Publ. RIMS, Kyoto Univ. Ser. A
Vol. 4 (1968), pp. 373-385

On KMS Boundary Condition

By
Huzihiro ARAKI and Hideo MIYATA*

Abstract

An invariant state satisfying the Kubo-Martin-Schwinger condition is studied. It is shown that the decomposition of a given state into extremal invariant states yields states satisfying the KMS boundary condition if and only if the cyclic representation associated with the given state is η-abelian, and that, if this is the case, the decomposition coincides with the standard central decomposition. The structure of the cyclic representation when it is non η-abelian is analyzed and typical examples are given. One of the examples gives a case where the cyclic representation is G-abelian but not η-abelian.

§1. Introduction

The Gibbs ensemble in quantum statistical mechanics satisfies the Kubo-Martin-Schwinger (KMS) boundary condition and a general property of such a state has been discussed [1], [2], [3], [4]. It is known that the center of the relevant W^*-algebra is time translation invariant [4]. From this it follows that the standard central decomposition yields again invariant states satisfying the KMS boundary condition. It is then an interesting question whether any further decomposition is possible and meaningful.

There exists theorems on the possibility of a unique decomposition into extremal invariant states under various assumptions: (weakly) asymptotic abelian [5], [6], η-abelian [7], M-abelian [8], large [9], G-abelian [10].

In this paper we shall show that, for an invariant state satisfying the KMS boundary condition, a decomposition into extremal invariant states is possible and yields exclusively states satisfying the KMS

Received July 9, 1968.
* Department of Physics, Kyoto University.

boundary condition if and only if the cyclic representation associated with the given state is η-abelian. If that is the case, the decomposition coincides with the standard central decomposition.

If the restriction of the representation to the subspace of invariant vectors is abelian (G-abelian), then any decomposition finer than the central decomposition is shown to yield a state not satisfying the KMS boundary condition. If the restriction of the representation to the subspace of invariant vectors is a factor, then it must be a finite factor. If it is type I, the structure of the state is in a sense completely analyzed. It is a tensor product of two spaces, where the time translation acts only on one space in which the invariant vector is unique and on the other space the state is a trace. If the given state satisfies the KMS boundary condition, then the part of the representation algebra in the first space does not contain any time translation invariant observable. The general case is a combination of abelian and factor situations. Typical examples for the two cases are given. In particular, we have an example where the system is (non trivially) G-abelian but not η-abelian and the state is a factor state.

From the analysis of the present paper, we have the impression that the central decomposition is probably optimal "good" decomposition for states satisfying the KMS boundary condition, as was hinted in [11], footnote to proposition 4. If the stationary observables are not commutative, a further non-unique reduction is possible according to their expectation values, but this does not change the structure of time dependent part.

§ 2. Preliminaries

Let τ be a continuous representation of a locally compact group G by automorphisms of a C^*-algebra $\mathfrak{A}$ and φ be a $\tau(G)$ invariant state of $\mathfrak{A}$. Let the Hilbert space H_φ, the representation π_φ of $\mathfrak{A}$, the cyclic unit vector Ω_φ and the continuous unitary representation $U_\varphi(G)$ of G be canonically associated with φ, namely

$$(2.1)\qquad \varphi(A)=(\Omega_\varphi,\pi_\varphi(A)\Omega_\varphi)\,,\qquad A\in\mathfrak{A}$$

$$(2.2)\qquad U_\varphi(g)\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(g)A)\Omega_\varphi\,,\qquad A\in\mathfrak{A},\ g\in G\,.$$

Let E_0 be the projection on the subspace of H_φ consisting of all $U_\varphi(G)$ invariant vectors. Let

$$(2.3)\qquad R_1=(\pi_\varphi(\mathfrak{A})\cup U_\varphi(G))''$$

$$(2.4)\qquad R_2=(\pi_\varphi(\mathfrak{A})'\cup U_\varphi(G))''$$

$$(2.5)\qquad C_1=R_1'$$

$$(2.6)\qquad C_2=R_2'$$

$$(2.7)\qquad C_0=R_1'\cap R_2'.$$

The set of all $\tau(G)$ invariant central elements of $\pi(\mathfrak{A})''$ is C_0.

