(i_k \circ g_{k_1}(z), i_k \circ g_{k_1}(x)) > \varepsilon$. Then Proposition 2.7 implies that x is a conical limit point of G_0. We can argue in the same way even when all h_k are in the form $g_{k_2} \circ f^{-1} \circ g_{k_1}$.

This completes the proof. $\square$

Now for an element $g = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1} \circ g_1 \in G$, we write $g \le 0$ if either $g_1 \notin J_1$, or $g_1 \in J_1$ and $\alpha_1 < 0$; $g > 0$ if $g_1 \in J_1$ and $\alpha_1 > 0$; $g \ge 0$ if either $g_1 \notin J_2$, or $g_1 \in J_2$ and $\alpha_1 > 0$; and $g < 0$ if $g_1 \in J_2$ and $\alpha_1 < 0$.

Using this notation, we consider a coset decomposition of G with respect to J_m for $m = 1, 2$ as follows.

$$G = J_1 \cup \left(\bigcup_{l,k} a_{l,k} J_1 \right) \cup \left(\bigcup_{l,k} b_{l,k} J_1 \right),$$

$$G = J_2 \cup \left(\bigcup_{l,k} c_{l,k} J_2 \right) \cup \left(\bigcup_{l,k} d_{l,k} J_2 \right),$$

where $|a_{l,k}| = |b_{l,k}| = |c_{l,k}| = |d_{l,k}| = l$, $a_{l,k} \le 0$, $b_{l,k} > 0$, $c_{l,k} \ge 0$ and $d_{l,k} < 0$.

Following Maskit, set $T_{0,m} = G_0(B_m)$ for $m = 1, 2$ and $T_0 = T_{0,1} \cup T_{0,2}$. Let C_0 be the complement of T_0 in $\overline{\mathbf{R}}^n$. For $l > 0$, we set $T_{l,1} = \bigcup_k a_{l,k}(B_1)$ and $T_{l,2} = \bigcup_k c_{l,k}(B_2)$, where $|a_{l,k}| = |c_{l,k}| = l$, $a_{l,k} \le 0$ and $c_{l,k} \ge 0$. We denote $T_{l,1} \cup T_{l,2}$ by T_l, and let C_l be the complement of T_l in $\overline{\mathbf{R}}^n$. It is easy to prove that $\{T_n\}$ is a decreasing sequence with respect to the inclusion, that is, $T_0 \supset T_1 \supset T_2 \supset \cdots$.

Corollary 3.3. *Under the hypotheses of Theorem* 3.1, *if* (B_1, B_2) *is precisely invariant under* (J_1, J_2) *in* G_0, *then each* ∂B_m *is a strong* (J_m, G)*-block if and only if each* B_m *is a strong* (J_m, G_0)*-block and hence all the conclusions in Theorem* 3.1 *hold.*

Proof. By assumption, we know that Fr B_m is precisely invariant under J_m in G. Let x be a parabolic fixed point of J_1. Since Fr B_1 is precisely invariant under J_1 in G, we know that

$$\mathrm{Stab}_{J_1}(x) = \mathrm{Stab}_{G_0}(x) = \mathrm{Stab}_G(x).$$

Set $H = \mathrm{Stab}_G(x)$.

The "if" part. We first assume that each B_m is a strong (J_m, G_0)-block. Let x be a parabolic fixed point of J_1, where the rank of H is $\kappa < n$. Then there is a peak domain U centred at x for G_0. By making U smaller if necessary, we have the following conditions:

(1) $f(U)$ is a peak domain centred at $f(x)$ for G_0;
(2) $G_0(U) \cap f(U) = \emptyset$ by Lemma 2.9;
(3) $\overline{U} \backslash \{x\} \subset \Omega(G_0)$ and $f(\overline{U}) \backslash \{f(x)\} \subset \Omega(G_0)$ since $(U \cup f(U)) \cap \Lambda(G_0) = \emptyset$.

By conjugation, we may assume that $x = \infty$. Decompose $\mathbf{R}^n$ into $\mathbf{R}^\kappa \times \mathbf{R}^{n-\kappa}$. By Bieberbach's theorem, we may assume that $\mathrm{Stab}_G^*(\infty)$ is the maximal abelian subgroup of finite index in $\mathrm{Stab}_G(\infty)$ which appeared in Definition 2.5, so that for any $g \in \mathrm{Stab}_G(\infty)$, $g(z) = Az + \mathbf{b}$, where the rotation A leaves $\mathbf{R}^\kappa$ and $\mathbf{R}^{n-\kappa}$ invariant and the vector $\mathbf{b}$ lies in the subspace $\mathbf{R}^\kappa$, whereas if g lies in $\mathrm{Stab}_G^*(\infty)$, then its restriction to the subspace $\mathbf{R}^\kappa$ is a translation. Thus we have U in the form

$$U = \left\{ x \in \mathbf{R}^n : \sum_{i=\kappa+1}^{n} x_i^2 > t^2 \right\},$$

with $t > 0$.

