ON KLEIN'S COMBINATION THEOREM

BY
BERNARD MASKIT(1)

The combination theorem of Klein, *Der Prozess der Ineinanderschiebung*, in [4], was first stated essentially as follows. Let G_1 and G_2 be finitely generated discontinuous groups of Möbius transformations, and let D_1 and D_2 be fundamental domains for G_1 and G_2, respectively. Assume that the interior of D_1 contains the boundary and exterior of D_2, and that the interior of D_2 contains the boundary and exterior of D_1. Then G, the group generated by G_1 and G_2, is again discontinuous, and $D = D_1 \cap D_2$ is a fundamental domain for G.

Proofs of the above theorem appear in most of the standard references (for example, [2, pp. 56–59], [3, pp. 190–194]). These proofs use different definitions of "fundamental domain", none of which are very general.

In this paper, we give a new version of this theorem, which uses fundamental sets rather than fundamental domains, and which does not have the restriction that G_1 and G_2 are finitely generated. We also have somewhat different goemetric hypotheses. The new conditions involve the existence of a certain Jordan curve. In Chapter III there is an example which shows that such a condition is needed. This example corresponds, in the formulation given above, to the case that the boundaries of D_1 and D_2 are not disjoint.

One of the conclusions of the combination theorem is that G is the free product of G_1 and G_2. In [5] I proved a weak generalization of this theorem for the case that G is the free product of G_1 and G_2 with an amalgamated subgroup. In this paper, there is a stronger generalization, in which the amalgamated subgroup is cyclic.

In order to state the main theorem of this paper, certain definitions and notations are needed. These are given in Chapter I, which also contains some basic facts about Kleinian groups. Chapter II contains the formulation of the main theorem and the proof. Chapter III contains the example mentioned above.

It was proven in [5] that a rather wide class of Kleinian groups can be constructed from very simple groups using the construction that appears in this paper, and another construction. This other construction will be discussed in a subsequent paper.

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The two constructions mentioned above are also used by Nielsen-Fenchel [1] to construct all finitely generated Fuchsian groups, starting with certain "elementary" groups.

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I. Basic facts.

1. We denote the extended complex plane, or Riemann sphere, by Σ. A Möbius transformation is a mapping $z \to (az+b)(cz+d)^{-1}$, where a, b, c, d are complex numbers with $ad - bc \neq 0$.

A group G of Möbius transformations is said to be *discontinuous at the point* z, if there is a neighborhood U of z, such that $g(U) \cap U = \varnothing$, for all $g \in G$, other than the identity; i.e. if $x, y \in U$, $g \in G$, $g(x) = y$, then $x = y$ and $g = 1$.

A *Kleinian group* is a group of Möbius transformations which is discontinuous at some point z. Throughout this chapter, the letter G will denote a Kleinian group.

The following notation will be used throughout this paper. If $S \subset \Sigma$ and H is some set of Möbius transformations, then

$$\prod(S,H) = \bigcup_{h \in H} h(S).$$

2. Given a Kleinian group G, there are two natural subsets of Σ which one can associate with G. The *regular set* $R(G)$ consists of those points of Σ at which G is discontinuous. The *limit set* $L(G)$ consists of those points z for which there is a point z_0, and a sequence $\{g_n\}$, of *distinct* elements of G, with $g_n(z_0) \to z$.

One easily sees that these sets are invariant under G, that $R(G)$ is open, that $L(G)$ is closed, and that $R(G) \cap L(G) = \varnothing$.

3. A set D is called a *partial fundamental set* (*PFS*) for G, if

(a) $D \neq \varnothing$,

(b) $D \subset R(G)$, and

(c) $g(D) \cap D = \varnothing$, for all $g \in G$, $g \neq 1$.

If, in addition to (a), (b), and (c), D satisfies property

(d) $\prod(D,G) = R(G)$,

then D is called a *fundamental set* (*FS*) for G.

Conditions (c) and (d) are the usual conditions for a fundamental set. Condition (c) asserts that no two points of D are equivalent under G, and condition (d) asserts that every point of $R(G)$ is equivalent to some point of D.

We remark that, using the definition of discontinuity, condition (c) can be restated as follows: if $x, y \in D$, $g \in G$, $g(x) = y$, then $x = y$ and $g = 1$.