Theorem 2.1. The following equalities for von Neumann algebras on E_0H_φ hold:

$$(2.8)\qquad E_0(E_0\pi_\varphi(\mathfrak{A})E_0)'=(E_0R_1E_0)'E_0=R_1'E_0$$
$$=E_0\pi_\varphi(\mathfrak{A})'E_0=E_0R_2E_0$$

$$(2.9)\qquad E_0\pi_\varphi(\mathfrak{A})''E_0=E_0R_1E_0=(E_0\pi_\varphi(\mathfrak{A})'E_0)'E_0$$
$$=(E_0R_2E_0)'E_0=R_2'E_0\,.$$

Proof. Because of $U_\varphi(g)E_0=E_0$ and $U_\varphi(g)\pi_\varphi(\mathfrak{A})U_\varphi(g)^{-1}=\pi_\varphi(\mathfrak{A})$ for $g\in G$, we have

$$(2.10)\qquad E_0R_1E_0=(E_0\pi_\varphi(\mathfrak{A})''E_0)''E_0\,.$$

Since $U_\varphi(g)RU_\varphi(g)^{-1}=R$ for $R=\pi_\varphi(\mathfrak{A})$ implies the same for $R=\pi_\varphi(\mathfrak{A})'$,

$$(2.11)\qquad E_0R_2E_0=(E_0\pi_\varphi(\mathfrak{A})'E_0)''E_0\,.$$

Since E_0 belongs to R_1 and R_2, these are von Neumann algebras and

$$(2.12)\qquad (E_0R_1E_0)'E_0=R_1'E_0$$

$$(2.13)\qquad (E_0R_2E_0)'E_0=R_2'E_0\,.$$

Since R_1 contains $\pi_\varphi(\mathfrak{A})$, R_1' is contained in $\pi_\varphi(\mathfrak{A})'$ and

$$(2.14)\qquad E_0\pi_\varphi(\mathfrak{A})'E_0\supset R_1'E_0\,.$$

Finally, let $A\in\pi_\varphi(\mathfrak{A})''$, $B\in\pi_\varphi(\mathfrak{A})'$, $g\in G$. We have $U_\varphi(g)\pi_\varphi(\mathfrak{A})U_\varphi(g)^{-1}=\pi_\varphi(\mathfrak{A})$ and hence $U_\varphi(g)\pi_\varphi(\mathfrak{A})'U_\varphi(g)^{-1}=\pi_\varphi(\mathfrak{A})'$. Hence

(2.15) $$[\pi_\varphi(A), \pi_\varphi(\tau(g)B)] = 0\,.$$

Let $\psi_1, \psi_2 \in E_0 H_\varphi$. Then we have, from (2.15)

(2.16) $$(\psi_1, \pi_\varphi(A) U_\varphi(g) \pi_\varphi(B) \psi_2) = (\psi_1, \pi_\varphi(B) U_\varphi(g^{-1}) \pi_\varphi(A) \psi_2)\,.$$

Taking the Godement mean and using the mean ergodic theorem [8], we have

(2.17) $$(\psi_1, \pi_\varphi(A) E_0 \pi_\varphi(B) \psi_2) = (\psi_1, \pi_\varphi(B) E_0 \pi_\varphi(A) \psi_2)\,.$$

Hence

(2.18) $$[E_0 \pi_\varphi(A) E_0, E_0 \pi_\varphi(B) E_0] = 0\,.$$

Therefore

(2.19) $$E_0 \pi_\varphi(\mathfrak{A})' E_0 \subset (E_0 \pi_\varphi(\mathfrak{A}) E_0)'.$$

From (2.19), (2.14), (2.12) and the commutant of (2.10), we have (2.8). A similar calculation yields (2.9).

Theorem 2.2. A state φ is extremal in the set of all invariant states of $\mathfrak{A}$ if and only if R_1' is trivial (i.e. the set consisting of multiples of the identity operator).