If $\kappa = n-1$, then U is the union of two open sets U_1 and U_2, where $U_1 = \{x \in \mathbf{R}^n : x_n > t\} \subset B_1^\circ$ and $U_2 = \{x \in \mathbf{R}^n : x_n < -t\}$ is in the exterior of $B_1 \cup B_2$.

Claim 3. *We can choose U to be small enough so that $U_2 \subset A_0$ and $f(U_1) \subset A_0$.*

Proof. We need only to prove that by choosing U_2 small enough, no G_0-translates of B_1 or B_2 intersect U_2. Suppose, on the contrary, that there is a sequence $\{g_k(B)\}$ of distinct G_0-translates of B_1 or B_2 intersecting $\{x \in \mathbf{R}^n : x_n < -s\}$ for any large s $(s > 0)$, where $B = B_1$ or B_2. By taking a subsequence and interchanging the indices if necessary, we may assume that $B = B_1$. This means that the projections of $g_k(B_1)$ to the subspace $\mathbf{R}^{n-(n-1)}$ converge to ∞. We may assume that g_k lies in $G_0 - J_1$ since J_1 stabilises B_1. Then Lemma 3.2-(7) implies that $\mathrm{diam}(g_k(B_1)) \to 0$. Hence $g_k(y) \to \infty$ for all $y \in B_1$ since $\{g_k(B_1)\}$ accumulates at ∞. By Lemma 2.1 and by choosing a suitable subsequence of $\{g_k\}$ (still denoted by the same symbol), we have $g_k(y) \to \infty$ for all y with at most one exception, which must be a limit point of G_0. Since $U \subset \Omega(G_0)$, $g_k(y) \to \infty$ for all $y \in U$. Since $g_k(U) \cap U = \emptyset$, the projections of $g_k(U)$ to the subspace $\mathbf{R}^{n-(n-1)}$ are bounded. By Theorem 2.9, for some fixed $y_0 \in U$ and for each k, we can choose an element $j_k \in H$ so that all $j_k \circ g_k(y_0)$ lie in a bounded set. Since for each k, $\infty \notin g_k(B_1)$, $\infty \notin j_k g_k(B_1)$. Since $|(j_k g_k(y))|_n = |(g_k(y))|_n$ and the projections of $g_k(B_1)$ to the subspace $\mathbf{R}^{n-(n-1)}$ converge to ∞, we see that all the $j_k g_k$ are distinct and $\{j_k g_k(B_1)\}$ also accumulates at ∞. By Lemma 3.2-(7), $j_k g_k(y) \to \infty$ for all $y \in B_1$. By Lemma 2.1, $j_k g_k(y) \to \infty$ for all y except for a limit point of G_0 by passing to a subsequence if necessary. This is a contradiction since $\{j_k g_k(y_0)\}$ does not converge to ∞ and $y_0 \in \Omega(G_0)$. By a similar argument, we can assume that $f(U_1) \subset A_0$. This proves our claim. $\square$

Then for any $g \in G - G_0$,

$$g(U) \cap U = (g(U_1) \cap U_1) \cup (g(U_1) \cap U_2) \cup (g(U_2) \cap U_1) \cup (g(U_2) \cap U_2),$$

where $g(U_2) \cap U_2 = \emptyset$ since $U_2 \subset A_0$ and A_0 is precisely invariant under G_0 in G by Lemma 3.2-(8). By dividing the proof into three cases, we will show that $g(U) \cap U = \emptyset$ for any $g \in G - G_0$ when $\kappa = n-1$. Let $g = f^{\alpha_n} \circ g_{\alpha_n} \circ \cdots \circ f^{\alpha_1} \circ g_{\alpha_1} \in G - G_0$ be a normal form with length l $(l > 0)$.

Case 1. $g(U_1) \cap U_1 = \emptyset$ for any $g \in G - G_0$.