5. In this section we prove three basic lemmas about Kleinian groups.

LEMMA 1. *Let x be some point of Σ, and let $\{g_n\}$ be a sequence of distinct elements of G Then $A = \{z \mid g_n(z) \to x\}$ is both open and closed in $R(G)$.*

Proof. If $z \in A \cap R(G)$, then we can find a neighborhood U of z where $g(U) \cap U = \varnothing$ for every $g \in G$ other than the identity, and U is the interior of a circular disc. Then, for every n, $g_n(U)$ is again the interior of a circular disc. Since the g_n are all distinct, $g_n(U) \cap g_m(U) = \varnothing$ for all $n \neq m$, and so, in terms of the metric on the sphere, the area of $g_n(U)$ converges to zero. It follows that the diameter of $g_n(U)$ converges to zero, and so $g_n(y) \to x$, for every $y \in U$; i.e. A is open in $R(G)$.

Now let $z_k \to z \in R(G)$, where each $z_k \in A \cap R(G)$. Pick a neighborhood U of z, as above. For k sufficiently large, $z_k \in U$, and so, by the above argument, $z \in A$; i.e. A is closed in $R(G)$.

LEMMA 2. *Let $z_n \to z \in R(G)$, and let $\{g_n\}$ be a sequence of distinct elements of G, with $g_n(z_n) \to x$. Then $g_n(z) \to x$.*

Proof. As in the proof of Lemma 1, let U be a neighborhood of z where U is the interior of a circular disc, and $g(U) \cap U = \varnothing$ for all $g \in G$ other than the identity. Then, for n sufficiently large, $z_n \in U$. Since the diameter of $g_n(U)$ converges to zero, $g_n(z) \to x$.

The following lemma was essentially proven by Ford [2, pp. 39–41]:

LEMMA 3. *Let $z_0 \in R(G)$ and let $\{g_n\}$ be a sequence of distinct elements of G. Assume that $g_n(z_0) \to x$. Then there is a subsequence $\{g_{n_i}\}$ such that $g_{n_i}(z) \to x$ for all $z \in \Sigma$ with at most one exception.*

Proof. We can assume, without loss of generality, that z_0 is the point at infinity. Since $z_0 \in R(G)$, the isometric circles of all the elements of G are bounded. Let a_n, a'_n be the centers of the isometric circles of g_n, g_n^{-1}, respectively, and let r_n be the radius of these isometric circles. Each g_n then maps $\{z \mid |z - a_n| > r_n\}$ onto $\{z \mid |z - a'_n| < r_n\}$, and in particular, $g_n(z_0) = a'_n$. By assumption, $g_n(z_0) = a'_n \to x$. We now pick the subsequence $\{g_{n_i}\}$ so that $\{a_{n_i}\}$ converges to some point a.

Now let $z' \neq a$ be a some point of Σ. Since the $\{g_{n_i}\}$ are all distinct, the sequence r_{n_i} converges to zero. It follows then, that for n_i sufficiently large, $|z' - a_{n_i}| > r_{n_i}$, and so $|g_{n_i}(z') - a'_n| < r_{n_i}$. The result now follows, since $a'_n \to x$, and $r_{n_i} \to 0$.

II. The Combination Theorem.

6. The following theorem was proven in [5].

THEOREM 1. *Let G_1 and G_2 be Kleinian groups, and let H be a common subgroup of G_1 and G_2. Let D_1, D_2, Δ be PFS's for G_1, G_2, H, respectively. For $i = 1,2$, set $E_i = \prod(D_i, H)$. Assume that $D' = \operatorname{int}(E_1 \cap E_2 \cap \Delta) \neq \varnothing$, and that $E_1 \cup E_2 = R(G_1) \cup R(G_2)$. Then G, the group generated by G_1 and G_2, is Kleinian, D' is a PFS for G, G is the free product of G_1 and G_2 with amalgamated subgroup H, and, if we set $D = E_1 \cap E_2 \cap \Delta$, then $g(D) \cap D = \varnothing$, for all $g \in G$, $g \neq 1$.*

The principal aim of this paper is to find suitable conditions under which D is a FS when D_1, D_2 and Δ are. The precise statement is as follows:

THEOREM 2. *Let G_1 and G_2 be Kleinian groups. Let H be a common subgroup of G_1 and G_2 where H is either cyclic or consists only of the identity. Let D_1, D_2, Δ be FS's for G_1, G_2, H, respectively. For $i=1, 2$, set $E_i=\prod(D_i, H)$. Assume that $E_1 \cup E_2 = R(H)$, and that $\operatorname{int}(E_1 \cap E_2 \cap \Delta) \neq \varnothing$. Assume further that there is a simple closed curve γ, contained in $\operatorname{int}(E_1 \cup E_2) \cup L(H)$; γ is invariant under H; the closure of $\gamma \cap \Delta$ is contained in $\operatorname{int}(E_1 \cap E_2)$, and γ separates $E_1 - E_2$ and $E_2 - E_1$. Then G, the group generated by G_1 and G_2 is Kleinian, G is the free product of G_1 and G_2 with amalgamated subgroup H, and $D = E_1 \cap E_2 \cap \Delta$ is a FS for G.*

We already know, from Theorem 1, that G is Kleinian, that G is the free product of G_1 and G_2 with amalgamated subgroup H, and that D satisfies properties (a) and (c) in the definition of FS; i.e. $D \neq \varnothing$, and $g(D) \cap D = \varnothing$ for all $g \in G$, $g \neq 1$. In the remainder of this chapter, we will prove that D satisfies properties (b) and (d) in the definition of FS; i.e. $D \subset R(G)$, and $\prod(D, G) = R(G)$. This will then complete the proof of Theorem 2.

The proof of property (b) is mainly technical and simply involves chasing points around with appropriate applications of Lemma 2.

After having proven that D satisfies property (b), we will then know that $\prod(D, G) \subset R(G)$. We also will know that points not in $R(G_1)$, and points not in $R(G_2)$, cannot be in $R(G)$, and therefore that the images of these points under G also cannot be in $R(G)$. We will then show that *every other point z of Σ is uniquely determined by a nested sequence of images of γ under G, and that $z \in L(G)$.*

7. We first observe that E_1 and E_2 are invariant under H, and, in fact, they are *precisely invariant* under H; that is, if $g_1 \in G_1 - H$, then $g_1(E_1) \cap E_1 = \varnothing$, and similarly, if $g_2 \in G_2 - H$, then $g_2(E_2) \cap E_2 = \varnothing$. The proof of this fact is quite simple. If, for example, $x \in g_1(E_1) \cap E_1$, then there are points z_1, $z_2 \in D_1$, and there are elements h_1, $h_2 \in H$, so that $x = g_1 \circ h_1(z_1) = h_2(z_2)$. It follows at once that $g_1 = h_2 \circ h_1^{-1} \in H$.

We also observe that $E_1 \cap E_2 = \prod(D, H)$. For if $z \in E_1 \cap E_2$, then $z \in R(G_1) \cap R(G_2) \subset R(H)$, and so there is an $h \in H$, with $h(z) \in \Delta$. Then, since $E_1 \cap E_2$ is invariant under H, $h(z) \in E_1 \cap E_2 \cap \Delta = D$, and so $z \in \prod(D, H)$.

Conversely, if $z \in \prod(D, H)$, then $z = h(y)$, where $y \in D = E_1 \cap E_2 \cap \Delta$. Again, $E_1 \cap E_2$ is invariant under H, and so $z \in E_1 \cap E_2$.

Finally, we remark that since H is cyclic, $L(H)$ consists of either 0, 1, or 2 points.

These simple but important remarks will be used throughout this chapter without further mention.

8. In this section, we show that $D \cap \operatorname{bd}(E_1) \subset R(G)$.

Lemma 4. *Let $z_0 \in E_1 \cap \mathrm{bd}(E_1)$. Let $g \in G_1$, and let U be a connected neighborhood of z_0, where $U \subset R(G_1)$. If $g(U) \not\subset E_2$, then there is a point $x \in g(U) \cap \gamma$.*

Proof. Since $U \subset R(G_1)$, $g(U) \subset R(G_1)$, and so $g(U) \subset R(H)$. If $g(U) \not\subset E_2$, then there are points of $g(U)$ which are not in E_2, and, since $E_1 \cup E_2 = R(H)$, there are points of $g(U)$ which are in $E_1 - E_2$.

If $g \notin H$, then $g(z_0) \notin E_1$, and so $g(z_0) \in E_2 - E_1$.

If $g \in H$, then $g(z_0) \in \mathrm{bd}(E_1)$, and so $g(U)$ contains points not in E_1; i.e. $g(U)$ contains points of $E_2 - E_1$.