Proof. Let R_1' contain a nontrivial projection E. Then $E\Omega_\varphi \neq c\Omega_\varphi$ for any constant c because $\pi_\varphi(\mathfrak{A})\Omega_\varphi$ is dense in H_φ and E commutes with $\pi_\varphi(\mathfrak{A})$. Now

(2.20) $$\varphi_1 = (\Omega_\varphi, \pi_\varphi(A) E \Omega_\varphi) / \|E\Omega_\varphi\|^2$$

(2.21) $$\varphi_2 = (\Omega_\varphi, \pi_\varphi(A)(1-E)\Omega_\varphi) / \|(1-E)\Omega_\varphi\|^2$$

are $\tau(G)$ invariant states of $\mathfrak{A}$ and

(2.22) $$\varphi = \varphi_1 \lambda + \varphi_2 (1-\lambda)$$

(2.23) $$1 > \lambda = \|E\Omega_\varphi\|^2 > 0\,.$$

If φ is extremal, then φ_1 must be φ, namely

$$(\Omega_\varphi, \pi_\varphi(A) E \Omega_\varphi) = (\Omega_\varphi, \pi_\varphi(A) \Omega_\varphi)\lambda\,.$$

Since $\pi_\varphi(A)^* \Omega_\varphi$ is dense in H_φ, we have $E\Omega_\varphi = \lambda \Omega_\varphi$, which is a contradiction. Hence φ is extremal only if R_1' is trivial.

Next assume

(2.24) $$\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2\,, \qquad 0<\lambda<1$$

Then

(2.25) $$\varphi_1(A) \leqq \lambda^{-1}\varphi(A)$$

for all $A \geqq 0$. Hence by a standard argument

(2.26) $$\varphi_1(A) = (\Omega_\varphi, \pi_\varphi(A)B\Omega_\varphi)$$

where $B \in \pi_\varphi(\mathfrak{A})'$, $B \geqq 0$, $\|B\| \leqq \lambda^{-1}$. If φ_1 is $\tau(G)$ invariant in addition, we have

(2.27) $$U_\varphi(g)B\Omega_\varphi = B\Omega_\varphi$$

for $g \ni G$. This implies

(2.28) $$U_\varphi(g)BU_\varphi(g)^{-1} - B$$

is 0 on Ω_φ. Since $\pi_\varphi(\mathfrak{A})\Omega_\varphi$ is dense in H_φ and (2.28) is in $\pi_\varphi(\mathfrak{A})'$, we have

(2.29) $$U_\varphi(g)BU_\varphi(g)^{-1} = B\,.$$

Namely $B \in R_1'$. If R_1' is trivial, then $B = c\ \mathbf{1}$ and hence $\varphi_1 = \varphi$. Therefore, if R_1' is trivial, φ is extremal.

§ 3. KMS Boundary Condition

In the rest of this paper, the group G is the additive group of reals.

Definition 3.1. A $\tau(G)$ invariant state φ satisfies the KMS boundary condition if for any $\tilde{f}(p)$ of the class $\mathscr{D}$,

(3.1) $$\varphi(AB(f_0)) = \varphi(B(f_\beta)A)$$

(3.2) $$f_\alpha(t) \equiv \int dp\tilde{f}(p) \exp(-ip[t+i\alpha])$$

(3.3) $$B(f) \equiv \int (\tau(t)B)f(t)dt\,.$$

Corollary 3.2. If a G invariant state φ satisfies the KMS boundary condition, then the center of $\pi_\varphi(\mathfrak{A})''$ is elementwise G invariant;

(3.4) $$U_\varphi(g)AU_\varphi(g)^{-1} = A \quad \text{if} \quad A \in \pi_\varphi(\mathfrak{A})'' \cap \pi_\varphi(\mathfrak{A})'\,.$$

Proof. See [4].

Corollary 3.3. If a G invariant state φ satisfies the KMS boundary condition, then the vector state $\hat{\varphi}$ of $\pi_\varphi(\mathfrak{A})''$ defined by Ω_φ satisfies the KMS boundary condition relative to the $\tau(G)$ defined by $U_\varphi(g)$, $g\in G$.

Proof. See [4].

Theorem 3.4. If a $\tau(G)$ invariant state φ satisfies the KMS boundary condition, then the following conclusions hold: (1) Ω_φ is a trace vector for R_1', R_2', $E_0\pi_\varphi(\mathfrak{A})''E_0$, and $E_0\pi_\varphi(\mathfrak{A})'E_0$. (2) Ω_φ is a cyclic and separating vector for $E_0\pi_\varphi(\mathfrak{A})'E_0$ and for $E_0\pi_\varphi(\mathfrak{A})''E_0$ on E_0H_φ. (3) Ω_φ is separating for R_1'.

