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On knot Floer homology in double branched covers

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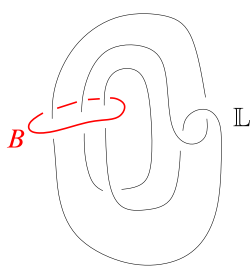
We define a link surgery spectral sequence for each knot Floer homology group for a knot, K, in a three manifold, Y. When K arises as the double cover of an unknot in S^3, and Y is the double cover of S^3 branched over a link, we relate the E^2–page to a version of Khovanov homology for links in an annulus defined by Asaeda, Przytycki and Sikora. Finally we examine the specific cases when the branch locus is a braid, and when it is alternating.

57M27; 57R58

1 Introduction

Let $A = \{z \mid 1 < |z| < 2\} \subset \mathbb{R}^2$, and let $\mathbb{L}$ be a link in $A \times [0,1] \subset \mathbb{R}^2 \times \mathbb{R}$. We will depict $\mathbb{L}$ through its projection into A along the second $\mathbb{R}$–factor. The complement of $\mathbb{L}$ in $A \times I$ is identified with the complement of $B \cup \mathbb{L}$ in S^3 where B is an unknot as depicted below, called the axis of $\mathbb{L}$. We will assume throughout that $\mathbb{L}$ intersects the spanning disc of B in an odd number of points. For example:



Let $\Sigma(\mathbb{L})$ be the double cover of S^3 branched over $\mathbb{L}$, and let $\widetilde{B}$ be the pre-image of B in $\Sigma(\mathbb{L})$. Then $\widetilde{B}$ is a null-homologous knot in $\Sigma(\mathbb{L})$ so we can try to compute its knot Floer homology groups (Ozsváth and Szabó [11])

$$\widehat{HFK}(\Sigma(\mathbb{L}), \widetilde{B}, i) = \bigoplus_{\{\mathfrak{s} \mid \langle c_1(\mathfrak{s}), [F]\rangle = 2i\}} \widehat{HFK}(\Sigma(\mathbb{L}), \widetilde{B}, \mathfrak{s})$$

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where $\underline{\mathfrak{s}}$ is a relative Spin^c structure for $\widetilde{B}$ and $[F]$ is the homology class of a pre-image of a spanning disc for B. A particularly interesting case occurs when $\mathbb{L}$ is a braid in $A\times[0,1]\cong S^1\times D^2$. Then, the pre-image of the open book of discs with binding B is an open book with binding $\widetilde{B}$.

To state our first theorem, we must adjust $\mathbb{L}$ by adding two copies of the center of A that are split from the remainder of $\mathbb{L}$, and call this new annular link $\mathbb{L}'$. Then $\Sigma(\mathbb{L}')\cong\Sigma(\mathbb{L})\,\#^2\,S^1\times S^2$ and $\widetilde{B}'$ is the knot $\widetilde{B}\,\#\,B(0,0)$ where $B(0,0)$ is the knot in $\#^2\,S^1\times S^2$ defined by a component of the Borromean rings after performing 0–surgery on the other two components. The connect sums alter the Heegaard Floer homologies in a defined manner, so we can recover the homologies for $(\Sigma(\mathbb{L}),\widetilde{B})$ from those of $(\Sigma(\mathbb{L}'),\widetilde{B}')$. We can now state:

Theorem 1.1 *Let $\mathbb{L}$ be a link in $A\times I\subset\mathbb{R}^2\times\mathbb{R}$ as above. Let $\mathbb{L}'$ be the adjusted version of $\mathbb{L}$. There is a spectral sequence whose E^2–term is isomorphic to the reduced Khovanov skein homology of the mirror, $\overline{\mathbb{L}}'$, in $A\times I$ with coefficients in $\mathbb{F}_2$, and which converges to:*

$$\bigoplus_{i\in\mathbb{Z}}\widehat{HFK}(\Sigma(\mathbb{L})\,\#^2\,(S^1\times S^2),\widetilde{B}\,\#\,B(0,0),i,\mathbb{F}_2)$$

In [13], P Ozsváth and Z Szabó constructed a spectral sequence that converged to $\widehat{HF}(Y)$ for Y a double branched cover of a link in S^3. This spectral sequence featured the reduced Khovanov homology of the mirror of the link as the E^2–term. The previous theorem is a generalization of the results in [13].

In the first half of this paper, we review the skein homology, first constructed by Asaeda, Przytycki and Sikora in [1], and examine its relationship to Khovanov homology. We then describe a spanning tree approach to computing the skein homology theory. Using the spanning tree approach, we analyze those $\mathbb{L}$ that admit an alternating projection into A. In Section 9, we prove the second main result of this paper:

Theorem 1.2 *Let $\mathbb{L}$ be a non-split alternating link in $A\times I$ intersecting the spanning disc for B in an odd number of points. Then for each k there is an isomorphism*

$$\widehat{HFK}(-\Sigma(\mathbb{L})\,\#^2\left(S^1\times S^2\right),\widetilde{B}\,\#\,B(0,0),k)\cong\bigoplus_{i,j\in\mathbb{Z}}H^{i;j,2k}(\mathbb{L})$$

where, for each Spin^c structure, the elements on the right side all have the same absolute $\mathbb{Z}/2\mathbb{Z}$–grading. Together these isomorphisms induce a filtered quasi-isomorphism from the E^2–page of the knot Floer homology spectral sequence to that of the skein *homology spectral sequence. Thus the knot Floer spectral sequence collapses after two steps. Furthermore, for any* $\mathfrak{s} \in \mathrm{Spin}^c(\Sigma(\mathbb{L}))$ *we have that*

$$\tau(\widetilde{B}, \mathfrak{s}) = 0$$

where $\widetilde{B}$ *is considered to be in* $\Sigma(\mathbb{L})$*.*

Here $\tau(K)$ is an invariant for a knot derived from knot Floer homology, which is a concordance invariant for $K \subset S^3$. In a sequel to this paper [16], we reprove the above theorem for $\mathbb{Z}$–coefficients and use it to analyze a class of fibered knots in certain three manifolds.

In the remainder of the paper, we describe the aforementioned spectral sequence and derive several of its consequences. In particular, we apply it to a question of O Plamenevskaya. In [15], O Plamenevskaya constructed a special element, $\widetilde{\psi}(\mathbb{L})$, in the Khovanov homology of a braid closure and showed that it is an invariant of the transverse isotopy class of the braid. She suggested that for certain knots, should this element survive in the spectral sequence from the Khovanov homology to $\widehat{HF}(\Sigma(\mathbb{L}))$, it would yield the Heegaard Floer contact invariant (Ozsváth and Szabó [12]) of the contact structure lifted from S^3 to the double branched cover branched over the transverse knot. Plamenevskaya's element appears naturally as a closed element in the skein homology, which provides it with a compelling interpretation. From these considerations we can supply some conditions guaranteeing that the contact invariant is trivial. This is explained in Section 8.

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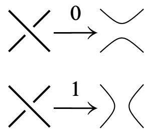
2 The reduced Khovanov skein homology of [1]

Throughout we will assume all coefficients are in $\mathbb{F}_2$. This section gives a brief description of a reduced form of the theory in [1] for categorifying the Kauffman bracket skein module for the I–bundle $A \times I$ and its relationship with the reduced Khovanov homology. We adjust the account in [1] to conform to that of Bar-Natan [3]. This alters the gradings from [1] to more directly related to Khovanov's original definition.

Let $D_{\mathbb{L}}$ be a projection of $\mathbb{L}$ into A. Pick an ordering $1, \ldots, c(D_{\mathbb{L}})$ for the crossings in $D_{\mathbb{L}}$. Let R be an element of $\{0, 1\}^{c(D_{\mathbb{L}})}$, then associate to R a collection of disjoint, simple, unoriented circles in A by resolving the crossings of P according to:



We denote the resulting diagram by $D_{\mathbb{L}}(R)$. Let $I(R)$ be

$$I(R) = \sum m_i \quad \text{where } R = (m_1, \ldots, m_{c(D_{\mathbb{L}})}).$$

Finally, call an unoriented circle resulting from the resolution *trivial* if it bounds a disc in A, and *non-trivial* if it does not.

An *enhanced Kauffman state* is a choice, R, of a resolution and a choice of symbol $\{+,-\}$ for each of the resulting circles. As usual in Khovanov homology, the enhanced states will be the generators of the chain groups. We define two bi-graded modules $V \cong \mathbb{F}v_+ \oplus \mathbb{F}v_-$ and $W \cong \mathbb{F}w_+ \oplus \mathbb{F}w_-$ where $\deg(v_+) = (1,1)$, $\deg(w_+) = (1,0)$ and $\deg(v_-) = -\deg(v_+)$, $\deg(w_-) = -\deg(w_+)$. If $D_{\mathbb{L}}(R)$ has m trivial circles and l non-trivial circles we associate to R the bi-graded module:

$$V_R(D_{\mathbb{L}}) = V^{\otimes l} \otimes W^{\otimes m}\{(I(R), 0)\}$$

We will refer to the first grading in the ordered pair as the q–grading and the second as the f–grading.

The r^{th} chain group, C_r, is then $\bigoplus_{\{R \mid I(R)=r\}} V_R(D_{\mathbb{L}})$. These will form the components of a complex, $\mathcal{C}$, and the Khovanov skein complex will be $\mathcal{C}[-n_-]\{(n_+ - 2n_-, 0)\}$ for some orientation on the link $\mathbb{L}$. The shift in $[\cdot]$ occurs in the dimension of the chain groups, whereas the shift (j,k) occurs in the bimodule gradings. This last set of shifts[1] will be called the *final* shifts. We will often only be interested in relative gradings, and so will sometimes ignore the final shifts. The complex before the final shifts will be called *unshifted*.

We now define the differential in the complex. We specify what happens to the enhanced Kauffman states when two circles merge when we change the resolution code at some crossing from 0 to 1 and what happens when a single circle divides in such a resolution. All symbols on circles unaffected by the change in resolution are likewise unaffected.

[1] We follow Bar-Natan's conventions on shifting.

This suffices to specify the differential as in [3]. The relevant maps for merging are:

$$w_+ \otimes w_+ \to w_+ \qquad v_+ \otimes v_+ \to 0$$
$$w_+ \otimes w_-,\, w_- \otimes w_+ \to w_- \qquad v_+ \otimes v_-,\, v_- \otimes v_+ \to w_-$$
$$w_- \otimes w_- \to 0 \qquad v_- \otimes v_- \to 0$$
$$v_\pm \otimes w_-,\, w_- \otimes v_\pm \to 0$$
$$w_+ \otimes v_\pm,\, v_\pm \otimes w_+ \to v_\pm$$

The relevant maps for dividing are

$$w_- \to w_- \otimes w_- \qquad v_+ \to v_+ \otimes w_- \qquad v_- \to v_- \otimes w_-$$
$$w_+ \to w_- \otimes w_+ + w_+ \otimes w_- \qquad w_+ \to v_+ \otimes v_- + v_- \otimes v_+$$

where the rule to apply to $w_\pm$ is determined by the topological type of the circles in the result (two trivial or two non-trivial circles). For any case not listed, the map is trivial. In particular, $w_- \to 0$ under division when the result of the division is two non-trivial circles. We provide a slight variation on the main result of Asaeda, Przytycki and Sikora in [1], applied to A.

Theorem 2.1 [1] *The tri-graded homology, $H^{*;**}(\mathbb{L})$, of the complex*

$$\mathcal{C}[-n_-]\{(n_+ - 2n_-, 0)\}$$

with the differential defined above is an invariant of the oriented link $\mathbb{L}$ in $A \times [0,1]$.