Therefore $g(U)$ is a connected open set containing points of $E_1 - E_2$ and points of $E_2 - E_1$. Since γ separates these two sets, γ passes through $g(U)$.

Lemma 5. *Let $z_0 \in D \cap \mathrm{bd}(E_1)$, and let $g \in G_1$, then there is a neighborhood U of z_0, with $g(U) \subset E_2$.*

Proof. Since $z_0 \in D$, $z_0 \in R(G_1)$, hence we can find a nested sequence $\{U_n\}$ of connected neighborhoods of z_0, where each $U_n \subset R(G_1)$. Now by Lemma 4, if $g(U_n) \not\subset E_2$, for every n, there is an $x_n \in g(U_n) \cap \gamma$. Then $\lim_{n\to\infty} x_n = g(z_0) \in \gamma$. This is a contradiction since $\gamma \subset \mathrm{int}(E_1 \cap E_2) \cup L(H)$, and $g(z_0) \notin L(H) \cup \mathrm{int}(E_1)$.

Lemma 6. *Let $z_0 \in D \cap \mathrm{bd}(E_1)$. Then there is a neighborhood U of z_0, with $g(U) \subset E_2$, for every $g \in G_1$.*

Proof. Assume not, and let $\{U_n\}$ be a nested sequence of connected neighborhoods of z_0, where each $U_n \subset R(G_1)$. Then, for each n, there is a $g_n \in G_1$, with $g_n(U_n) \not\subset E_2$. By Lemma 5, we can pick out a subsequence so that the g_n are all distinct, and in fact, since E_2 is invariant under H, we can pick the subsequence so that the g_n represent distinct elements of H/G_1.

By Lemma 4, for each n, there is a point $x_n \in g_n(U_n) \cap \gamma$. Since U_n is open we can assume that $x_n \notin L(H)$. We define a sequence $h_n \in H$, by $h_n(x_n) = y_n \in \Delta \cap \gamma$. We now choose a subsequence which we again call by the same name, so that $y_n \to y \in \mathrm{closure}(\gamma \cap \Delta) \subset \mathrm{int}(E_1 \cap E_2) \subset R(G_1)$.

We now have that $y_n \to y \in R(G_1)$, $g_n^{-1} \cdot h_n^{-1}(y_n) \to z_0$, and the elements $g_n^{-1} \cdot h_n^{-1}$ are all distinct. Hence, by Lemma 2, $g_n^{-1} \circ h_n^{-1}(y) \to z_0$, and so $z_0 \notin R(G_1)$, which is a contradiction.

Lemma 7. $D \cap \mathrm{bd}(E_1) \subset R(G)$.

Proof. Let $z_0 \in D \cap \mathrm{bd}(E_1)$. Then, by Lemma 6, we can find a neighborhood U of z_0 with the following properties:

(i) $g(U) \subset E_2$ for all $g \in G_1$, and

(ii) $g(U) \cap U = \varnothing$, for all $g \in G_1$, $g \neq 1$.

Now let $g \neq 1$ be any element of G. If $g \in G_1$, then we already know by property (ii) that $g(U) \cap U = \varnothing$. If $g \notin G_1$, then we can write $g = g_n \circ g_{n-1} \circ \cdots \circ g_2 \circ g_1$, where $g_{2j} \in G_2$, $g_{2j-1} \in G_1$, and for $j > 1$, $g_j \notin H$.

By property (i), $g_1(U) \subset E_2$. Then, since $g_2 \notin H$, $g_2 \circ g_1(U) \subset E_1 - E_2$. Similarly $g_3 \circ g_2 \circ g_1(U) \subset E_2 - E_1$, and so on. Hence $g(U) \subset E_1 - E_2$ if n is even, and $g(U) \subset E_2 - E_1$ if n is odd.

If n is even, then $U \subset E_2$, and $g(U) \subset E_1 - E_2$, hence $g(U) \cap U = \emptyset$.

Now let n be odd and assume that $g(U) \cap U \neq \emptyset$. We have already observed that $g(U) \cap U = \emptyset$ for $g \in G_1$, and so $n \geqq 3$. We have that $g_n \circ \cdots \circ g_1(U) \cap U \neq \emptyset$, or, equivalently $g_{n-1} \circ \cdots \circ g_1(U) \cap g_n^{-1}(U) \neq \emptyset$. Since n is odd, $g_{n-1} \circ \cdots \circ g_1(U) \subset E_1 - E_2$, $g_n^{-1} \in G_1$, and so, by property (i), $g_n^{-1}(U) \subset E_2$. Hence $g_{n-1} \circ \cdots \circ g_1(U) \cap g_n^{-1}(U) = \emptyset$.