Proof. If $B\in R_2'\subset\pi_\varphi(\mathfrak{A})''$, then $B(t)=B$. Further

$$\int f_\alpha(t)dt = 2\pi\tilde{f}(0) \tag{3.5}$$

is independent of α. Hence by the KMS boundary condition for $\hat{\varphi}$

$$\hat{\varphi}(AB) = \hat{\varphi}(BA),\quad A\in R_2',\quad B\in\pi(\mathfrak{A})'' \tag{3.6}$$

where $\hat{\varphi}$ is the normal extension of φ to $\pi_\varphi(\mathfrak{A})''$.

In particular, $\hat{\varphi}$ is a trace on R_2'. Since $E_0\in(R_2')'$, $E_0\Omega_\varphi=\Omega_\varphi$ and $E_0R_2'=E_0\pi_\varphi(\mathfrak{A})''E_0$, $\hat{\varphi}$ is a normal trace on $E_0\pi_\varphi(\mathfrak{A})''E_0$ with the trace vector Ω_φ. Since Ω_φ is cyclic for $E_0\pi_\varphi(\mathfrak{A})''E_0$, it must be separating, and it must also be a cyclic and separating trace vector for the commutant of $E_0\pi_\varphi(\mathfrak{A})''E_0$ on E_0H_φ. Hence Ω_φ is a cyclic and separating trace vector for $E_0\pi_\varphi(\mathfrak{A})'E_0=R_1'E_0$ in H_φ. (See Theorem 2.1) Since $A\in R_1'\to AE_0\in R_1'E_0$ is an isomorphism, Ω_φ is also a faithful trace vector for R_1'.

Theorem 3.5. For a $\tau(G)$ invariant state φ satisfying the KMS boundary condition, the following conditions are equivalent:

(i) R_1' is trivial,

(ii) φ is extremal among $\tau(G)$ invariant states of φ,

(iii) $\dim E_0H_\varphi=1$,

(iv) $\mathrm{M}(\pi_\varphi(\tau(t)A))=\varphi(A)\mathbf{1}$, $A\in\mathfrak{A}$ where M is the Godement mean.

The same equation holds if M is replaced by and invariant mean η.

Proof. The equivalence of (i) and (ii) is given in Theorem 2.2. If R_1' is trivial, then $E_0\pi_\varphi(\mathfrak{A})'E_0$ is trivial and has a cyclic (and separating trace) vector Ω_φ on E_0H_φ. Hence E_0H_φ must be one dimensional. Hence (i) implies (iii). Conversely, if E_0H_φ is one dimensional, E_0R_1' is trivial and hence R_1', which is isomorphic to E_0R_1', must be one dimensional. Hence (iii) implies (i). Finally we have

$$
\begin{aligned}
(3.7)\qquad & \mathrm{M}\{\varphi(B_1[\tau(t)A]B_2(f_0))\} \\
&= \mathrm{M}\{\varphi(B_2(f_\beta)B_1\tau(t)A\} \\
&= (\Omega_\varphi, \pi_\varphi(B_2(f_\beta)B_1)E_0\pi_\varphi(A)\Omega_\varphi) \\
&= \varphi(B_2(f_\beta)B_1)\varphi(A) = \varphi(B_1B_2(f_0))\varphi(A)
\end{aligned}
$$

where we have used (iii) in the last step. Hence (iii) implies (iv). Conversely, (iv) implies that $\pi_\varphi(\mathfrak{A})$ is η-abelian. It is known that if $\pi_\varphi(\mathfrak{A})$ is η-abelian, then the weak cluster property (iv) implies (i), (ii), (iii).

Theorem 3.6. Let $\mathfrak{A}$ be separable. In order that a $\tau(G)$ invariant state φ satisfying KMS boundary condition is an integral of a family of $\tau(G)$ extremal invariant states satisfying KMS boundary condition, it is necessary and sufficient that $\pi_\varphi(\mathfrak{A})$ is η-abelian. If $\pi_\varphi(\mathfrak{A})$ is η-abelian, the central decomposition of the state φ coincides with the decomposition of φ into extremal G invariant states.