If $f \circ g \circ f^{-1} \in G_0$, then there is an element $j \in G_0$ with $g = f^{-1} \circ j \circ f$. Since g is a normal form, $j \notin J_2$. Thus $j \notin \mathrm{Stab}_{G_0}(f(x))$ since $\mathrm{Stab}_{G_0}(f(x)) = \mathrm{Stab}_{J_2}(f(x))$, and hence $j \circ f(U_1) \cap f(U_1) = \emptyset$ for $f(U)$ is a peak domain centred at $f(x)$ for G_0. Therefore, $g(U_1) \cap U_1 = \emptyset$ for this case. If $f \circ g \circ f^{-1} \notin G_0$, then $f(g(U_1) \cap U_1) = f \circ g \circ f^{-1}(f(U_1)) \cap f(U_1) \subset f \circ g \circ f^{-1}(A_0) \cap A_0 = \emptyset$ for A_0 is precisely invariant under G_0 in G.

Case 2. $g(U_1) \cap U_2 = \emptyset$ for any $g \in G - G_0$.

If $g_1 \in J_1$ and $\alpha_1 < 0$ or $g_1 \notin J_1$, then $g(U_1) \subset g(B_1^\circ) \subset T_n^\circ \subset T_0^\circ$. It follows that in this case $g(U_1) \cap U_2 = \emptyset$. If $\alpha_1 > 0$ and $g_1 \in J_1$, then there is an element $h_1 \in J_2$ with $f \circ g_1 = h_1 \circ f$. Thus $g = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1 - 1} \circ h_1 \circ f$ is a normal form of length l. If $l > 1$, then $f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1 - 1}$ is a normal form of length $l - 1$ and $g(U_1) = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1 - 1} \circ h_1 \circ f(U_1) \subset f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1 - 1} \circ h_1(A_0) = f^{\alpha_n} \circ g_n \circ \cdots \circ f^{\alpha_1 - 1}(A_0) \subset T_0^\circ$ by Lemma 2.6. If $l = 1$, then $g = g_2 \circ f \circ g_1 = g_2 \circ h_1 \circ f$ and $g(U_1) \cap U_2 = g_2 \circ h_1(f(U_1)) \cap U_2 \subset g_2 \circ h_1(f(U)) \cap U = \emptyset$ by the second assumption for U and $f(U)$. Thus for this case, $g(U_1) \cap U_2 = \emptyset$.

Case 3. $g(U_2) \cap U_1 = \emptyset$ for any $g \in G - G_0$.

Since $g \notin G_0$, $g^{-1} \notin G_0$ and $g(U_2) \cap U_1 = g(U_2 \cap g^{-1}(U_1)) = \emptyset$ by Case 2.

These discussions show that U is precisely invariant under H in G, i.e., U is a peak domain centred at x for G and x is a parabolic vertex of G.

If $\kappa < n - 1$, then we can assume that U lies in B_1° or in the exterior of B_1 and B_2. If $U \subset B_1^\circ$, then we may assume that $f(U) \subset A_0$ by the same argument as in Claim 3. It follows that $g(U) \cap U = \emptyset$ for all $g \in G - G_0$ by similar discussions as in Case 1. If U is in the exterior of $B_1 \cup B_2$, we may assume that $U \subset A_0$ by similar discussions as in Claim 3. Thus $g(U) \cap U \subset g(A_0) \cap A_0 = \emptyset$ for all $g \in G - G_0$. In either case, we can choose U small enough so that U is a peak domain for G. Thus x is a parabolic vertex of G. We thus have shown that $\mathrm{Fr}\, B_1$ is a strong (J_1, G)-block.

We now consider $\mathrm{Fr}\, B_2$. Let x be a parabolic fixed point of J_2 in $\mathrm{Fr}\, B_2$. Then $f^{-1}(x)$ is a parabolic fixed point of G in $\mathrm{Fr}\, B_1$. Since $\mathrm{Fr}\, B_1$ is a strong (J_1, G)-block, $f^{-1}(x)$ is a parabolic vertex of G. Thus x is a parabolic vertex of G and $\mathrm{Fr}\, B_2$ is a strong (J_2, G)-block.