9. Since G_1, D_1, E_1, and G_2, D_2, E_2 appear symmetrically in the hypotheses of Theorem 2, we can interchange these everywhere in §8, and so prove

Lemma 8. $D \cap \mathrm{bd}(E_2) \subset R(G)$.

In order to complete the proof that $D \subset R(G)$, it remains only to show that those points of D which are on the boundary of Δ are contained in $R(G)$.

Lemma 9. $(D - (\mathrm{bd}(E_1) \cup \mathrm{bd}(E_2)) \cap \mathrm{bd}(\Delta)) \subset R(G)$.

Proof. If z_0 is a point of $D - (\mathrm{bd}(E_1) \cup \mathrm{bd}(E_2))$, then we easily see that $z_0 \in \mathrm{int}(E_1 \cap E_2)$. Now, we can find a neighborhood U of z_0, such that $g(U) \cap U = \emptyset$, for all $g \in H$, and $U \subset \mathrm{int}(E_1 \cap E_2)$. Since $g(E_1 \cap E_2) \cap (E_1 \cap E_2) = \emptyset$ for all $g \in G - H_1$ we see at once that $g(U) \cap U = \emptyset$, for all $g \in G$, $g \neq 1$, and so $z_0 \in R(G)$.

Putting together Theorem 1 and Lemmas 7, 8, and 9, we have

Lemma 10. $D \subset R(G)$.

10. To complete the proof of this theorem, we have to show that $\prod(D, G) = R(G)$. We have already shown that $\prod(D, G) \subset R(G)$.

Set $R_0 = \prod(D, G)$, $L_1 = \prod(\Sigma - R(G_1), G)$, $L_2 = \prod(\Sigma - R(G_2), G)$, $T = \Sigma - (R_0 \cup L_1 \cup L_2)$. For $i = 1, 2$, $L_i \cap R(G) = \emptyset$, and so, if R_0 were properly contained in $R(G)$, $T \cap R(G)$ would not be empty. We will in fact show that $T \subset L(G)$.

Let z_0 be some point of T. Then $z_0 \notin L_1$ and so, in particular, $z_0 \in R(G_1)$. Therefore there is a $g_1 \in G_1$ with $g_1(z_0) \in D_1$. Since R_0, L_1, and L_2 are invariant under G, T is invariant under G, and so $g_1(z_0) \in T$. We next observe that $g_1(z_0) \notin E_2$, for $E_1 \cap E_2 \subset R(G)$.

Since $g_1(z_0) \in T$, $g_1(z_0) \notin L_2$, and so there must be a $g_2 \in G_2$, with $g_2 \circ g_1(z_0) \in D_2$. As before, $g_2 \circ g_1(z_0) \notin E_1$, and so $g_2 \notin H$.

We can continue in this manner to get a sequence $\{g_n\}$ of elements of G with the following properties:

(1) $g_{2i} \in G_2$,

(2) $g_{2i+1} \in G_1$,

(3) $g_i \notin H$, $i > 1$,

(4) $g_n \circ \cdots \circ g_1(z_0) \in D_1 - E_2$, if n is odd,
(5) $g_n \circ \cdots \circ g_1(z_0) \in D_2 - E_1$, if n is even,
(6) $g_n \circ \cdots \circ g_1(z_0) \in T$.
We set $j_n = (g_n \circ \cdots \circ g_1)^{-1}$ and $\gamma_n = j_n(\gamma)$.

Since $\gamma \subset E_1 \cap E_2 \cup L(H)$, $\gamma \cap T = \varnothing$. T is invariant under G, and so $\gamma_n \cap T = \varnothing$, for every n. Since $z_0 \in T$, z_0 lies in the interior of one of the topological discs bounded by γ_n. Let S_n be the closed disc, bounded by γ_n, which contains z_0.