Proof. If $\pi_\varphi(\mathfrak{A})$ is η-abelian, then any $\tau(G)$ invariant factor state of $\pi_\varphi(\mathfrak{A})$ is extremal. On the other hand, the KMS boundary condition implies that the center of $\pi_\varphi(\mathfrak{A})''$ commutes with $U_\varphi(t)$ and hence it is contained in R_1'. Therefore, if $\pi_\varphi(\mathfrak{A})$ is η-abelian and φ satisfies the KMS boundary condition, then φ is a factor state if and only if it is an extremal invariant state.

Now consider a general state φ, which satisfies the KMS boundary condition. Let F be any central projection and consider the new state

$$
(3.8)\qquad \varphi_F(A) \equiv (\Omega_\varphi, \pi_\varphi(A)F\Omega_\varphi)/\|F\Omega_\varphi\|^2 .
$$

Then φ_F is $\tau(G)$ invariant because F commutes with $U_\varphi(t)$. Further, since $\pi_\varphi(A)''$ satisfies the KMS condition, we have

$$\begin{aligned}\varphi_F(AB(f_0)) &= \phi(AB(f_0)F) \\ &= \phi(AFB(f_0)) = \phi(B(f_\beta)AF) \\ &= \varphi_F(B(f_\beta)A)\,.\end{aligned} \tag{3.9}$$

Hence φ_F is orthogonal to $AB(f_0)-B(f_\beta)A$ for any $\tilde{f}\in\mathcal{D}$. This implies that the factor components in the central decomposition are $\tau(G)$ invariant and satisfies the KMS condition almost everywhere.

Thus if $\pi_\varphi(\mathfrak{A})$ is η-abelian, the central decomposition of φ is (after a possible modification of measure zero components) a unique decomposition into extremal $\tau(G)$ invariant states and the resulting factor states satisfy the KMS boundary condition.

We now come to the converse. If a $\tau(G)$ invariant extremal state φ_ξ satisfies the KMS boundary condition, then $\pi_{\varphi_\xi}(\mathfrak{A})$ is η-abelian. Let h_α be an M-filter giving an invariant mean η. Then we have

$$\lim_\alpha \varphi_\xi(C_1[B,\, A(h_\alpha)]C_2) = 0 \tag{3.10}$$

for each ξ, C_1, C_2, B, $A\in\mathfrak{A}$, where

$$A(h_\alpha) = \int \tau(t)Ah_\alpha(t)dt\,. \tag{3.11}$$

Since

$$|\varphi_\xi(C_1[B,\, A(h_\alpha)]C_2)| \leq 2||C_1||\,||C_2||\,||B||\,||A|| \tag{3.12}$$

we have by a theorem on bounded convergence

$$\lim_\alpha \varphi(C_1[B,\, A(h_\alpha)]C_2) = 0 \tag{3.13}$$

for

$$\varphi = \int \varphi_\xi d\mu(\xi) \tag{3.14}$$

where μ is a nonnegative measure with the total measure 1. Hence $\pi_\varphi(\mathfrak{A})$ is η-abelian. Since

$$\varphi_\xi(A-\tau(t)A) = 0 \tag{3.15}$$

$$\varphi_\xi(AB(f_0)-B(f_\beta)A) = 0 \tag{3.16}$$

for all ξ, we have the same for φ. Hence φ satisfies the KMS boundary condition Q.E.D.

Remark. Consider a state φ which satisfies the KMS boundary condition and for which $\pi_\varphi(\mathfrak{A})$ is η-abelian.

From the KMS condition, we have

$$
\begin{aligned}
(3.17)\qquad & \mathrm{M}\varphi(C_1[B(f_0),\ \tau(t)A]C_2(f_0'))\\
&= \mathrm{M}\varphi(C_2(f_\beta')C_1B(f_0)\tau(t)A-B(f_\beta)C_2(f_\beta')C_1\tau(t)A)\\
&= \eta\varphi(\text{the same})\\
&= \eta\varphi(C_1[B(f_0),\ \tau(t)A]C_2(f_0')) = 0\,.
\end{aligned}
$$

Since $B(f_0)$, $B\in\mathfrak{A}$, $\tilde{f}\in\mathscr{D}$ is uniformly dense in $\mathfrak{A}$ and the same for $C_2(f_0')$, we see that $M\{[\pi_\varphi(A),\ \pi_\varphi(\tau(t)B)]\}=0$ for any $A, B\in\mathfrak{A}$, in weak sense.