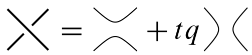
Proof Let $S(D_{\mathbb{L}})$ be the set of enhanced states and define for $S \in S(D_{\mathbb{L}})$ the following statistics:

$$\tau(S) = \#\{\text{positive trivial circles}\} - \#\{\text{negative trivial circles}\}$$
$$\Psi(S) = \#\{\text{positive non-trivial circles}\} - \#\{\text{negative non-trivial circles}\}$$
$$J(S) = I(S) + \tau(S) + \Psi(S)$$

Let $S_{i;jk}(D_{\mathbb{L}})$ be the subset of $S(D_{\mathbb{L}})$ with $I(S) = i$, $J(S) = j$, and $\Psi(S) = k$. Define $C^{i;jk}(D_{\mathbb{L}})$ to be the free Abelian group generated by $S_{i;jk}(D_{\mathbb{L}})$. It is shown in [1] that the maps above preserve the j and k values of an enhanced state, and increase i by 1. In addition, they define a differential on $\bigoplus_{i\in Z} C^{i;jk}(D_{\mathbb{L}})$ with j and k fixed (actually, this is proved with $J'(S) = I(S) + 2\,\tau(S)$, but as the differential does not change k, the proof applies here as well). The homology in [1] is invariant under the Reidemeister II and III moves. However, in an annulus and with the shifts from a choice of orientation on the link, the theory we have outlined is also invariant under the Reidemeister I move. As with translation from Viro's notation to Bar-Natan's, the shifts at the end are also necessary to pin down an invariant grading for the second Reidemeister move, but the relative graded theory is invariant regardless. □

Definition 2.2 *The grading on* $C^{*;**}$ *provided by* $I(S)$ *will be called the* homological grading*; that induced by* $J(S)$ *will be called the* quantum grading*, and that provided by* $\Psi(S)$ *will be called the* Alexander grading *(this name anticipates later sections).*

Let $\mathcal{B}(A) \cong \{0, 1, \dots, \}$ be the set of all link diagrams in A with no crossings or trivial components, identified with the number of non-trivial components. Using the rules



$$L \cup \bigcirc = (q+q^{-1})L$$

we can associate an element of $\mathbb{Z}[q^{\pm 1}, t, \mathcal{B}(A)]$ to any diagram of $\mathbb{L}$, denoted $[\mathbb{L}]$. If we map the monoid $\mathcal{B}(A)$ to $\mathbb{Z}[q^{\pm 1}, x^{\pm 1}]$ by $1 \to qx + q^{-1}x^{-1}$ we get a map $\phi\colon \mathbb{Z}[q^{\pm 1}, t, \mathcal{B}(A)] \to \mathbb{Z}[q^{\pm 1}, t, x^{\pm 1}]$. After orienting $\mathbb{L}$, let

$$V(t,q,x) = t^{n_-} q^{n_+ - 2n_-} \phi(\mathbb{L}),$$

where $n_\pm$ denotes the number of positive and negative crossings. $V(t,q,x)$ equals $\sum_{k \in \mathbb{Z}} q_{k,\mathbb{L}} x^k$ where:

$$q_{k,\mathbb{L}} = \sum_{i,j} t^i q^j \operatorname{rk}_{\mathbb{F}}\big(H^{i;jk}(\mathbb{L})\big)$$

The Euler characteristic for the skein homology is then $V(-1,q,x)$ and is an isotopy invariant of $\mathbb{L}$ in $A \times I$. On the other hand $V(-1,q,1)$ is the Jones polynomial as described by Khovanov (see also [3]).

There is also a reduced version of this theory. We mark the circle in $D_{\mathbb{L}}$ that is closest to the center, at the point intersecting the spanning disc for B. Every diagram $D_{\mathbb{L}}(R)$ inherits this marking. Note that the marked circle in the resolved diagrams may be either trivial or non-trivial. The reduced homology is then defined to be the homology of the sub-complex generated by those enhanced states with a $-$ sign on the marked circle. In the reduced theory, we will omit the grading contributions from the v_- or w_- on the marked circle. The reduced chain groups are denoted by $\widetilde{V}(D_{\mathbb{L}})$ and the overall homology by $\widetilde{H}^{i;jk}$.

Lemma 2.3 *For each* j*, there is a spectral sequence whose* E^1*–term is*

$$\bigoplus_{i,k} H^{i;jk}(\mathbb{L})$$

and that converges to $\bigoplus_i Kh^{i,j}(\mathbb{L})$*, where* $Kh^{i,j}(\mathbb{L})$ *is the usual Khovanov homology for the embedding* $\mathbb{L} \to A \times [0,1] \to S^3$*. This statement also applies to the reduced theory.*

Proof The entire construction has been performed so that by ignoring the distinction between trivial and non-trivial circles we obtain the Khovanov chain groups, ie, if we use $\mathbb{L} \to A \times [0,1] \to \mathbb{R}^2 \times [0,1]$ as an embedding of $\mathbb{L}$ in S^3 and ignore the axis. In this case we neglect the Alexander grading and treat $v_\pm$ and $w_\pm$ the same. The maps defining the differential above are almost those for the Khovanov homology, with the exception of a few terms which have been dropped. These terms are boxed below:

$$\begin{array}{ll} v_+ \to v_+ \otimes w_- + \boxed{v_- \otimes w_+} & \qquad v_+ \otimes v_+ \to \boxed{w_+} \\ v_+ \otimes w_-, w_- \otimes v_+ \to \boxed{v_-} & \qquad w_- \to \boxed{v_- \otimes v_-} \end{array}$$

Each of these terms preserves the quantum grading and increases the homological grading by 1, but decreases the Alexander grading by 2. Thus, the axis can be seen as filtering the Khovanov homology with the Alexander grading providing the filtration index, and with the E^1–term of the corresponding spectral sequence being the Khovanov skein homology. Since the maps in the spectral sequence also preserve the sub-complex used for the reduced theory, this conclusion applies to the reduced homology as well. □

Lemma 2.4 *Let $\overline{\mathbb{L}}$ be the mirror of $\mathbb{L}$. Then there is an isomorphism*

$$H^{i;jk}(\mathbb{L}) \cong H_{-i;-j,-k}(\overline{\mathbb{L}})$$

where $H_{i;jk}$ is the corresponding cohomology group. Over a field $\mathbb{F}$, the last group is also isomorphic to

$$H_{\mathbb{F}}^{-i;-j,-k}(\overline{\mathbb{L}}).$$

Furthermore, the spectral sequence converging to Khovanov homology on $H^{*;**}(\mathbb{L})$ is filtered-chain-isomorphic to that induced on the cohomology groups $H_{*;**}(\overline{\mathbb{L}})$, and, over a field, to that on $H^{*;**}(\overline{\mathbb{L}})$ through the isomorphisms above.

Proof Each state for a projection of $\mathbb{L}$ defines a state for the projection of $\overline{\mathbb{L}}$ found by reflecting all the crossings. We map a state S for $\mathbb{L}$ to the state for $\overline{\mathbb{L}}$ with the same collection of circles but with the opposite sign assigned to each circle. The 0–resolved crossings of $\mathbb{L}$ used in S are then 1–resolved in for the state for $\overline{\mathbb{L}}$, and vice-versa for the 1–resolved crossings in S. Thus, the gradings for S are mapped $i \to c(\mathbb{L}) - i$, $j \to c(\mathbb{L}) - j$, and $k \to -k$ in the unshifted theories for each link. Examining the differential between two states shows that the differential for $\overline{\mathbb{L}}$ is the differential for the cohomology of $\mathbb{L}$. Furthermore, after the final shifts we have

$$(i,j,k) \to (i - n_-, j + n_+ - 2n_-, k) \quad \text{for } \mathbb{L}$$

and $(c-i, c-j, -k) \to (c-i-n_+, c-j+n_- - 2n_+, -k)$, where n_- and n_+ refer to $\mathbb{L}$. This last triple equals $(-(i-n_-), -(j+n_+-2n_-), -k)$. For coefficients in a field, standard homological algebra implies that:

$$H^{\mathbb{F}}_{i;jk}(\overline{\mathbb{L}}) \cong H^{i;jk}_{\mathbb{F}}(\overline{\mathbb{L}})$$

A similar examination of the terms giving rise to the spectral sequence shows that these map to the terms in the spectral sequence on the cohomology. □

Since the Alexander grading filters the Khovanov complex, we can define for any element $\xi \in Kh^{i,j}(\mathbb{L})$ a number:

$$T_{\mathbb{L}}(\xi) = \min\{k\colon \xi \in \mathrm{Im}\big(H_*\big(\textstyle\bigoplus_{l \le k} C^{i;jl}\big) \to Kh^{i,j}(\mathbb{L})\big)\}$$

If $\mathbb{L}$ defines an unknot when embedded in S^3, these numbers satisfy a relation similar to one satisfied by the τ–invariant in knot Floer homology (Ozsváth and Szabó [10]), and with an almost identical proof.

Lemma 2.5 *Assume $\mathbb{L}$, considered in S^3, is an unknot and let $\overline{\mathbb{L}}$ be its mirror image. Let $\mathbf{u}_\pm$ be the generators of the Khovanov homology of the unknot in q–gradings ± 1, respectively. Then:*

$$T_{\mathbb{L}}(\mathbf{u}_{\pm 1}) = -T_{\overline{\mathbb{L}}}(\mathbf{u}_{\mp 1})$$

Proof Let $\mathcal{F}_{j;s} = \bigoplus_{i;k \le s} C^{i;jk}(\mathbb{L})$ and let $C_j = \bigoplus_{i,k} C^{i;jk}(\mathbb{L})$. Since the differential preserves the q–grading, j, there is a short exact sequence:

$$0 \longrightarrow \mathcal{F}_{j;s} \xrightarrow{I_s} C_j \xrightarrow{P_s} Q_{j;s} \longrightarrow 0$$

where $Q_{j;s}$ is the quotient complex, $C_j/\mathcal{F}_{j;s}$. Now $\bigoplus_j H_*(C_j) = \mathbb{Z}\mathbf{u}_+ \oplus \mathbb{Z}\mathbf{u}_-$, and $T_{\mathbb{L}}$ measures the first s for which the map, I_{s*}, in the corresponding homology long exact sequence, includes $\mathbf{u}_\pm$ in its image, relative to the q–grading.

There is a duality isomorphism $D\colon H^{i;j}(U) \to H_{-i;-j}(\overline{U})$, $D(\mathbf{u}_\pm) = \mathbf{u}_\mp$, on the Khovanov homologies which is induced by the symmetric pairing

$$a_+ \otimes a_- \xrightarrow{m} a_- \xrightarrow{\epsilon} 1$$

where $\epsilon\colon A \to \mathbb{Z}$ is the counit for the Frobenius algebra underlying Khovanov homology. In particular, $a_+ \to \langle a_+, \cdot\rangle = a_-^*$. This can be extended to V as well, and corresponds to changing the markers on each of the circles in an enhanced state. It thus induces a map on the skein homology spectral sequences. Checking the effect on the differential establishes the following commutative square:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_{+1,s}(U) & \xrightarrow{\;I_s\;} & C_1(U) & \xrightarrow{\;P_s\;} & \mathcal{Q}_{+1;s}(U) & \longrightarrow & 0 \\
 & & \big\downarrow D & & \big\downarrow D & & \big\downarrow D & & \\
0 & \longrightarrow & \mathcal{Q}^*_{-1;-s-1}(\overline{U}) & \xrightarrow{\;P^*_{-s-1}\;} & C^*_{-1}(\overline{U}) & \xrightarrow{\;I^*_{-s-1}\;} & \mathcal{F}^*_{-1,-s-1}(\overline{U}) & \longrightarrow & 0
\end{array}$$

If $\mathbf{u}_+$ is in the image of $I_{s,*}$, then $\mathbf{u}_-^*$ is in the image of P^*_{-s-1}. In particular, $I^*_{-s-1}(\mathbf{u}_-^*) = 0$. But then there is no element in $\mathcal{F}_{-1,-s-1}$ that maps to $\mathbf{u}_-$ and so $-s-1 < T_{\overline{\mathbb{L}}}(\mathbf{u}_-)$. One such choice of s is $s = T_{\mathbb{L}}(\mathbf{u}_+)$, from which we conclude that $-T_{\mathbb{L}}(\mathbf{u}_+) \leq T_{\overline{\mathbb{L}}}(\mathbf{u}_-)$. If $\mathbf{u}_+$ is not in the image of $I_{s,*}$—ie, $s < T_{\mathbb{L}}(\mathbf{u}_+)$—then $I^*_{-s-1}(\mathbf{u}_-^*) \neq 0$. Choose some element on which this image pairs non-trivially and is uniformly in q–grading -1. This element must then map in homology to $\mathbf{u}_-$ and $-s-1 \geq T_{\overline{\mathbb{L}}}(\mathbf{u}_-)$ for all such s. In particular, we may take $s = T_{\mathbb{L}}(\mathbf{u}_+)-1$ and conclude that $-T_{\mathbb{L}}(\mathbf{u}_+) \geq T_{\overline{\mathbb{L}}}(\mathbf{u}_-)$. These two inequalities imply that $-T_{\mathbb{L}}(\mathbf{u}_+) = T_{\overline{\mathbb{L}}}(\mathbf{u}_-)$. The same argument with $j = -1$ handles the other case. □

Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two links in $A \times [0,1]$. Let $\mathbb{L} = \mathbb{L}_1 | \mathbb{L}_2$ be the link in $A \times [0,1]$ where $A = \{z : 1 \leq |z| \leq 3\}$ and $\mathbb{L}_1$ lies in $A_1 = \{z : 1 \leq |z| \leq 2\} \times [0,1]$ while $\mathbb{L}_2$ lies in $A_2 = \{z : 2 \leq |z| \leq 3\} \times [0,1]$. Then we can prove:

Lemma 2.6 *With coefficients in a field, $\mathbb{F}$, there is an isomorphism:*

$$H^{i;jk}(\mathbb{L}) \cong \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j \\ k_1+k_2=k}} H^{i_1;j_1k_1}(\mathbb{L}_1) \otimes H^{i_2;j_2k_2}(\mathbb{L}_2)$$

Furthermore, if $\xi_1 \in Kh^{i_1;j_1}(\mathbb{L}_1)$ and $\xi_2 \in Kh^{i_1;j_1}(\mathbb{L}_1)$ then:

$$T_{\mathbb{L}}(\xi_1 \otimes \xi_2) = T_{\mathbb{L}_1}(\xi_1) + T_{\mathbb{L}_2}(\xi_2)$$

Proof The states $S_{i;jk}(L_1|L_2)$ decompose according to their projections into A_1 and A_2. Consequently, the chain group for the projection of $\mathbb{L}$ is a tensor product of chain groups, and the various indices add but are otherwise independent. This applies also to the differential, where the terms in the differential decompose into a sum of those induced by resolution changed in A_1 and those induced by resolution changes in A_2. Consequently, the complexes are tensor product complexes whose homology, over a field, is as described by the Künneth formula above. The last statement follows by noting that the same conclusions apply to the original Khovanov homology in this setting. Thus, to have $\xi_1 \otimes \xi_2$ arising in a filtration level for the first time requires that each factor also arises in some corresponding summand $F_{s_1}(L_1) \otimes F_{s_2}(L_2)$. □

Finally, if there is a non-trivial component, L_1, split from the rest of $\mathbb{L}$, it can be made to lie in the diagram without crossing any other strand of $\mathbb{L}$. L_1 survives unchanged in every resolution; thus, marking it induces a marking on a non-trivial circle for every resolution. When the number of intersections of $\mathbb{L}$ with the spanning disc for B is odd, then the reduced skein homology of this configuration has the form $\widetilde{H}(\mathbb{L} - L_1) \otimes V$. This choice shifts the complex by $\{(-1,-1)\}$, so we will always shift at the end to compensate. Thus, the final shift will be $[-n_-]\{(n_+ - 2n_- + 1, +1)\}$ for this marking convention. For the mirror of $\mathbb{L}$, the isomorphisms above will then map this marking to a $+$; however, as the component does not interact with the rest of the diagram we can change it to a $-$ with the only change being how we perform the final shifts.

3 Spanning tree complex

As with Khovanov homology, the skein homology for links in $A \times I$ with connected projections admits another presentation in terms of the spanning trees for the knot diagram. We follow S Wehrli's argument [19] for Khovanov homology in establishing this result, but see also Champanerkar and Kofman [4] for an alternate approach.

Informally, Wehrli tells us to take a diagram $D_{\mathbb{L}}$ and enumerates its crossings $C_{\mathbb{L}} = \{c_1, \ldots, c_n\}$. We then proceed to resolve the crossings in order, in both possible ways, skipping those crossings for which one or other resolution results in a disconnected diagram, but resolving in both ways those for which both the resolutions are connected. These are grouped into a tree by the different choices of resolution at each of the crossings. For an example see [19].

More formally, Wehrli's algorithm results in a rooted binary tree of diagrams resulting from resolving a subset of the crossings of $\mathbb{L}$. We now give a precise description of this tree. To describe a binary, rooted tree we specify the value at the root, r, of the tree and then two rooted sub-trees: the L–tree and R–tree, whose roots are joined to r by its two edges. To do this for $D_{\mathbb{L}}$ we start with the projection, D, of some link, sitting in A, equipped with an ordering subset of its crossings, $C = \{c_{i_1}, \ldots, c_{i_m}\}$ with $i_1 < i_2 < \cdots < i_m$. We assume that D is connected as a 4–valent graph. From this data Wehrli's algorithm produces a rooted, binary tree, $T(D, C)$. The root of $T(D, C)$ is the diagram D. We then find the first crossing $c_{i_k} \in C$ for which both resolutions of D, $D_0(c_{i_k})$ and $D_1(c_{i_k})$, are connected as 4–valent graphs. The L–tree is the tree $T(D_0(c_{i_k}), \{c_{i_{k+1}}, \ldots, c_{i_n}\})$, while the R–tree is $T(D_1(c_{i_k}), \{c_{i_{k+1}}, \ldots, c_{i_n}\})$.

When there is no c_{i_k} for which both resolutions are connected (as 4–valent graphs), then the tree consists solely of the root D. Take note that we drop from the data all those crossings where one or other resolution is disconnected. We then recursively apply this recipe to the new data. Since C is finite, $T(D,C)$ will be a finite tree.

As with any rooted binary tree, the leaves of $T(D,C)$ are well-ordered. Each leaf corresponds to a sequence of L's and R's that describe the path from the root to that leaf. The leaves are ordered by using the lexicographic ordering on these sequences induced by asserting $L < R$.

Definition 3.1 *The resolution tree of $D_{\mathbb{L}}$ is the tree $T(D_{\mathbb{L}}, C_{\mathbb{L}})$ where $C_{\mathbb{L}}$ is* all *the crossings in the diagram $D_{\mathbb{L}}$.*

Following [19], we can describe some properties of the leaves of $T(D_{\mathbb{L}}, C_{\mathbb{L}})$. They are diagrams D that arise from resolving some of the crossings of $D_{\mathbb{L}}$ and which satisfy the following:

(1) The diagram D is connected, ie, the underlying 4–valent graph in A is a connected graph.
(2) For any crossing c in D one of the resolved diagrams $D_0(c)$ or $D_1(c)$ is disconnected.

It is shown in Appendix B that these diagrams correspond to unknots, and the diagram can be simplified in $\mathbb{R}^2$ to a standard unknot diagram using only the first Reidemeister move (this is implicit in [19], but we will need the slight generalization that is proven in the appendix). We will call these twisted unknots. Furthermore, due to the connectedness properties, for each leaf diagram there is a unique way to smooth each of the remaining crossings to get an unknot embedded in the plane. In addition, given a smoothing, S, of all the crossings in $D_{\mathbb{L}}$, which produces a single circle, there is a unique leaf D_S in $T(D_{\mathbb{L}}, C_{\mathbb{L}})$ which smooths to it: namely, start at the root $D_{\mathbb{L}}$ and look at how S smooths the crossing used to form the L and R–trees, then follow the branch corresponding to the resolution in S. We repeat this process at each node of the tree until we come to a leaf. Following [19], let $K_1(D_{\mathbb{L}})$ be the set of those smoothings of all the crossings of $D_{\mathbb{L}}$ with only one component, then $S \leftrightarrow D_S$ is a one-to-one correspondence between the elements of $K_1(D_{\mathbb{L}})$ and the leaves of $T(D_{\mathbb{L}}, C_{\mathbb{L}})$. Finally, let $r(D, D')$ be the number of 1–resolutions required to smooth crossings in a diagram D to obtain the diagram D'. For instance $r(D_{\mathbb{L}}, S)$ is the number required to smooth all the crossings of $D_{\mathbb{L}}$ to obtain $S \in K_1$ and $r(D_{\mathbb{L}}, D_S)$ is the number required to smooth to the leaf D_S, that is, the number of R–branches used in the binary tree.

Note Below the twisted unknot D_S will be assumed to have an orientation, chosen arbitrarily, so that the final shifts for the skein homology are defined.

When considered in A, the diagram D_S may not be able to be simplified to a standard unknot diagram, due to the unavailability of RI–moves which cross outside the annulus (ie, the twisted unknot's linking with B prevents some RI–moves). However, we may use the first Reidemeister move alone to simplify D_S to D'_S, a special twisted unknot, where all the remaining twisting links with B (see Appendix B). D'_S is a diagram easily reducible to one of the special form in Figure 1, where each n_i records the number of half twists in each twisting region, using only RII–moves to remove opposite crossings in the horizontal twist regions. We can thus reduce to the special twisted diagrams in Figure 1. The homologies of these knots will form the building blocks of the spanning tree complex for the Khovanov skein homology. Finally, note that since we require the number of intersections with the spanning disc to be odd, the diagram D'_S must be non-trivial, and thus link B.

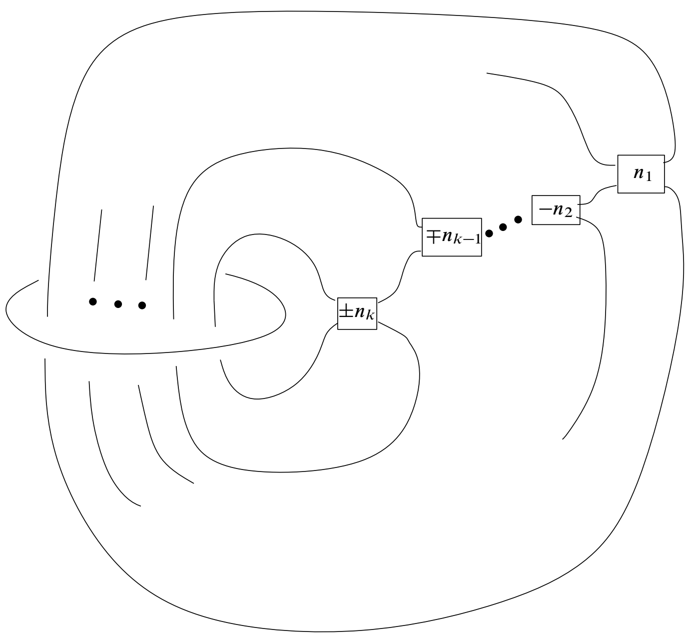


Figure 1: The special class of unknots that act as the base cases for the spanning tree complex

In simplifying from D_S to D'_S, we employ the first Reidemeister move. The *unshifted* skein complexes behave in the following way under an RI–move that does not escape A:

$$\mathcal{C}(\not\!\bigcirc) \cong \mathcal{C}(\supset)\{(-1,0)\} \oplus B_1 \qquad \mathcal{C}(\times\!\!\!\bigcirc) \cong \mathcal{C}(\supset)[1]\{(2,0)\} \oplus B_2$$

where B_1 and B_2 are contractible. It does not matter whether the RI–move involves trivial or non-trivial components in the complete smoothings. The shifts can be computed from the invariance of the theory after the final shifts are performed. For the first RI–move above, the left side would need to be shifted $[0]\{(1,0)\}$ further than the right side, due to the extra positive crossing. Thus the right side should be shifted $[0]\{(-1,0)\}$ to correspond to the left side in the unshifted complex. A similar argument applies to the second RI–move depicted. In addition, if D' is obtained from D by using an RII–move to remove two crossings, then

$$\overline{C}^{*;**}(D) \cong \overline{C}^{*;**}(D')[1]\{(1,0)\}$$

since one positive and one negative crossing have been removed.

With this observation we may proceed analogously to [19] to obtain the following proposition:

Lemma 3.2 (cf [19]) *Let $\mathbb{L} \subset A \times I$ have a connected diagram, D, in A. Then there is a decomposition $\overline{C} \cong \mathcal{A} \oplus \mathcal{B}$ where $\overline{C}$ is the unshifted version of the skein complex, $\mathcal{B}$ is contractible and $\mathcal{A}$ is given by*

$$\bigoplus_{S \in K_1(D)} H^{*;**}(D_S)[-w(D_S)]\{(-2w(D_S),0)\}[r(D,S)]\{(r(D,S),0)\}$$

where $w(D_s)$ is the writhe of D_s, and $r(D,S)$ is the number of 1–smoothings necessary in resolving the diagram for $\mathbb{L}$ to get the complete resolution S.