LEMMA 11. *$S_n \supset S_{n+1}$ for every n.*

Proof. Let $z \in \gamma \cap E_1 \cap E_2$. Assume first that n is odd. Then $j_{n+1}^{-1}(z_0) \in E_2 - E_1$ and $j_{n+1}^{-1} \circ j_n(z) = g_{n+1} \circ g_n \circ \cdots \circ g_1 \circ g_1^{-1} \circ \cdots \circ g_n^{-1}(z) = g_{n+1}(z) \in E_1 - E_2$, since $g_{n+1} \in G_2 - H$. Therefore γ separates $j_{n+1}^{-1} \circ j_n(z)$ and $j_{n+1}^{-1}(z_0)$. Applying j_{n+1} to each of the expressions in this statement, we find that $j_{n+1}(\gamma) = \gamma_{n+1}$ separates $j_n(z)$ and z_0; i.e., $j_n(\gamma)$ lies outside S_{n+1}, except for the at most two points of γ which lie in $L(H)$. This shows that $S_n \supset S_{n+1}$ for n odd.

Interchanging G_1, E_1, and G_2, E_2, in the above argument, we get $S_n \supset S_{n+1}$ for n even.

Now let $S = \bigcap_n S_n$, and let $\bar{S}$ be the boundary of S.

LEMMA 12. If $L(H) = \varnothing$, *then $\bar{S}$ consists of at most one point.*

Proof. Let $x \in \bar{S}$. Then x can be realized as the limit of a sequence x_n, where $x_n \in \mathrm{bd}(S_1) = \gamma_n$. Set $y_n = j_n^{-1}(x_n)$. The sequence $\{y_n\}$ has at least one limit point y. Since $L(H) = \varnothing$, $\gamma \subset \mathrm{int}(E_1 \cap E_2) \subset R(G)$, and so $y \in R(G)$. By Lemma 2, $j_n(y) \to x$, and then by Lemma 1, $j_n(z) \to x$ for every $z \in \gamma$.

If x' were a different point of $\bar{S}$, then we would have equally well that $j_n(z) \to x'$, for every $z \in \gamma$, which is an obvious contradiction.

LEMMA 13. *If $L(H)$ consists of two points, then $\bar{S}$ contains at most two points.*

Proof. The proof involves picking appropriate subsequences so that various things converge. To avoid cumbersome notation, we will call the subsequence by the same name as the original sequence.

Let z, z' be the two points of $L(H)$. We first pick a subsequence so that $j_n(z) \to x$, $j_n(z') \to x'$. Now assume that there is some point $x'' \in \bar{S}$, $x'' \neq x, x'$. Since $x'' \in \bar{S}$, there is a sequence $x_n \in \gamma_n$, with $x_n \to x''$. Since $x'' \neq x, x'$, we can pick a subsequence so that, for every n, $x_n \neq j_n(z), j_n(z')$. Then $y_n = j_n^{-1}(x_n) \in R(H)$, and so there is an $h_n \in H$, with $w_n = h_n(y_n) \in \Delta$. Observe that w_n in fact lies in $D \cap \gamma$, for γ is invariant under H. We again choose a subsequence so that $w_n \to w$. One of the hypotheses of Theorem 2 is that closure $(D \cap \gamma) \subset \mathrm{int}(E_1 \cap E_2)$, and so $w \in R(G)$.

We now have $w_n \to w \in R(G)$, and $j_n \circ h_n^{-1}(w_n) \to x''$. Hence, by Lemma 2, $j_n \circ h_n^{-1}(w) \to x''$. Now, by Lemma 3, we can choose a subsequence so that $j_n \circ h_n^{-1}(t) \to x''$, for all $t \in \Sigma$, with at most one exception. In particular, either $j_n \circ h_n^{-1}(z) \to x''$, or $j_n \circ h_n^{-1}(z') \to x''$. However, z and z' are fixed points of elements of H, and so we have that either $j_n(z) \to x''$, or $j_n(z') \to x''$. We have reached a contradiction since $j_n(z) \to x$, $j_n(z') \to x'$, and we have assumed that $x'' \neq x, x'$.

LEMMA 14. *If $L(H)$ consists of one point, then $\bar{S}$ contains at most one point.*

As in the proof of Lemma 13, the proof here consists in appropriately choosing subsequences. We again use the same notation for both the subsequence and the original sequence.