In the course of the above proof, we have obtained

Corollary 3.7. Let $\mathfrak{A}$ be separable and φ be a $\tau(G)$ invariant state of $\mathfrak{A}$ satisfying the KMS boundary condition, the central decomposition of φ yields factor states which are $\tau(G)$ invariant and satisfies the KMS boundary condition. The same holds for any partial central decomposition (namely the diagonalization of a sub-algebra of the center).

Corollary 3.8. Let $\mathfrak{A}$ be separable and φ be a $\tau(G\times G_1)$ invariant state of $\mathfrak{A}$ which satisfies the KMS boundary condition with respect to the one parameter group G. Further assume that $\pi_\varphi(\mathfrak{A})$ is M-abelian or η-abelian for amenable group or weakly asymptotically abelian for a non compact group or large, with respect to G_1. Then the decomposition of φ into extremal G_1 invariant state yields $\tau(G)$ invariant state satisfying the KMS boundary condition with respect to G.

Proof. This follows from the previous Corollary and the known fact that R_1 is in the center of $\pi_\varphi(\mathfrak{A})''$ if $\tau_\varphi(\mathfrak{A})$ is M-abelian or η-abelian or weakly asymptotically abelian or large.

§ 4. Non η-abelian Case

We now analyze the general structure when $\pi_\varphi(\mathfrak{A})''$ is not η-abelian. By Corollary 3.7, the central decomposition always yields a $\tau(G)$ invariant state satisfying the KMS boundary condition, our problem is reduced to a factor state φ which is $\tau(G)$ invariant and satisfies the KMS boundary condition. For this case we have two steps of possible decomposition towards extremal invariant states. First step is the decomposition according to the central part $R_1'E_0 \cap R_2'E_0$. If this part is understood, we may proceed to the case where $R_1'E_0$ is a factor on E_0H_φ. These two steps will be discussed here, together with typical examples. The first step necessarily yields a state not satisfying the KMS condition. In the second case, one can find the structure more explicitly. Our example also shows a case where $\pi_\varphi(\mathfrak{A})''$ is G abelian but not η-abelian.

Theorem 4.1. Let R_a be a sub-W^*-algebra of $(R_1'E_0) \cap (R_2'E_0)$ in E_0H_φ. If R_a is not contained in CE_0 (C is the center of $\pi_\varphi(\mathfrak{A})''$) then a decomposition of φ into $\tau(G)$ invariant states diagonalizing R_a necessarily yields some states which do not satisfy the KMS boundary condition.

Proof. Let $\hat{F}$ be a projection in R_a. Since it is in $R_2'E_0$, there exists a projection F in R_2' such that $FE_0 = \hat{F}$. We note that F commutes with E_0 and $F \in \pi_\varphi(\mathfrak{A})'' \supset R_2'$. Now we assume that

$$\varphi_F(A) = (\Omega_\varphi, AF\Omega_\varphi), \qquad A \in \pi_\varphi(\mathfrak{A})'' \tag{4.1}$$

satisfies the KMS boundary condition and derive the conclusion that F commutes with any $\pi_\varphi(B)$, $B \in \mathfrak{A}$ and hence is in the center of $\pi_\varphi(\mathfrak{A})''$. (If $\varphi_F(A)$ satisfies the KMS condition for $A \in \pi_\varphi(\mathfrak{A})$, then it satisfies the same for $A \in \pi_\varphi(\mathfrak{A})''$ by Corollary 3.3.)

We have

$$\varphi_F(AB(f_0)) = \varphi_F(B(f_\beta)A) . \tag{4.2}$$

We note the previous result (eq. (3.6)) that

$$\phi(AF) = \phi(FA) , \qquad A \in \pi_\varphi(\mathfrak{A})'' \tag{4.3}$$

(4.1) implies

$$\begin{aligned} \varphi(AB(f_0)F) &= \varphi(B(f_\beta)AF) \\ &= \varphi(AFB(f_0)) . \end{aligned} \tag{4.4}$$

We set $A = A_1(f_\beta')A_2$. Then we have

$$\varphi(A_2[B(f_0), F]A_1(f_0')) = 0\,. \tag{4.5}$$

Since $\pi_\varphi(\mathfrak{A})''\Omega_\varphi$ is dense in H_φ, and $C(f_0)$ is uniformly dense in $\pi(\mathfrak{A})''$, we have the desired conclusion:

$$[B, F] = 0\,, \qquad B \in \pi_\varphi(\mathfrak{A})''. \tag{4.6}$$

Remark 4.2. After the reduction is made by diagonalizing the center of $R_1'E_0$, we obtain φ which is still a trace on $R_1'E_0$ and $R_2'E_0$. Then $R_1'E_0$ and $R_2'E_0$ are now factors of either type I_n or type II_1. For type I_n situation, we have the following theorem. We expect a similar structure for type II_1 case.