Proof (cf [19]) For any crossing c, the unshifted chain complex $\overline{C}^{*;j*}(D)$ is isomorphic to a mapping cone $\mathrm{MC}(\overline{C}^{*;j*}(D_0) \to \overline{C}^{*;j*}(D_1)[1]\{(1,0)\})$. Applying this to the crossings used to build the resolution tree constructs a filtered complex out of iterated mapping cones, filtered by the L, R–binary structure: ie, those leaves which have an L earlier in the resolution process occur higher in the filtration. We will call this the tree filtration. At this stage, the skein homology complexes have the structure of

$$\bigoplus_{S \in K_1(D)} \overline{C}^{*;**}(D_S)[r(D,D_S)]\{(r(D,D_S),0)\}$$

where D_S is one of the leaf diagrams in the resolution tree. Suppose we have to perform n_{II} RII, $n_{I,+}$ positive RI–moves, and $n_{I,-}$ negative RI–moves to get to one of the diagrams in Figure 1.

We now use the formulas for RI and RII moves noted above:

$$\begin{aligned}&\overline{C}^{*;**}(D_S)\\ &\qquad\cong \overline{C}^{*;**}(D'_S)[n_-(D_S,D'_S)+n_{II}]\{((-n_+ + 2n_-)(D_S,D'_S)+n_{II},0)\} \oplus B_S\end{aligned}$$

where B_S is contractible and $n_\pm(D_S, D'_S)$ is the number of positive/negative crossings in D_S lost in simplifying to D'_S. The final shifts show that:

$$\overline{C}^{*;**}(D'_S) \cong C^{*;**}(D'_S)[n_-(D'_S)]\{((-n_+ + 2n_-)(D'_S), 0)\}$$

Over $\mathbb{Z}/2\mathbb{Z}$, the mapping cones will "commute" with direct sums with contractible complexes (see [19]), so the tree filtered chain complex contracts to a complex with underlying groups

$$\bigoplus_{S\in K_1(D)} C^{*;**}(D'_S)[n_-(D_S)]\{((-n_+ + 2n_-)(D_S), 0)\}[r(D, D_S)]\{(r(D, D_S), 0)\}$$

where we compose the shifting and use that each negative crossing in D_S is either removed by an RI–move, an RII–move, or is included in the final shifts for D'_S and likewise for positive crossings. However, $r(D, D_S) = r(D, S) - n_+(D_S)$, since in order to get a connected complete resolution we must resolve through the crossings, which gives a 1 for the positive twists and a 0–resolution for the negative twists. So $r(D, D_S) + n_-(D_S) = r(D, S) - w(D_S)$ and $r(D, D_S) - n_+ + 2n_- = r(D, S) - 2w(D_S)$, which correspond to the shifts in the statement of the lemma. Taking the E^0–page for the tree filtration yields a chain complex with underlying chain groups

$$\bigoplus_{S\in K_1(D)} H^{*;**}(D'_S)[-w(D_S)]\{(-2w(D_S), 0)\}[r(D, S)]\{(r(D, S), 0)\}$$

Of course, after the final shifts, the homology $H^{*;**}(D_S) \cong H^{*;**}(D'_S)$ since these two represent links in $A \times I$ which are isotopic. Note that this also applies if we think of the k–gradings as a filtration on the reduced Khovanov complex; we then have a bifiltered complex since the contractions do not change the k–grading. □

We can say a little more concerning the unknots in Figure 1. Since these are unknots, their Khovanov homologies are composed of $\mathbb{F}\mathbf{u}_+$ in homological and q–grading $(0, 1)$ and $\mathbb{F}\mathbf{u}_-$ in $(0, -1)$. We compute the numbers, $T_{\mathbb{L}}(\mathbf{u}_{\pm 1})$, for these unknots.

Proposition 3.3 *For the special twisted unknots with diagrams $D_{\mathbb{L}}$ in Figure 1, let $T_\pm$ be the number of left/right-handed twist regions in Figure 1, and let $T(D_{\mathbb{L}})$ denote $T_- - T_+$. Then:*

$$T_{\mathbb{L}}(\mathbf{u}_{\pm 1}) = T(D_{\mathbb{L}}) \pm 1$$

For the alternating unknots in this family, $T_{\mathbb{L}}(\mathbf{u}_{\pm 1}) = \pm 1$.

Proof These unknots are isotopic to the standard planar unknot using only RI–moves. M Jacobsson provides rules for mapping closed elements in the Khovanov cube of a

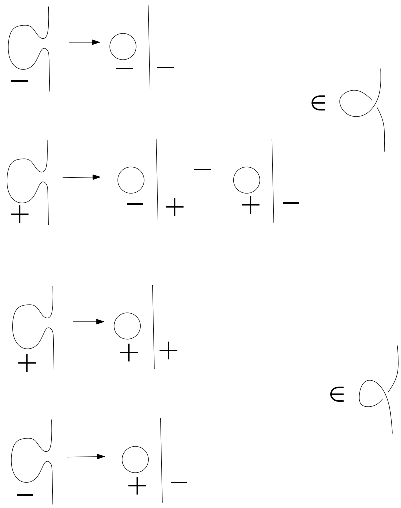


Figure 2: Rules for transferring generators when an RI move is applied: The particular twist is represented on the far right. Note that these maps are chain maps inducing isomorphisms on the Khovanov homologies [7].

link to those of the link with a single RI–move, which in our notation are as in Figure 2, which induce isomorphisms on the Khovanov homology. We can use these moves to try to compute $T_{\mathbb{L}}(\mathbf{u}_{\pm 1})$. As a first step, we exhibit a specific generator which will produce $\mathbf{u}_{\pm 1}$ in homology. The maximal value of k needed to obtain this generator in $\bigoplus_{l \leq k} C^{i;jk}$ will then be an upper bound on $T_{\mathbb{L}}(\mathbf{u}_{\pm 1})$.

We will proceed by induction. If $\mathbb{L}$ has a diagram, $D_{\mathbb{L}}$, in A given by a single embedded core circle, then the theorem is true, since $T_{\pm} = 0$. Assume that $D_{\mathbb{L}}$ is a diagram as in Figure 1 and that $T_{\mathbb{L}}(\mathbf{u}_{\pm}) = T(D_{\mathbb{L}}) \pm 1$ as in the conclusion of the proposition. We will use the moves in Figure 2 to show that the conclusion of the theorem is also true if we add a new twisted band in the innermost region of L, whose final loop goes around the core of the annulus; see Figure 3. Since $\mathbb{L}$ has $T_{\mathbb{L}}(\mathbf{u}_{\pm}) = T(\mathbb{L}) \pm 1$, there are linear combinations of generators for $\mathbf{u}_{\pm}$ in filtration levels $\leq T(\mathbb{L}) \pm 1$ with some element in filtration level $T(\mathbb{L}) \pm 1$ that are closed and generate $\mathbf{u}_{\pm}$ in the Khovanov homology. Consider the enhanced Kauffman states for the generators in these linear combinations. The arc which will be twisted to form the band lies on a circle in one of these states. This circle can be decorated with either a $+$ or a $-$, and the twisting can either be right-handed or left-handed. We consider each of these cases.

Right-handed twisting For right-handed twisted bands we use the lower pair of rules in Figure 2. Given a (simplified) linear combination of enhanced Kauffman states

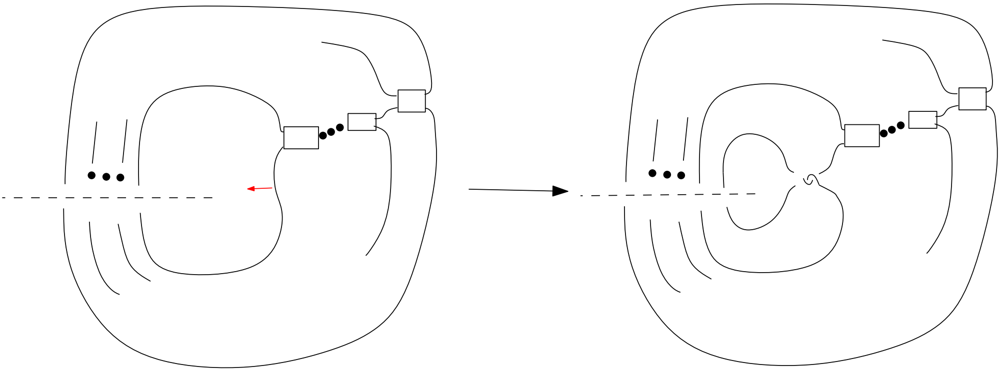


Figure 3: The new twisted band will occur in the direction of the small red arrow on the left. It will appear as in the right with some twisting. The dashed line indicates where the unknot defining the annulus would go. We will twist as if the diagram does not sit in an annulus, and then place the annulus in after we have finished the twisting.

γ in the Khovanov complex for $D_{\mathbb{L}}$, we divide $\gamma = \gamma^+ + \gamma^-$ into those state with $\pm$ decorations on the innermost non-trivial circle. Let P be the map given by the following rules: For each state we add a circle with a $+$. Thus $P(\gamma^{\pm}) = \gamma^{\pm} \otimes c^+$ where we have extended linearly. For each twist, we apply P until the last, where the circle will be non-trivial. Thus we obtain for the image of γ a linear combination $\gamma \otimes c_1^+ \otimes \cdots \otimes c_{n_- - 1}^+ \otimes d^+$, where d represents a non-trivial circle. Since the last twist produces a non-trivial circle in A, its decoration alters the Alexander grading by adding 1. In other words, if the enhanced Kauffman state for $D_{\mathbb{L}}$ was in Alexander grading k, then that obtained from our rules for $D_{\mathbb{L}'}$ will be in Alexander grading $k+1$. Applying this to the summands of γ produces a linear combination representing $\mathbf{u}_{\pm}$ for $D'_{\mathbb{L}}$ with highest Alexander grading terms in $T(D_{\mathbb{L}}) \pm 1 + 1$. As we have added a new right-handed twist region, we can conclude that $T(\mathbb{L}') = T(\mathbb{L}) + 1$, and hence that $T_{\mathbb{L}'}(\mathbf{u}_{\pm}) \leq T(\mathbb{L}') \pm 1$. Note that no cancellation occurs in the new linear combination since the decorations on the circles in γ are unaltered. If there was no cancellation in γ, there will be none in $P^k(\gamma)$.

Left-handed twisting For left-handed twisted bands, we look at the two cases where we are extending a $(+)$–marker or a $(-)$–marker. For a $(-)$–marker, the same argument applies as in the right handed twisting case, $P_-^k(\gamma) = \gamma \otimes c^- \otimes \cdots \otimes d^-$, except the top diagram shows that each new circle will receive a $(-)$–marker. Thus, the last core circle in the new enhanced Kauffman states will receive a $(-)$–marker and the new states in our linear combination will occur in Alexander grading decreased by 1.