Let z be the one point of $L(H)$. We first choose a subsequence so that $j_n(z) \to x$. We now assume that there is a point $x' \neq x$ in $\bar{S}$. Then there is a sequence $x_n \in \gamma_n$ where $x_n \to x'$. Let $y_n = j_n^{-1}(x_n)$. Since $x' \neq x$, we can choose a subsequence so that $y_n \neq z$ for all n. Since $y_n \neq z$, $y_n \in R(H)$ and so there is an $h_n \in H$, with $w_n = h_n(y_n) \in \Delta$. Since $y_n \in \gamma$, $w_n \in \gamma$, and so $w_n \in D \cap \gamma$. We choose another subsequence so that $w_n \to w$. Since $w_n \in D \cap \gamma$, $w \in R(G)$. We are now in a position to apply Lemma 2, for $w_n \to w \in R(G)$, and $j_n \circ h_n^{-1}(w_n) = x_n \to x'$. Hence $j_n \circ h_n^{-1}(w) \to x'$. Since z is the fixed point of all elements of H, we also have that $j_n \circ h_n^{-1}(z) \to x$.

Now let $h \neq 1$ be some fixed element of H. We set $j_n' = j_n \circ h_n^{-1}$, $h_n = j_n' \circ h \circ (j_n')^{-1}$, $j_n'(w) = u_n$, $j_n' \circ h(w) = u_n'$, $j_n'(z) = v_n$. Then h_n is a *parabolic* Möbius transformation in G with fixed point v_n, and $h_h(u_n) = u_n'$. We also know that $v_n \to x$, $u_n \to x' \neq x$, and, by Lemma 1, $u_n' \to x'$.

We can assume, without loss of generality that none of the points x, v_n, u_n, u_n' is the point at infinity. Then each h_n can be uniquely represented by a matrix, with determinant $+1$, of the form

$$h_n \sim \begin{bmatrix} 1 + p_n v_n & -p_n v_n^2 \\ p_n & 1 - p_n v_n \end{bmatrix}.$$

Since $h_n(u_n) = u_n'$, we have

$$u_n' = [(1 + p_n v_n)u_n - p_n v_n^2][p_n u_n + 1 - p_n v_n]^{-1}.$$

Solving for p_n, we obtain

$$p_n = (u_n - u_n')(u_n - v_n)^{-1}(u_n' - v_n)^{-1}.$$

Since u_n, $u_n' \to x'$, $v_n \to x \neq x'$, we see at once that $p_n \to 0$. Then, since the v_n are all bounded, the sequence of transformations $\{h_n\}$ converges to the identity. It follows then that G is not discontinuous, and we have reached a contradiction. Hence $\bar{S}$ contains at most one point.

Lemmas 12, 13, and 14 exhaust all possible cases and show that $\bar{S}$ contains at most two points. Since $\bar{S}$ is the boundary of the intersection of a nested sequence of closed discs, it follows at once that $\bar{S} = S = \{z_0\}$ where z_0 is the point of T

with which we started. Since $j_n(z) \to z_0$ for every $z \in \gamma$, we have shown that $z_0 \in L(G)$, which is the desired conclusion. The proof of Theorem 2 is now complete.

11. It should be remarked that, in the case that H consists only of the identity, the fact that we are dealing with *Möbius transformations* enters into the proof of Theorem 2 only through Lemmas 1 and 2. These lemmas remain equally valid for groups of Möbius transformations on the n-sphere S^n. (One regards S^n as Euclidean n-space with the one point compactification, and the Möbius transformations are generated by reflections in hyperplanes, rotations, translations, and inversions in hyperspheres.) One easily sees that, for this case, the proof of Theorem 2 remains valid if one sets $\Sigma = S^n$, and γ is an embedded S^{n-1} which separates S^n into two n-discs.

As a particular application of the above remark, we mention that, again for the case that H consists only of the identity, we need not restrict ourselves to *orientation preserving* transformations of the Riemann sphere, and we can allow transformations of the form

$$z \to (a\bar{z} + b)(c\bar{z} + d)^{-1}.$$

III. An example.

12. In this chapter we give an example which shows that the hypotheses of Theorem 2 cannot be substantially weakened. In this example G_1 and G_2 are both cyclic, and H consists only of the identity.

Let G_1 be the group generated by the transformation $A: z \to 3z$. One sees at once that

$$D_1 = \{z \mid 1 \leqq |z| < 3\}$$

is a FS for G_1. For G_2 we take the group generated by the transformation $B: z \to (-2z + 5)(z - 2)^{-1}$. One easily sees that

$$D_2 = \{z \mid |z - 2| \geqq 1, |z + 2| > 1, z \neq 1\} \cup \{z = -3\}$$

is a FS for G_2.