Theorem 4.3. If Ω_φ is a trace vector on factors of type I_n, $R_1'E_0$ and $R_2'E_0$, then $\pi_\varphi(\mathfrak{A})''$ have the following structure:

$$H_\varphi = H_1 \otimes H_2 \tag{4.7}$$

$$\Omega_\varphi = \Omega_1 \otimes \Omega_2 \tag{4.8}$$

$$\pi_\varphi(\mathfrak{A})'' = Q_1 \otimes Q_2 \tag{4.9}$$

$$R_1' = \mathbf{1} \otimes Q_2' \tag{4.10}$$

$$U_\varphi(t) = U(t) \otimes \mathbf{1} \tag{4.11}$$

where $U(t)$ is a continuous unitary representation of G on H_1, Ω_1 is the unique $U(t)$ invariant vector in H_1 and Ω_2 is a trace vector on Q_2 and Q_2'. If φ satisfies the KMS boundary condition, then $R_2' = 1 \otimes Q_2$, namely there is no stationary observable in Q_1.

Proof. We know that R_1' is isomorphic to $R_1'E_0$. Let E_2 be the central carrier of E_0 in R_2. Then $K = E_2R_2'$ is isomorphic to $R_2'E_0$ and is a weakly closed subalgebra of R_2'. Thus we have two type I_n factors R_1' and K which commutes with each other. Then $(R_1' \cup K)''$ is a factor of type I_{n^2} and $A \to AE_0$ is an isomorphism of $R_1' \cup K$ onto $(R_1'E_0 \cup KE_0)'' = B(E_0H_\varphi)$, where E_0 is in the commutant of $R_1' \cup K$. It is easily checked that $(R_1' \cup K)' \cap \pi_\varphi(\mathfrak{A})'' \equiv \hat{Q}_1$ have the property $\pi_\varphi(\mathfrak{A})'' = \hat{Q}_1 K$. Hence we have the structure of (4.7), (4.9), (4.10), where $1 \otimes Q_2 = K$ and $Q_1 \otimes 1 = \hat{Q}_1$. Since φ is irreducible for $(Q_2 \cup Q_2')''$, (4.8) follows. Since φ is a trace on K and R_1', Ω_2 must be a trace on Q_2 and Q_2'. Since $(K \cup R_1')''\Omega$ coincides with $E_0 H_\varphi$, Q_2 is invariant under $U_\varphi(t)$, which implies (4.11), and Ω_1 must be a uniqne invariant vector (up to a constant) on H_1.

We now show that if Ω_1 satisfies the KMS boundary condition and some A in Q_1 commutes with $U(t)$, then A must be a multiple of the identity. First, the uniqueness of invariant vector and $U(t)A\Omega_\varphi = \Omega_\varphi$ implies $A\Omega_\varphi = \lambda\Omega_\varphi$ for a scalar λ. Next we have from the KMS condition

$$(4.12) \qquad (\Omega_1, B_1 A B_2(f_0)\Omega_1) = (\Omega_1, B_2(f_\beta)B_1 A\Omega_1) = \lambda(\Omega_1, B_2(f_\beta)B_1\Omega_1) = \lambda(\Omega_1, B_1 B_2(f_0)\Omega_1).$$

Therefore, we have $A = \lambda\mathbf{1}$.

Example 4.4. (4.7)$\sim$(4.11), where Q_2 and Q_2' may be any finite factor, gives an examples, which are not G-abelian. In this case any further decomposition with respect to R_1' which is obviously non unique yields states which differ with respect to stationary observables Q_2 but which are essentially the same for the time dependent part Q_1. In general Q_1 can contain also stationary observables for which Ω_1 is necessarily an eigen state.