However, for a $(+)$–marker the story is more complicated:

$$P_+^k(\mathbf{x}_+) = \mathbf{x}_- \otimes P_+^{k-1}(c^+) + \mathbf{x} \otimes P_-^{k-1}(c^-) = \mathbf{x}_- \otimes P_+^{k-1}(c^+) + \mathbf{x}_+ \otimes c_1^- \otimes \cdots \otimes d^-$$

where we use $\mathbf{x}_\pm$ to represent the state with decoration $\pm$ on the innermost non-trivial circle. Expanding the P_+ operator requires a linear combination, but we will obtain:

$$P_+^k(\mathbf{x}_+) = \mathbf{x}_- \otimes c_1^- \otimes \cdots \otimes c_i^- \otimes \cdots \otimes d^+ \\ + \sum_{i=1}^{k-1} \mathbf{x}_- \otimes c_1^- \otimes \cdots \otimes c_i^+ \otimes \cdots \otimes d^- + \mathbf{x}_+ \otimes c_1^- \otimes \cdots \otimes d^-$$

If $\mathbf{x}_+$ is in Alexander grading k, then the first and last term are in grading $k-1$, whereas the middle sum consists of terms in grading $k-2$. When we apply this to $\gamma = \gamma^-$, we shift every term from Alexander grading k to Alexander grading $k-1$. Since there is some term with grading $T(D_{\mathbb{L}}) \pm 1$ we obtain some term in $P^k(\gamma)$ with grading $T(D_{\mathbb{L}}) \pm 1 - 1 = T(D_{\mathbb{L}'}) \pm 1$. If $\gamma^+ \neq 0$, then there is the possibility of some cancellation: however, the first terms of P^k applied to each summand in γ^+ has a $+$ on d and thus can't cancel with any other term in the sums above or from γ^-. If the term in γ in Alexander grading $T(D_{\mathbb{L}}) \pm 1$ occurs only in γ^- then all the terms in $P^k(\gamma^+)$ occur in Alexander grading $< T(D_{\mathbb{L}}) \pm 1 - 1$, whereas if there is a term in γ^+ in Alexander grading $T(D_{\mathbb{L}}) \pm 1$ we have constructed a non-canceling term in $P^k(\gamma^+)$ in grading $T(D_{\mathbb{L}}) \pm 1 - 1$. In either case, $P^k(\gamma)$ when simplified has summands in Alexander grading $T(D_{\mathbb{L}}) \pm 1 - 1 = T(D_{\mathbb{L}'}) \pm 1$ and lower. Consequently, all the terms in a closed linear combination of generators for $D_{\mathbb{L}'}$ representing $\mathbf{u}_\pm$ in the homology occur in Alexander gradings $T(D_{\mathbb{L}'}) \pm 1$. Therefore, $T_{\mathbb{L}'}(\mathbf{u}_\pm) \leq T(D_{\mathbb{L}'}) \pm 1$.

In every case $T_{\mathbb{L}'}(\mathbf{u}_{\pm 1}) \leq T_-(D_{\mathbb{L}'}) - T_+(D_{\mathbb{L}'}) \pm 1$. However, the argument also applies to $\overline{\mathbb{L}'}$ and we know that $T_{\overline{\mathbb{L}'}}(\mathbf{u}_{\mp 1}) = -T_{\mathbb{L}'}(\mathbf{u}_{\pm 1})$. In the mirror image there are $T_+(D_{\mathbb{L}'})$ left-handed regions and $T_-(D_{\mathbb{L}'})$ right-handed regions. Hence, $T_{\overline{\mathbb{L}'}}(\mathbf{u}_{\mp 1}) \leq T_+(D_{\mathbb{L}'}) - T_-(D_{\mathbb{L}'}) \mp 1$. Replacing the left side with $-T_{\mathbb{L}}(\mathbf{u}_{\pm 1})$ gives $T_{\mathbb{L}}(\mathbf{u}_{\pm 1}) \geq T_-(D_{\mathbb{L}'}) - T_+(D_{\mathbb{L}'}) \pm 1$, and the result follows. The final statement is simply a reflection of the even number of twist regions, alternating between handedness, when there are an odd number of strands. □

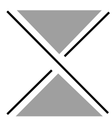
4 Results for the skein homology of alternating links

The goal of this section is to use the spanning tree presentation of the skein homology to prove the following theorem:

Theorem 4.1 *Let $\mathbb{L}$ be an alternating link in $A \times I$ intersecting the spanning disc for B in an odd number of points. Then the Khovanov skein homology $H^{i;jk}(\mathbb{L})$ is trivial unless $k - j + 2i = \sigma(\mathbb{L})$. Thus the homology is determined by the Euler characteristic $V(-1, q, x) = (-1)^{n_-} q^{n_+ - 2n_-} \phi([\mathbb{L}])$, defined in Section 1, and the signature of the oriented link $\sigma(\mathbb{L})$, thought of as embedded in S^3.*

We will follow [19] in calculating the Khovanov-type homology of an alternating configuration. In [19] spanning trees are used to provide a simplified proof of E S Lee's result concerning alternating links, [8]. This theorem describes the result of computing the spectral sequence for the axis filtration: the homology will be supported on the lines $j - 2i = -\sigma(L) \pm 1$. It is towards a variation of this result that we now aim.

Assume that $\mathbb{L}$ admits an alternating projection to $A = \{z \big| 1 < |z| < 2\} \subset \mathbb{C}$ which is connected as a subset of A. Let B be the intersection of $A \times I \subset \mathbb{C} \times \mathbb{R}$ with the half-plane $\{(z, t) \mid \arg z = \pi\}$. We maintain the assumption that $\mathbb{L}$ intersects the spanning disc for B in an odd number of points; however, we will relax this when it is to our advantage. We will bi-color the *plane* according to the following convention:



For any $\mathbb{L}$, regardless of the parity of intersecting the spanning disc, we define $M(\mathbb{L})$ to be the number $N_W - N_B$ where N_W is the number arcs in the projection of B to A coming from intersection with the white regions and N_B is the number of arcs coming from intersection with the black regions. When $\mathbb{L}$ intersects the spanning disc in an odd number of points, $M(\mathbb{L}) = 0$; for an even number of points, $M(\mathbb{L}) = \pm 1$. This number does not change under Reidemeister moves applied to $\mathbb{L}$, nor does it change when crossings of $\mathbb{L}$ are resolved. Furthermore, all the projections, D_S, in the spanning tree complex will be alternating. We start with a lemma concerning the twisted unknots in Figure 1.

Lemma 4.2 *For each alternating twisted unknot in Figure 1 the non-zero homology groups $H^{i;jk}(\mathbb{L})$ are supported on $k - j + 2i = M(D_S)$.*

We will show that diagrams of the special form above have the property that $H^{i;jk}$ satisfies $k - j + 2i = M(D_S)$, and that the last number is determined by the type of crossing on the outermost boundary. We start with the following cases:

(1) $\mathbb{L}$ as a single non-trivial unknot has this property. Its homology is 0 unless $(i; j, k) = \pm(0; 1, 1)$, and those have homology $\mathbb{F}$. But then $k - j + 2i = 0 = M$ since there is one black and one white region.

(2) If $H^{i;jk}(D) \not\cong 0$ implies $k-j+2i = C$, where C is a constant, then $D \cup N$, where N is a disjoint non-trivial circle, has $H^{i;jk}(D \cup N) \not\cong 0$ when $k-j+2i = C$. This follows from $H^{i;jk}(D \cup N) \cong H^{i;jk}(D) \otimes V$.

(3) The closures of braid generators $\sigma_1 \in B_2$ and $\sigma_1^{-1} \in B_2$, where σ_1 gives a positive crossing, have the property that $k-j+2i = M(D)$. This requires a computation. For σ_1^{-1} the shifted complex has homology

$$H^{i;jk} \cong \begin{cases} \mathbb{F}_{-1} & \text{if } (j,k) = (-3,0) \\ \mathbb{F}_0 & \text{if } (j,k) = (-3,-2), (-1,0), (1,2) \end{cases}$$

where the subscript denotes i, and each element has $k-j+2i = +1$. Furthermore, $N_W = 2$ and $N_B = 1$, so $M(D) = 1$. For the closure of σ_1 we obtain:

$$H^{i;jk} \cong \begin{cases} \mathbb{F}_1 & \text{if } (j,k) = (3,0) \\ \mathbb{F}_0 & \text{if } (j,k) = (-1,-2), (1,0), (3,2) \end{cases}$$

and $k-j+2i = -1 = M(D)$.

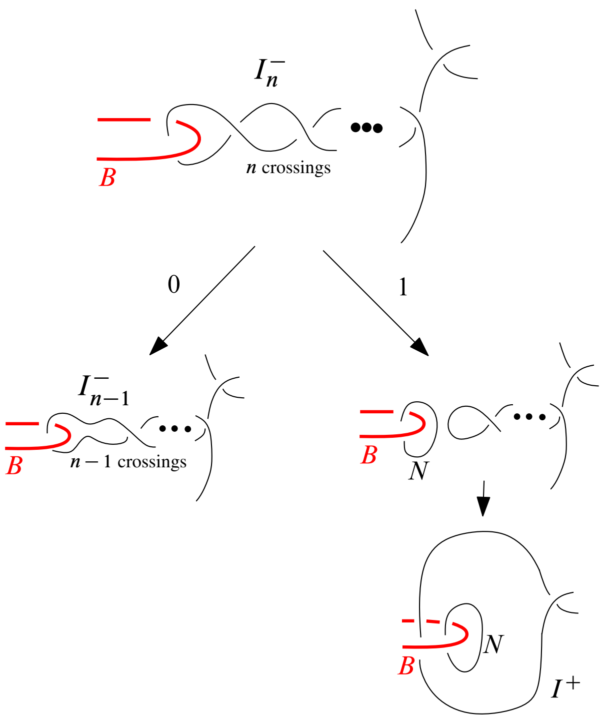


Figure 4: A depiction of I_n^- for $n > 1$, and the corresponding I^+, as it occurs in the resolution tree for the innermost crossing

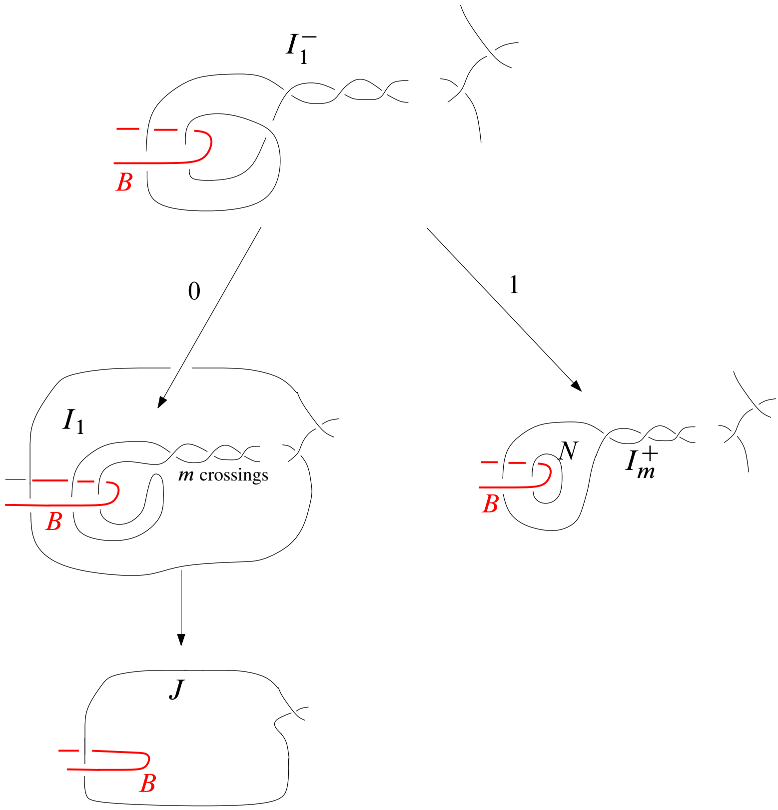


Figure 5: A depiction of I_1^-, I^+ and J^-, as they occur in the resolution tree for the innermost crossing

The nontrivial unknot and the closures of σ_1 and σ_1^{-1} are the base cases for our induction. Note that each of these represent the unknot in S^3 and thus have signature 0. We now assume that we have a twisted unknot as in Figure 1, which has $\sigma(\mathbb{L}) = 0$, since these are also unknots in S^3. In the following argument, the reader should refer to Figure 4 and Figure 5 to clarify the notation. We start by assuming that near the inner point where B crosses the plane the twisting is right-handed. Assume that there are $n > 1$ negative crossings, and call this knot I_n^-. If we 0–resolve the innermost crossing we obtain I_{n-1}^-, while if we 1–resolve the crossing we obtain $N \cup I^+$.