The "only if" part. We assume that $\mathrm{Fr}\, B_1$ is a strong (J_1, G)-block. For any parabolic fixed point $x \in \mathrm{Fr}\, B_1$ of G_0, if the rank of $\mathrm{Stab}_{G_0}(x)$ is $\kappa < n$, then so is $\mathrm{Stab}_G(x)$ for $\mathrm{Stab}_G(x) = \mathrm{Stab}_{J_1}(x) = \mathrm{Stab}_{G_0}(x)$. Then there is a peak domain U centred at x for G, which is also a peak domain for G_0. Therefore B_1 is a strong (J_1, G_0)-block since $B_1^\circ \subset \Omega(G_0)$. If $x \in \mathrm{Fr}\, B_2$ is a parabolic fixed point of G_0, where the rank of $\mathrm{Stab}_{G_0}(x)$ is $\kappa < n$, then $f^{-1}(x) \in \mathrm{Fr}\, B_1$ is a parabolic fixed point of G_0 with rank k. Since B_1 is strong, there is a peak domain U centred at $f^{-1}(x)$ for G_0. Then $f(U)$ is a peak domain centred at x for G_0. This shows that B_m is a strongly (J_m, G_0)-block for each m $(m = 1, 2)$.

If we assume that $\mathrm{Fr}\, B_2$ is a strong (J_2, G)-block, then by the reasoning similar to the above, we can show that each B_m is a strong (J_m, G_0)-block. □

4. Applications

4.1. The statement of Theorem 4.1. Following [19] or [20], we denote by $PSL(2, \Gamma_n)$ the n-dimensional Clifford matrix group. Then $PSL(2, \Gamma_n)$ is isomorphic to $M(\overline{\mathbf{R}}^n)$ (cf. [1]).

We assume that $n = 3$. We denote the standard basis of $\mathbf{R}^3$ by 1, e_1 and e_2. Each element $x \in \mathbf{R}^3$ is expressed as

$$x = x_1 + x_2 e_1 + x_3 e_2.$$

We set

$$j_1 = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad j_3 = \begin{pmatrix} e_1 & 1 \\ 0 & -e_1 \end{pmatrix},$$
$$j_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & -10e_2 \\ 0 & 1 \end{pmatrix}$$

and

$$J_1 = J_2 = \langle j_1, j_2, j_3 \rangle, \quad G_0 = \langle j_4, J_1 \rangle, \quad G_1 = \langle f \rangle \quad \text{and} \quad G = \langle G_0, G_1 \rangle.$$

By the definition of Clifford algebra, j_1, j_2, j_3, j_4 and f act on $\mathbf{R}^3$ as follows.

$$j_1(x) = -x_1 - x_2 e_1 + x_3 e_2, \quad j_2(x) = (x_1 + 1) + x_2 e_1 + x_3 e_2,$$
$$j_3(x) = -x_1 + (1 - x_2) e_1 + x_3 e_2,$$

$$j_4(x) = \frac{-x_1 + x_2 e_1 + x_3 e_2}{x_1^2 + x_2^2 + x_3^2}, \quad f(x) = x_1 + x_2 e_1 + (x_3 - 10) e_2,$$

where $x = x_1 + x_2 e_1 + x_3 e_2$.

Then we have the following.

THEOREM 4.1. *G is geometrically finite.*

We shall prove this theorem in the remainder of the paper.

4.2. Several propositions.

PROPOSITION 4.2. $\mathrm{Stab}_{G_0}(\infty) = J_1 = J_2$, *which means that J_m $(m = 1, 2)$ is a geometrically finite subgroup of G_0.*

Proof. We can see that $J_1 = J_2 \subset \mathrm{Stab}_{G_0}(\infty)$. Now take any $g \in \mathrm{Stab}_{G_0}(\infty)$. Then

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Since $ad = 1$ and a, d are Gaussian integers, we may assume that $a = d = 1$ or $a = e_1$ and $d = -e_1$.

If $a = e_1$ and $d = -e_1$, then

$$g = j_1 \begin{pmatrix} 1 & -e_1 b \\ 0 & 1 \end{pmatrix},$$

where $-e_1 b$ is also a Gaussian integer. Therefore, we only need to consider the case when $a = d = 1$, i.e., $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where b is a Gaussian integer. We can put $b = \alpha + e_1\beta$, where α, β are integers. Then

$$g = j_2^{\alpha} \circ (j_1^{-1} \circ j_3)^{-\beta}. \qquad \square$$

By the statements of Section 5 in [10], we see that

Proposition 4.3. (1) *G_0 is geometrically finite;*
(2) *$\Lambda(G_0) = G_0(\infty) \cup \{$the conical limit points of $G_0\}$;*
(3) *∞ is a parabolic vertex of G_0 and U is a peak domain of ∞, where*

$$U = \{x \in \mathbf{R}^3 : x_3^2 > 16\}.$$

Set

$$\mathrm{Fr}\, B_1 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 5\} \cup \{\infty\}, \quad B_1 = \{x \in \mathbf{R}^3 : x_3 \geq 5\} \cup \{\infty\},$$
$$\mathrm{Fr}\, B_2 = \{x \in \mathbf{R}^3 : x_3 = -5\} \cup \{\infty\}, \quad B_2 = \{x \in \mathbf{R}^3 : x_3 \leq -5\} \cup \{\infty\},$$
$$A = \overline{\mathbf{R}}^3 \backslash (B_1 \cup B_2) \quad \text{and} \quad A_0 = A \backslash G_0(B_1 \cup B_2).$$