Example 4.5. Let I be a finite subset of reals. Let H_1 and H_2 be both $L_2(I)$, namely the Hilbert space of $\{f(x); x \in I\}$ with $(f, g) = \sum f(x)^* g(x)$. We define a basis e_x by $e_x(y) = 0$ for $x \neq y$ and $e_x(x) = 1$. We define $U(t)$ on $H_\varphi = H_1 \otimes H_2$ by $U_t(t)(e_x \otimes e_y) = e^{it(x-y)}(e_x \otimes e_y)$. $\pi_\varphi(\mathfrak{A})$ is defined as $B(H_1) \otimes \mathbf{1}$. We then see that $e_x \otimes e_x$, $x \in I$ span $E_0 H_\varphi$. An operator in $B(H_1)$ can be represented by a matrix $\mathrm{A}(x, y)$. Then R_1 consists of all $A \otimes \mathbf{1}$ for which $\mathrm{A}(x, y) = 0$ if $x \neq y$ and the $\mathrm{A}(x, x)$ are arbitrary. R_2 consists of $\mathbf{1} \otimes A$ with the same A. $R_1 E_0 = R_2 E_0$ is abelian and hence we have G-abelian property. We set

$$(4.13) \qquad \Omega_\varphi = \sum e^{-\beta x} e_x \otimes e_x$$

$$(4.14) \qquad \varphi(A) = (\Omega_\varphi, A\Omega_\varphi) = \sum_x \mathrm{A}(x, x) e^{-\beta x}.$$

Then Ω_φ is cyclic for $\pi_\varphi(\mathfrak{A})$, invariant under $U(t)$ and φ satisfies the KMS boundary condition;

$$(4.15)\qquad F_1(t) \equiv \varphi(AB(t)) = \sum_{x,y} \mathrm{A}(x, y)\mathrm{B}(y, x)e^{-\beta x}e^{it(y-x)}$$

$$(4.16)\qquad F_2(t) \equiv \varphi(B(t)A) = \sum_{xy} \mathrm{B}(x, y)\mathrm{A}(y, x)e^{-\beta x}e^{it(x-y)}$$

$$= \sum_{x'y'} \mathrm{A}(x', y')\mathrm{B}(y', x')e^{-\beta x'}e^{i(t+\beta)(y'-x')}$$

$$(4.17)\qquad F_2(t) = F_1(t+i\beta)\,.$$

However $\pi(\mathfrak{A})$ is a factor and its center is trivial. From our theorem, $\pi(\mathfrak{A})$ cannot be η-abelian. In fact

$$(4.18)\qquad \mathrm{M}\{[A, B(t)]\}(x, y) = \mathrm{A}(x, y)(\mathrm{B}(y, y) - \mathrm{B}(x, x))$$

which is non zero for some A unless E_0BE_0 is a multiple of the identity operator on E_0H.

References

[1] Haag, R., N. M. Hugenholtz and M. Winnink, Comm. Math. Phys. **5** (1966), 215–236.

[2] Hugenholtz, N. M., Comm. Math. Phys. **6** (1967), 189–193.

[3] Robinson, D., Comm. Math. Phys. **3** (1968), 337–348.

[4] Araki, H., Publ. RIMS Kyoto Univ. Ser. A, **4** (1968), 361–371.

[5] Ruelle, D., Comm. Math. Phys. **3** (1966), 133–150.

[6] Kastler, D. and D. W. Robinson, Comm. Math. Phys. **3** (1966), 151–180.

[7] Doplicher, S., R. V. Kadison, D. Kastler and D. W. Robinson, Comm. Math. Phys. **6** (1967), 101–120.

[8] Doplicher, S. and D. Kastler, Comm. Math. Phys. **7** (1968), 1–20.

[9] Størmer, E., Comm. Math. Phys. **5** (1967), 1–22.

[10] Lanford, E. and D. Ruelle, J. Math. Phys. **8** (1967), 1460–1463.

[11] Araki, H., Progr. Theoret. Phys. **32** (1964), 844–854.

[12] Winnink, M., An application of C^* algebras to quantum statistical mechanics of systems in equilibrium, Groningen Thesis, 1968.

Note added in proof:

From (3.7), it follows that h_α in the proof of Theorem 3.6. can be taken to be a sequence h_n which is $1/n$ in $[0, n]$ and 0 outside. The authors are indebted to Dr. Winnink for the following comments: Theorem 3.6. follows easily from results in [12]. Theorem 4.1 overlaps with Theorem 3.6.