G_1 and G_2, with FS's D_1 and D_2, satisfy some of the hypotheses of Theorem 2; $D_1 \cup D_2 = \Sigma$, and $\operatorname{int}(D_1 \cap D_2) \neq \varnothing$, but there is no Jordan curve lying in the interior of $D_1 \cap D_2$ which separates $D_1 - D_2$ and $D_2 - D_1$. We wish to show that $D = D_1 \cap D_2$ is not a FS for G, the group generated by G_1 and G_2. Since the hypotheses of Theorem 1 are satisfiied, we know that G is discontinuous, and in fact we can prove that $D \subset R(G)$, and that D is a PFS for G.

To prove that D is not a FS for G, we observe that the extended real axis R is invariant under G, and that $R \cap D = \varnothing$. Hence it suffices to show that $R \cap R(G) \neq \varnothing$.

In order to show that $R \cap R(G) \neq \varnothing$, we define new FS's D_1' and D_2' for G_1 and G_2 respectively. D_1' and D_2' are shown in Figures 1 and 2. These sets are

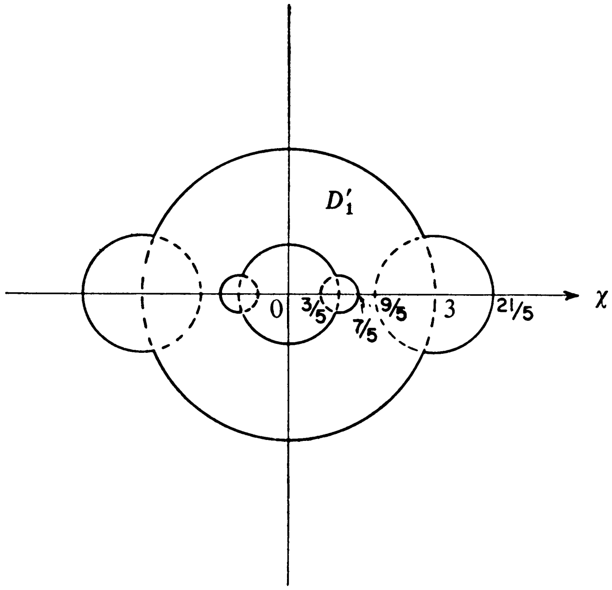

FIGURE 1

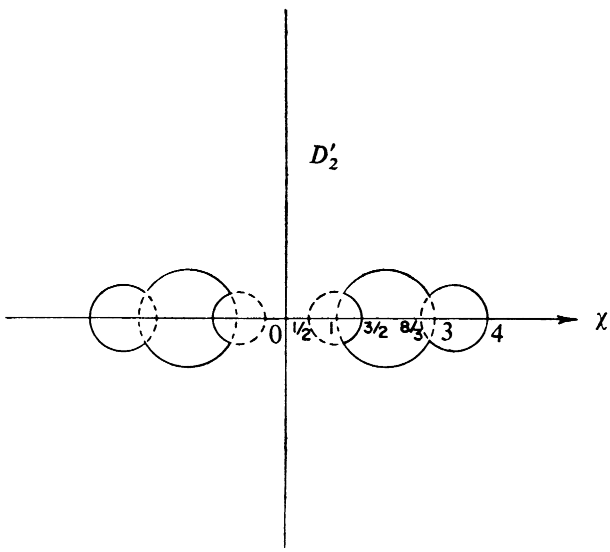

FIGURE 2

bounded by arcs of circles orthogonal to the real axis. The inner boundary of D_1' is bounded by $|z| = 1$, the circles passing through $3/5$ and $7/5$, and the circle passing through $-3/5$ and $-7/5$. The outer boundary is obtained from the inner boundary by applying the transformation A. The boundary of D_2 in the right half plane is bounded by $|z-2|=1$, the circle passing through 1/2 and 3/2, and the circle passing through 8/3 and 4. The other boundary of D'_2 is obtained by applying B to this boundary. One easily sees that $\operatorname{int}(D'_2)$ is symmetric with respect to the imaginary axis. One also easily sees that G_1 and G_2 with FS's D'_1 and D'_2 satisfy the hypotheses of Theorem 2, and that, for example, the open interval $(4, 21/5)$ lies in $\operatorname{int}(D'_1 \cap D'_2)$.

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Institute for Advanced Study,
Princeton, New Jersey