Let $[s]\{(t,0)\}$ be the contribution to the final shift of I_n^- arising from the crossings not involved in this twist region. Let H^* denote the skein homology after shifting. There is then a long exact sequence:

$$\cdots \longrightarrow H^*(I^+ \cup N)[-s]\{(-t,0)\}[n-1]\{(2n-2,0)\}[1]\{(1,0)\} \longrightarrow H^*(I_n^-)[-s+n]\{(-t+2n,0)\} \longrightarrow H^*(I_{n-1}^-)[-s+n-1]\{(-t+2n-2,0)\} \longrightarrow \cdots$$

where the sequence arises from $0 \to 1$–resolution maps in the unshifted complexes. The additional shifts for $H^*(I^+ \cup N)$ come from its arising in the 1–resolution and from the additional negative crossings introduced by the RI–moves when simplifying to I^+. Those for $H^*(I_{n-1}^-)$ come from the negative crossings remaining in the resolved diagram. The two internal arrows are degree preserving. If $k-j+2i = C$ for the skein homology for I_{n-1}^- then applying the shifts in the long exact sequence we have $k' = k$, $j' = j-t+2n-2$ and $i' = i-s+n-1$. Thus

$$k'-j'+2i' = C+t-2n+2-2s+2n-2 = C+t-2s.$$

Reversing the shifts in the sequence for $H^*(I_n^-)$ gives $i = i'+s-n$, $j = j'-t+2n$ and $k = k'$. Elements mapping non-trivially to the I_{n-1}^-–term satisfy

$$k-j+2i = C+t-2s-2n+t+2n+2s = C$$

in the skein homology for I_n^-. By assumption $C = M(I_{n-1}^-)$, and $M(I_n^-) = M(I_{n-1}^-)$ since there has been no change in the black/white region count. On the other hand, $M(I^+) = M(I_n^-) - 1$ since we have lost the interior region, necessarily white by our crossing assumption. The addition of N does not change $k-j+2i$, so if $k-j+2i = M(I^+)$ in the skein homology for I^+, we see that the terms in the unshifted complex for I^+ used in the long exact sequence satisfy

$$k'-j'+2i' = M(I^+)+t-1-2n+2+2(-s+1+n-1) = M(I^+)+t-2s+1.$$

For those elements that map into I_n^-, applying the shifts to get the skein homology for I_n^- produces elements with $j = j'+t-2n$ and $i = i'+s-n$, which implies $k-j+2i = M(I^+)+1 = M(I_n^-)$. Every element in the image of $H^*(I^+ \cup N)$ will have the property in the lemma. Thus by induction, the property will be true also for I_n^-.

This leaves the case where $n=1$. The 1–resolution occurs in the same way and we may draw the same conclusion. However, for the 0–resolution a large collapse can occur. If I^+ has $m \geq 1$ positive crossings in the next region, the 0–resolution allows us to untwist all of these until we get to J^-. The complex for J^- is thus shifted by $\{(-m,0)\}$ when injected into that for I_1^-. This implies that $k-j+2i$ increases by m in the unshifted complexes. In the shifted complexes, I_1^- is shifted $[-1]\{(m-2,0)\}$ more than J^-. That shift reduces $k-j+2i$ by $2-m-2 = -m$. Thus after the final shift there is a difference of 0. But note that the resolution eliminates both a black and a white region and thus leaves $M(J^-) = M(I_1^-)$. All told, if $k - j + 2i = M$ holds for the knots with fewer crossings and the innermost crossing is negative then it also holds for I_n^-.

A similar argument can be deployed for the case where the innermost crossing is positive. Alternately we can appeal to the symmetry under reflection to switch the two cases. Since this switches the black and white regions, it also multiplies M by -1. □

Thus for every unknot in the collection depicted in Figure 1 we have $k - j + 2i = M(D_S)$ for every generator in the homology. In particular, (j,k) determines i. Note that this conclusion remains valid if we add a single marked non-trivial circle. It also remains true if we shift by $[-w]\{(-2w, 0)\}$. As with the original proofs of the alternating links property, the value of $r(S)$ is the same for every complete smoothing in $K_1(\mathbb{L})$, depending only on the number of black regions and the crossings joining them. So all the generators for the spanning tree model of the unshifted homology satisfy $k - j + 2i = r(S)$ after the $[r(S)]\{(r(S), 0)\}$ shifts and the odd number of intersections. The final shift of the diagram for $\mathbb{L}$ is $[-n_-]\{(n_+ - 2n_-, 0)\}$ and produces generators satisfying $k - j + 2i = r(S) - n_+$. From [8], we have that $r(S) - n_+ = \sigma(\mathbb{L})$. Thus, after the final shifting, every generator in the spanning tree complex satisfies $k - j + 2i = \sigma(\mathbb{L})$, but since the differential preserves (j,k) and increases i, it must therefore be trivial. Consequently the spanning tree complex is also the homology. For those generators which survive the spectral sequence to the Khovanov homology, we also have that $j - 2i = -\sigma(\mathbb{L}) \pm 1$. Thus, for these generators, $k = \pm 1$.

A comment about supports Wehrli's argument produces an unshifted chain complex, which has the same chain groups for $l + r = i$ and $2l + r \pm 1 = j$ where $r = r(S)$ is constant. Thus $j - 2i = -r \pm 1$, which when shifted yields $j - 2i = -\sigma(\mathbb{L}) \pm 1$. For a given q–grading, j, there are two i–gradings differing by 1. Thus there can still be non-zero terms in the differential, which may result in torsion or vanishing homology groups, and thus the homology is at most supported on these lines. In our case, these groups are distinguished by their k–value, which is also preserved by the differential. The issue of torsion returns in the spectral sequence, but it is known that at most 2^r–torsion occurs for alternating knots [18], and so working over $\mathbb{F}_2$ will correct it.

5 The knot Floer homology spectral sequence

We now leave the Khovanov skein homology to recall some results in knot Floer homology. In the next section we will begin relating these two theories. The two will intertwine further in later sections.

Let $L = L_1 \cup \cdots \cup L_n$ be a framed link in a three manifold Y. Following section 4 of [13] we let $R = (m_1, \ldots, m_n)$ where $m_i \in \{0, 1, \infty\}$ and $Y(R)$ be the result of $\mathrm{fr}(L_i) + m_i\,\mu_i$–surgery on each L_i where μ_i is the meridian of L_i and ∞–surgery is μ_i–surgery. We let $0 < 1 < \infty$ define a lexicographic ordering on $\{0, 1, \infty\}^n$ and call I' an immediate successor of I if, as in [13], all the m'_j are the same as m_j except for one where $m'_i > m_i$, excluding the case $(m'_i, m_i) = (\infty, 0)$. Then to each immediate successor I' of I there is a map

$$F_{R<R'}\colon \widehat{HF}(Y(R)) \longrightarrow \widehat{HF}(Y(R'))$$

arising from the associated 2–handle additions.

According to section 8 of [11], 2–handle additions attached in a manner algebraically unlinked from a knot induce maps on the levels of the knot Floer homology. Viewed differently, the knot turns the chain map above into a filtered morphism for the filtered chain groups defining the knot Floer complex. The "top" levels of these filtered morphisms form exact sequences that specialize to the skein exact sequence for crossing changes. Following these thoughts leads to:

Theorem 5.1 *Let $\mathcal{L} = L_1 \cup \cdots \cup L_n$ be a framed link in (Y, K), with K a null-homologous knot bounding a surface S, such that $L_s \cap S = 0$ for all s. For each integer k there is a spectral sequence such that:*

1. *The E^1–page is $\bigoplus_{R \in \{0,1\}^n} \widehat{HFK}(Y(R), K, k)$.*
2. *The d_1–differential is obtained by adding all $\widehat{F}_{R<R'}$ where R' is an immediate successor of R.*
3. *All the higher differentials respect the dictionary ordering of $\{0, 1\}^n$.*
4. *The spectral sequence eventually collapses to a group isomorphic to*

$$\widehat{HFK}(Y, K, k).$$

The proof can be found in Roberts [17], along with more details concerning the link surgery spectral sequences. In fact, more can be deduced from the arguments described above. Namely, there is a filtered chain complex whose chain groups are given by $\widehat{CFK}(Y(R), K)$, considered as $\widehat{CF}(Y(R))$ with the filtration induced by K, with differential induced by counts of higher polygons, which is filtered 1–quasi-isomorphic to the filtered chain complex $\widehat{CFK}(Y,K)$. With this additional data we may also capture the spectral sequence from $\bigoplus_{k\in\mathbb{Z}} \widehat{HFK}(Y,K,k)$ converging to $\widehat{HF}(Y)$ through the surgery spectral sequence.

More specifically, in [17], following [13], let $\mathcal{L}$ be a framed link in a closed, oriented three manifold Y, and let K be a null-homologous knot in Y bounding an embedded surface S. We define $X(Y,K)$ to be the complex [17] whose chain groups are

$$\bigoplus_{R\in\{0,1\}^{|\mathcal{L}|}} \widehat{CFK}(Y(R),K)$$

with $\widehat{CFK}(Y(R),K)$ in homological grading $I(R)=\sum m_i$ when $R=(m_1,\dots,m_{|\mathcal{L}|})$. When $R'>R$ in the sense that, as vectors,

$$R'-R\geq\vec{0},$$

there is a filtered map

$$D_{R,R'}\colon \widehat{CFK}(Y(R),K)\longrightarrow \widehat{CFK}(Y(R'),K)$$

described below, such that for $x\in\widehat{CFK}(Y(R),K)$,

$$D(\mathbf{x})=\sum_{R'\geq R} D_{R,R'}(\mathbf{x})$$

satisfies $D^2=0$, and D preserves the knot filtrations from K. The knot filtration preserving part of D induces a differential in each knot filtration level.

The maps $D_{R,R'}$ are defined by picking a bouquet for $\mathcal{L}$ and finding a Heegaard diagram subordinate to the bouquet and K simultaneously. We let $D_{R,R'}=\sum D_{R<R_1<\cdots<R'}$ where we sum over all sequences $R,R_1,\dots,R'$ where each entry is the immediate successor of the preceding entry. The map $D_{R<R_1<\cdots<R'}(\mathbf{x})$ is:

$$\sum_{(\mathbf{y},\phi)\in\mathcal{S}} \#\mathcal{M}(\phi)\,\mathbf{y}$$

where

$$\mathcal{S}=\left\{(\mathbf{y},\phi)\,\middle|\,\mathbf{y}\in\widehat{CF}(\alpha,\eta_k),\phi\in\pi_2(\mathbf{x},\Theta^+_{\eta_1\eta_2},\dots,\Theta^+_{\eta_{k-1}\eta_k},\mathbf{y}),\mu(\phi)=0,n_w(\phi)=0\right\}$$

See [17] or [13] for more details. Note that for ϕ as envisioned in the summation $\mathcal{F}(\mathbf{y})-\mathcal{F}(\mathbf{x})=-n_z(\phi)$ since the maps preserve the knot Floer filtration. Here (Σ,α,η_1) is the Heegaard diagram for $Y(R)$ obtained from the diagram subordinate to the bouquet, and (Σ,α,η_i) is the diagram for $Y(R_i)$, while (Σ,α,η_k) is a diagram for $Y(R')$. In particular, the difference between η_i and η_{i+1} is the framing on one component of the surgery bouquet. All other curves are small Hamiltonian isotopes of those in η_i. All admissibility requirements can be arranged so that this map is well-defined.

In this paper, we apply these spectral sequences in the following situation. Let $\mathbb{L}$ intersect a spanning disc for B generically in an odd number of points. Let R be a complete resolution of the crossings in $\mathcal{P}$, a projection of $\mathbb{L}$. Of the closed curves in the resolved diagram $\mathcal{P}(R)$, some number, m, are geometrically split from the axis, B. The remainder, l, form an unlink each of whose components link the axis one time. For such a link of unknots, the double branched cover is easily computed to be $\#^{l+m-1} S^1 \times S^2$. Moreover, $\widetilde{B}(R)$ is still a knot since each unknot which is split from B intersects a disc generically an even number of times. This knot is $\#^{(l-1)/2} B(0,0) \subset \#^{l+m-1} S^1 \times S^2$ (and the unknot in $\#^m S^1 \times S^2$ if $l = 1$) where $B(0,0) \subset S^1 \times S^2 \# S^1 \times S^2$ is the knot obtained by performing 0–surgery on any two of the three components of the Borromean rings. Hence, by calculations in [11]:

$$\widehat{HFK}(\widetilde{B}) \cong V^{\otimes(l-1)} \otimes W^{\otimes m}$$
$$\text{where } V \cong \mathbb{Z}_{(\frac{1}{2},\frac{1}{2})} \oplus \mathbb{Z}_{(-\frac{1}{2},-\frac{1}{2})} \quad \text{and} \quad W \cong \mathbb{Z}_{(\frac{1}{2},0)} \oplus \mathbb{Z}_{(-\frac{1}{2},0)}$$

Here the first term in the subscript is the rational grading, whereas the second term is the filtration. Since $l-1$ is even, the filtration levels are integers. Furthermore, this homology is entirely supported in the trivial Spin^c structure. In addition, there are no higher differentials in the spectral sequence from the direct sum of knot Floer homology groups to $\widehat{HF}$. All we have done is compartmentalize the Heegaard Floer homology of $\#^{l+m-1} S^1 \times S^2$ in a manner reflecting the filtration induced by the knot $\widetilde{B}$. Without the filtration information we recover the which group associated to R in [13].