Proposition 4.4. *Each B_m is a (J_m, G_0)-block $(m = 1, 2)$.*

Proof. Obviously, $\Lambda(J_m) = \{\infty\}$ and $B_m \cap \Omega(J_m) = B_m \cap \Omega(G_0) = B_m \backslash \{\infty\}$. By Propositions 4.2 and 4.3, we know that $B_m \cap \Omega(G_0)$ is precisely invariant under J_m in G_0. $\square$

Proposition 4.5. *$A_0 \neq \emptyset$.*

Proof. Since $B_1^{\circ} \cup B_2^{\circ} \subset \Omega(G_0)$ by Lemma 3.2-(3), we have $\Lambda(G_0) \subset A_0 \cup G_0(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2)$. On the other hand, $\Lambda(G_0) \cap G_0(\mathrm{Fr}\, B_1 \cup \mathrm{Fr}\, B_2) = G_0(\Lambda(J_1) \cup \Lambda(J_2)) = G_0(\infty)$. An easy computation shows that $\pm\sqrt{3}$ are fixed points of a loxodromic element $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \in G_0$, they are conical limit points of G_0 and are not G_0-equivalent to ∞. Therefore, $\pm\sqrt{3} \in A_0$. $\square$

Proposition 4.6. *B_1 and B_2 are jointly f-blocked.*

Proof. By Propositions 4.3 and 4.4, we know that $(B_1 \cap \Omega(G_0), B_2 \cap \Omega(G_0))$ is precisely invariant under (J_1, J_2) in G_0. By computation, $f(\bar{\mathbf{R}}^3 \backslash B_1) = B_2^\circ$ and $fJ_1f^{-1} = J_2$. Combining these with Proposition 4.4, we see that B_1 and B_2 are jointly f-blocked. $\square$

Proposition 4.7. *Set*

$$D_0 = \left\{ x \in \mathbf{R}^3 : -\frac{1}{2} < x_1 \leq \frac{1}{2}, 0 < x_2 \leq \frac{1}{2}, |x| \geq 1 \right\} \Big\backslash (A_1 \cup A_2 \cup A_3),$$

where $A_1 = \{x \in \mathbf{R}^3 : x_2 = 0, -\frac{1}{2} \leq x_1 \leq 0\}$, $A_2 = \{x \in \mathbf{R}^3 : x_2 = \frac{1}{2}, -\frac{1}{2} \leq x_1 \leq 0\}$, *and* $A_3 = \{x \in \mathbf{R}^3 : |x| = 1, -\frac{1}{2} \leq x_1 \leq 0\}$. *Then D_0 is maximal.*

Proof. It is obvious that D_0 is a fundamental set for G_0. Since $D_0 \cap B_m$ is a fundamental set for the action of J_m on B_m and $f(D_0 \cap \mathrm{Fr}\, B_1) = D_0 \cap \mathrm{Fr}\, B_2$, D_0 is maximal. $\square$

Proposition 4.8. *$\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block $(m = 1, 2)$.*

Proof. It is obvious that the rank of $\mathrm{Stab}_G(\infty)$ is 3. It follows that ∞ is a parabolic vertex of G. Obviously, $G_0 \cap G_1 = \{id\}$. By Theorem 3.1, $G = \langle G_0, G_1 \rangle = G_0 *_f$, G is discrete and $\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block $(m = 1, 2)$. $\square$

Now we are ready to prove Theorem 4.1.

4.3. The proof of Theorem 4.1. Since G_0 is geometrically finite, each B_m is a strong (J_m, G_0)-block. On the other hand, by Proposition 4.8, each $\mathrm{Fr}\, B_m$ is a strong (J_m, G)-block $(m = 1, 2)$. By Theorem 3.1, G is geometrically finite.

From the proof of Theorem 4.1, we can easily get the following corollary.

Corollary 4.9. *B_m is not precisely invariant under J_m in G_0.*

Remark 4.1. The group G in Theorem 4.1 does not satisfy the condition that "B_m $(m = 1, 2)$ is precisely invariant under J_m in G_0", which is required in Theorem 1.1.

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