Now let R' be another resolution of $\mathcal{P}$ where $R' > R$, so some number of 0–resolved crossings in R will be 1–resolved in R' and all the other crossings will be resolved identically. An arc in the plane joining the two strands at a resolved crossing lifts to a circle in $\Sigma(\mathcal{P}(R))$, and the effect of changing from a 0–resolution to a 1–resolution is to remove a solid torus neighborhood of the circle and glue it in with a different framing, corresponding to the addition of a four-dimensional 2–handle. If we take the complete 0–resolution of $\mathcal{P}$, the lift of the arcs for each crossing define a link $L_1, \ldots, L_s$ in $\Sigma(\mathcal{P}(R))$. We can encode the 1–resolution at the i^{th} crossing by the framing of L_i required by the aforementioned handle addition. Each of the L_i comes from an arc disjoint from the spanning disc of B, and thus L_i will be disjoint from the double cover of this spanning disc, which we take as the spanning surface S for the knot $\widetilde{B}$. Thus, $L_i \cap S = 0$, and we may apply Theorem 5.1.

In the following section, we will establish filtered isomorphisms between

$$\widehat{HFK}(Y(R), K)$$

and copies of $V^{\otimes(l-1)} \otimes W^{\otimes m}$ used in the skein homology theory. We then compute the d_1–differentials for the spectral sequence induced by the I–filtration, corresponding to the number of surgeries, and herein called the homological filtration. This allows us to identify the E^1–page with the skein homology complex. Analogous to the results in [13] we may then conclude:

Proposition 5.2 *Let $\mathbb{L}$ be a link in $A \times I \subset \mathbb{R}^2 \times \mathbb{R}$ as above. There is a spectral sequence whose E^2–term is isomorphic to the reduced Khovanov skein chain complex of $\overline{\mathbb{L}}'$ in $A \times I$ with coefficients in $\mathbb{F}_2$ and which converges to*

$$\bigoplus_{k \in \mathbb{Z}} \widehat{HFK}(\Sigma(\mathbb{L}) \#^2 (S^1 \times S^2), \widetilde{B} \# B(0,0), k, \mathbb{F}_2).$$

By splitting according to the filtration data we can obtain the slightly stronger result:

Proposition 5.3 *There is a spectral sequence whose E^2–term is isomorphic to the subcomplex of the reduced Khovanov skein complex of $\overline{\mathbb{L}}'$ generated by the enhanced states with $\Psi(S) = 2k$ and which converges to $\widehat{HFK}(\Sigma(\mathbb{L}) \#^2 (S^1 \times S^2), \widetilde{B} \# B(0,0), k)$.*

In fact, by taking the direct sum of all the groups for the knot Floer homologies over all the resolutions we can obtain a bi-filtered complex, filtered by the pair (I, Ψ), where the E^1 term corresponds to the filtration of the bi-filtered reduced Khovanov homology complex. Using the graded objects for just the Ψ filtration and taking their homology produces the first proposition above. The additional terms in the maps in the Khovanov complex induce maps in the E^2–level of the spectral sequence using the Ψ filtration, since these correspond to terms in the filtered cobordism maps between the Heegaard Floer homologies. These maps fit together to provide a filtered version of the spectral sequence in section 4 of [13] with K inducing the filtration. Additional pages ultimately calculate the Heegaard Floer homology of the branched double cover.

More can be concluded from the proof outlined above and the homological algebra in Appendix A.

Lemma 5.4 *For each $r \geq 1$, the E^r–page of the spectral sequence induced by the knot Floer filtration from $\widetilde{B}$, which starts with*

$$\bigoplus_{k \in \mathbb{Z}} \widehat{HFK}(\Sigma(\mathbb{L}) \#^2 (S^1 \times S^2), \widetilde{B} \# B(0,0), k, \mathbb{F}_2)$$

and converges to $\widehat{HF}(\Sigma(\mathbb{L})\#^2\ (S^1\times S^2))$*, is quasi-isomorphic to the* E^r*–page for the* Ψ*–filtered complex* $X(\mathbb{L})$*, computed using the maps induced from the link surgeries spectral sequence.*

Thus, the values of Ψ will filter the Heegaard Floer homology groups in a manner replicating the Alexander filtration (namely, the associated graded groups will be isomorphic).

6 The E^1–page is isomorphic to the skein homology of the mirror

First, we will establish an isomorphism between the skein homology chain groups and the E^1–page thought of as a group. We then compute the d_1–differential for this representation of the chain groups. However, if we try to define an isomorphism

$$\Phi_B(R)\colon \widetilde{V}_{\mathbb{L}}(R) \xrightarrow{\cong} \widehat{HFK}(\Sigma(\mathcal{P}(R)), \widetilde{B})$$

for any complete resolution R, a slight mismatch arises: the knot Floer homology of the binding implicitly corresponds to marking a non-trivial circle. This cannot always be arranged in the skein homology theory. We rely upon a trick to resolve this problem: we introduce two non-trivial circles into $\mathbb{L}$ that link B once and otherwise do not interact with the diagram. These should be considered innermost circles. We always mark the innermost one (we need two to keep the binding connected) and since this circle does not include any crossings, it will be the marked circle throughout. The link with the addition of these two circles will be denoted $\mathbb{L}'$.

The effect on the double cover of changing $\mathbb{L}$ to $\mathbb{L}'$ is to replace $\Sigma(\mathbb{L})$ with

$$\Sigma(\mathbb{L})\,\#^2\, S^1\times S^2$$

and to replace $\widetilde{B}$ with $\widetilde{B}\,\#\,B(0,0)$. We see this by shrinking the two new components to nearby meridians of B and then examining the double cover of a small ball that includes them and an arc on B. The effect of these connect sums on the Heegaard Floer homology is well understood. In particular, since $B(0,0)$ induces an entirely collapsed spectral sequence for the Heegaard Floer homology, we will be able to read off any information about the knot Floer homology of $\widetilde{B}$ from that of $\widetilde{B}\,\#\,B(0,0)$.

With this alteration, we may now define the isomorphism. Order the circles in $\mathcal{P}(R)$ by the marked circle first, then all the non-trivial circles, then all the trivial circles. An element of $\widetilde{V}(\mathcal{P}(R), B)$ is encoded as $+\otimes v_{\pm}^1\otimes\cdots\otimes w_{\pm}^n$ and is mapped to $\gamma_{i_1}\cdots\gamma_{i_k}\cdot\Theta^+$ where $\{i_1,\ldots,i_k\}$ are the indices for the minus signs on non-marked circles, γ_j is the first homology class dual to the j^{th} sphere, and Θ^+ is the highest degree generator of $\widehat{HF}(\Sigma(\mathcal{P}(R)))$. In particular, a representative for γ_j in $\mathbb{F}_2$–homology can be found by lifting an arc between the marked circle and the j^{th} circle.

To explicitly compute the d_1–differential, we associate maps in the knot Floer homology to the changes in the resolution code at a single crossing. In our case, these maps become maps between filtered groups. We will work backwards from the unfiltered maps in [13].

First, we note that the resolution changes occur in three ways: between circles split from the axis, between circles linking the axis and between circles of mixed linking. The first occur precisely as in [13] due to the local nature of the surgeries in the double cover and the connected sum decomposition of the covering manifolds. In particular, the maps for the filtered theory are just the maps for the unfiltered theory tensored with the identity on the tensor products of the V–vector spaces. Hence, they reflect the differential of the reduced Khovanov homology.

Now consider a resolution change joining two circles that link the axis. In the double cover, this corresponds to a cobordism that involves 0–surgery on a curve that is homologically non-trivial and intersects only those spheres intersecting the binding. Such a circle is isotopic to a circle in a fiber of the open book determined by $\# B(0,0)$ before connect summing with extra $S^1 \times S^2$'s. Moreover, since the circle is the lift of an arc between two branch points, it is homologically non-trivial in the fiber. Ignoring the choice of basis implicit in the above description, we can calculate the effect of such a surgery by looking at the standard picture of $B(0,0)$ and doing 0–surgery on a meridian of one of the 0–surgered components of the Borromean rings. When we connect sum with copies of $B(0,0)$ we obtain a diffeomorphic picture to the one described above. We then use homology classes to pin down the maps in the original picture. In the unfiltered version, the model calculation uses the following long exact sequence (which must split as depicted due to ranks and gradings).

$$\cdots \xrightarrow{-\frac{1}{2}} \mathbb{F}_1 \oplus \mathbb{F}_0^2 \oplus \mathbb{F}_{-1} \xhookrightarrow{-\frac{1}{2}} \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{-\frac{1}{2}} \dashrightarrow^{0} \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{-\frac{1}{2}} \xrightarrow{-\frac{1}{2}} \cdots$$

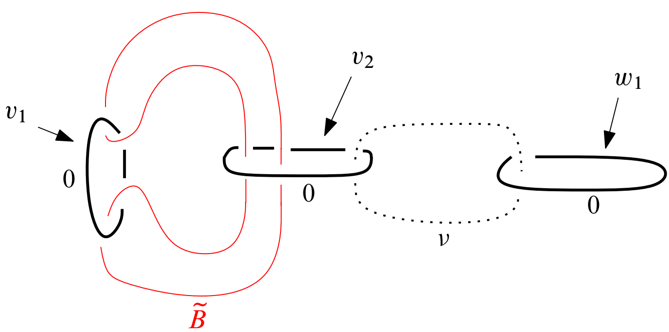
In the identification with Khovanov homology, the $\mathbb{F}_1$–term corresponds to $v_+ \otimes v_+$ and it thus maps to w_+. The term mapping to $\mathbb{F}_{-\frac{1}{2}}$ in the surjection is the image under ν_2 of $\mathbb{F}_1$ where ν_2 is the meridian we *do not* surger. Meanwhile the image of ν_1 is annihilated. Transferring back to the basis from the resolution, this tells us that $\gamma_1 + \gamma_2$ generates the kernel, and γ_1 and γ_2 are mapped isomorphically to γ'. Transferring back further to Khovanov's notation, we get $\gamma_1 \to v_- \otimes v_+ \to w_- \leftarrow \gamma'$ and $v_+ \otimes v_- \to w_-$.

For the filtered version, we obtain the model long exact sequence, which filters the above one:

$$\begin{array}{ccccccc}
 & \longrightarrow \mathbb{F}_1 & & & \mathbb{F}_{\frac{3}{2}} \longrightarrow & & \\
\cdots \longrightarrow & \mathbb{F}_0 \oplus \mathbb{F}_0 & & \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{-\frac{1}{2}} & & \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{\frac{1}{2}} & \longrightarrow \cdots \\
 & \longrightarrow \mathbb{F}_{-1} & & & \mathbb{F}_{-\frac{1}{2}} \longrightarrow & &
\end{array}$$

The first term is the knot Floer homology of $B(0,0)$; the second is the knot Floer homology of the unknot in $S^1 \times S^2$, which we obtain after the 0–surgery on the meridian; the third term is the result of $+1$–surgery on the meridian, $B(0,-1)$ in the notation of [11]. The grading and ranks again determine the filtered maps on the first page. When we join two curves which link the axis, we obtain one which does not link the axis. This can be seen by considering the possible winding numbers for the result: 0 or 2. However, the result is a Jordan curve in the plane and thus cannot have winding number 2 about the origin. Working back through the basis transformations as before, these correspond in our notation to the maps $v_+ \otimes v_+ \to 0$, $v_+ \otimes v_- \to w_-$, $v_- \otimes v_+ \to w_-$ and $v_- \otimes v_- \to 0$.

Finally, the model calculation in the cases of joining a linked with an unlinked circle corresponds to the map in the following diagram:



The surgery circle, ν, annihilates $\gamma_1 + \gamma_2$ again in mod-2 homology. The result of the resolution change is now a circle that links the axis. The relevant cobordism map is from $B(0,0) \# S^1 \times S^2 \to B(0,0)$ and corresponds to $v_+ \otimes w_+ \to v_+$, $v_+ \otimes w_- \to 0$, $v_- \otimes w_+ \to v_-$ and $v_- \otimes w_- \to 0$. This can be seen from the following graded exact sequence:

$$\begin{array}{cccc} \longrightarrow \mathbb{F}_{\frac{1}{2}} \oplus \mathbb{F}_{\frac{3}{2}} \hookrightarrow \mathbb{F}_1 & & \mathbb{F}_1 \longrightarrow \\ \cdots \longrightarrow \mathbb{F}^2_{-\frac{1}{2}} \oplus \mathbb{F}^2_{\frac{1}{2}} \hookrightarrow \mathbb{F}^2_0 & & \mathbb{F}^2_0 \longrightarrow \cdots \\ \longrightarrow \mathbb{F}_{-\frac{3}{2}} \oplus \mathbb{F}_{-\frac{1}{2}} \hookrightarrow \mathbb{F}_{-1} & & \mathbb{F}_{-1} \longrightarrow \end{array}$$

where $\mathbb{F}_{\frac{3}{2}}$ corresponds to $v_+^1 \otimes v_+^2 \otimes w_+$ and is mapped to $v_+^1 \otimes v'_+$, taking into account both 0–framed knots in the Borromean rings. Note that a w_- always forces the map to be 0.

Due to the introduction of the two new components we do not need to examine what happens if one of the circles is the marked circle: a division or merging never includes the marked circle.

Similar considerations, or duality, establish the maps for the case of splitting a circle into two circles. Note that the above maps are from 1–resolutions to 0–resolutions. This force us to use the mirror of $\mathbb{L}$ in establishing the relationship between the knot Floer homology of $\widetilde{B}$ and the reduced skein homology.

Proposition 6.1 *Let $\mathcal{P}$ be a projection for $\overline{\mathbb{L}}' \cup B$. Let R be a choice of resolution for each crossing of $\overline{\mathbb{L}}'$. Then there is an isomorphism:*

$$\Phi_B(R)\colon\ \widetilde{V}(\mathcal{P}(R), B) \xrightarrow{\cong} \widehat{HFK}(\Sigma(\mathcal{P}(R)), \widetilde{B})$$

Let R' be a resolution found by changing a single smoothing in R from 0 to 1. Then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{V}(\mathcal{P}(R), B) & \xrightarrow{d_{\overline{\mathbb{L}}}} & \widetilde{V}(\mathcal{P}(R'), B) \\ \Big\downarrow \Phi_B(R) & & \Big\downarrow \Phi_B(R') \\ \widehat{HFK}(\Sigma(\mathcal{P}(R)), \widetilde{B}) & \xrightarrow{\widehat{F}_{R<R'}} & \widehat{HFK}(\Sigma(\mathcal{P}(R')), \widetilde{B}) \end{array}$$

where $\widehat{F}_{R<R'}$ is the cobordism map for the knot Floer homologies induced by the surgery corresponding to the resolution change, and $d_{\overline{\mathbb{L}}}$ is the differential in the skein homology. This square is a $\mathbb{Z}$–direct sum of squares where the index corresponds to the filtration of the knot Floer homology and the k–index in the skein homology (with $\mathcal{F} = \frac{k}{2}$ after the final shifts).

This proposition establishes that each component of the d_1–differential, coming from a change of resolution on a single crossing, corresponds under our isomorphism to the term in the skein homology differential for the corresponding change of resolution for the mirror diagram. Consequently, the E^1–page of the link surgery spectral sequence corresponds to the skein homology in each knot Floer filtration level.

7 Transverse links, open books and contact invariants

First, we note that:

Theorem 1 *Any transverse link is transversely isotopic to a braid closure. Furthermore, two braids represent transversally isotopic links if an only if one can be obtained from the other by conjugations in the braid group, positive Markov moves and their inverses.*

This is the culmination of work by Bennequin for the first part, and by V Ginzburg, S Orevkov, and N Wrinkle, who independently proved the second part. We will replace the contact structure with an open book. The standard contact structure on S^3 is supported by the open book with unknotted binding and discs for pages. In the braid picture, this corresponds to including the axis of the braid, which is an unknot. When we take a branched cover of a transverse link, the contact structure lifts to a contact structure in the cover where we use a Martinet contact neighborhood of the transverse link. In the open book picture, this contact structure is supported by the pre-image of the open book, whose fibers are now more complicated, but whose binding is the lift of the axis. This follows since the lifted contact structure remains C^0–close to the pages of the open book, and transverse to the binding. We call this contact structure ξ. The contact structure on $\#^2\,(S^1 \times S^2)$ induced by the fibered knot $B(0,0)$ will be denoted ξ_0.

For a braid, O Plamenevskaya [15; 14] defines a cycle, $\widetilde{\psi}(\mathbb{L})$, in the reduced Khovanov homology chain group. First she resolves all the crossings in the direction of the oriented braid. This constructs the maximal number of non-trivial loops in the skein algebra perspective. She then labels every one of the unmarked strands with a $-$ and the marked strand with a $+$. This enhanced state is closed in the reduced Khovanov homology theory [14].

Let $\mathbb{L}$ be a braid whose closure is the transverse link.

Theorem 7.1 *Suppose $\mathbb{L}$ intersects the spanning disc for B an odd number of times. Then the element $\widetilde{\psi}(\mathbb{L}')$ is closed in the skein Khovanov homology and represents the unique homology class with minimal Ψ–grading. Under the correspondence with the*

E^2–term of the spectral sequence converging to knot Floer homology, it maps to an element which survives the spectral sequence and generates:

$$\widehat{HFK}(-\Sigma(\mathbb{L})\,\#^2\,(S^1\times S^2),\, \widetilde{B}\,\#\,B(0,0), -1-g(\widetilde{B}))\cong \mathbb{F}_2$$

Upon mapping this last group into $\widehat{HF}(-\Sigma(\mathbb{L})\,\#^2\,(S^1\times S^2))$, $\widetilde{\psi}(\mathbb{L}')$ corresponds to the contact element $c(\xi\,\#\,\xi_0)$.

Note The correspondence at the end is not the same as first mapping ψ into the reduced Khovanov homology and then considering the spectral sequence from it to the Heegaard Floer homology of the branched double cover.

Proof There is only one element in the skein chain group that has

$$\Psi = -2g(\widetilde{B}\,\#\,B(0,0))$$

and that is Plamenevskaya's element. For if b is the braid index of $\mathbb{L}'$, then Euler characteristic calculations imply that $1-2g(\widetilde{B}\#B(0,0)) = 2-b$ and thus Ψ must equal $1-b$. This can only happen when all the crossings are resolved in the direction of the link so that there are b non-trivial circles and precisely one circle (the marked one) is adorned with a $+$ sign. $\widetilde{\psi}(\mathbb{L}')$ is characterized as the unique enhanced state with minimal value for $\Psi(S)$ and thus generates the homology in this f–grading. This enhanced state survives in the spectral sequence for the knot Floer homology of the binding and yields in the limit a generator of $\widehat{HFK}(-\Sigma(\mathbb{L}'), \widetilde{B}\,\#\,B(0,0), -g(\widetilde{B}\,\#\,B(0,0)))\cong\mathbb{F}$ since it is the only generator in the filtration level.

The branched cover of B over $\mathbb{L}'$ is $\widetilde{B}\,\#\,B(0,0)$ which supports the contact structure $\xi\,\#\,\xi_0$. The contact element $c(\xi\,\#\,\xi_0)$ is the image in $\widehat{HF}(-\Sigma(\mathbb{L}'))$ of the generator of $\widehat{HFK}(-\Sigma(\mathbb{L}'), \widetilde{B}\,\#\,B(0,0), -g(\widetilde{B}\,\#\,B(0,0)))$. Lemma 5.4 guarantees that this generator corresponds to the $-g(\widetilde{B}\,\#\,B(0,0))$–level of the associated graded group for $\widehat{HF}(-\Sigma(\mathbb{L}'))$. This level is either $\cong\mathbb{F}$ or $\cong 0$ depending upon whether the contact element vanishes. Thus, Plamenevskaya's element converges to the contact element in the Heegaard Floer homology (with $\mathbb{F}_2$–coefficients). □

Corollary 7.2 *Under the correspondence in the previous theorem, $\widetilde{\psi}(\mathbb{L})$ corresponds to $c(\xi)\in\widehat{HF}(-\Sigma(\mathbb{L}),\mathbb{F}_2)$.*

Proof If $\mathbb{L}$ intersects the spanning disc for B an even number of times, use a positive Markov move to increase the number of strands by 1. $\widetilde{\psi}(\mathbb{L})$ is mapped to $\widetilde{\psi}(\mathbb{L}_+)$ [14] under this move. Meanwhile, in the double cover this corresponds to positively stabilizing the open book, and thus does not change the contact invariant. Renaming $\mathbb{L}_+$ by $\mathbb{L}$, we may now assume $\mathbb{L}$ intersects the spanning disc an odd number of times.

Furthermore, $\widetilde{\psi}(\mathbb{L})$ clearly corresponds to $\widetilde{\psi}(\mathbb{L}')$ in a precise way. Using the previous theorem we have that $\widetilde{\psi}(\mathbb{L}')$ maps to $c(\xi \# \xi_0)$ under the spectral sequence and $c(\xi \# \xi_0) = c(\xi) \otimes c(\xi_0)$. Adding the two meridional strands tensors both homologies with $V^{\otimes 2}$. Thus, $c(\xi)$ in the knot Floer homology of $\widetilde{B}$ corresponds to Plamenevskaya's element in the skein homology of $\mathbb{L}$ since both are altered in the same formal manner by the introduction of the new strands. □

We now turn to proving a the non-vanishing result mentioned in the introduction. We begin with a lemma:

Lemma 7.3 *Let $\mathcal{C}$ be a bifiltered complex over a field. Then up to isomorphism there is a unique bifiltered complex $\mathcal{C}'$ such that:*

(1) *$\mathcal{C}'$ is bifiltered chain homotopy equivalent to $\mathcal{C}$.*
(2) $\mathcal{C}'_{ij} \cong H_*(\mathcal{C}_{ij})$
(3) *The differential $d' = \sum d'_{ij}$ on $\mathcal{C}'$ has $d'_{00} = 0$, and induces the same spectral sequences for both filtrations.*

Proof Use the cancellation lemma as per sections 4 and 5 of Rasmussen's thesis, but only for those elements with the same bifiltration indices. □

We note that since the knot Floer spectral sequence for $\#^k B(0,0)$ collapses at E^2, the use of the above lemma for the I–filtration means that $\bigoplus_j \mathcal{C}'_{ij}$ is isomorphic to the knot Floer homology for the summands in the cube complex corresponding to that I–value. In particular, there are no differentials keeping I fixed, and reducing Ψ. For lack of a better name, we will also call this reduced complex $X(\mathbb{L})$, or just X. As a result, $E^1_I(X) \cong X$ for the filtration from I. Since X is bi-filtered chain homotopy equivalent to $X(S)$, it too is quasi-isomorphic to the chain complex for $\widehat{CF}(-\Sigma(\mathbb{L}))$ by a Ψ–filtered map.

We begin with a little notation: we let X_j be the sub-complex of X with $\Psi \leq j$. Likewise, let K_j be the sub-complex of the reduced Khovanov homology with the same condition. Now the I–filtration—from the flattened cube—filters these sub-complexes and their quotient complexes.

Corollary 7.4 *Suppose there exists a n such that:*

(1) *$\psi(\mathbb{L})$ is exact in K_n.*
(2) *The I–induced spectral sequence on X_n/X_{-2g} collapses at E^2.*

Then $c(\xi) = 0$.

The second condition, of course, makes some complex computed from the knot Floer chain groups isomorphic to the corresponding complex computed from the skein Khovanov chain groups.[2]

Proof Suppose $\psi(\mathbb{L})$ has the bifiltration value $(I_\psi, -2g)$. If we try to compute the homology of X_n using the I–filtration, then ψ generates the only group in the Ψ–filtration level $-2g$. Since ψ is exact in K_n, there is some element with I–filtration $I_\psi - 1$ whose differential in K_n is ψ (recall the differential *increases* I–values). This element, ν, may be a linear combination of elements with many different Ψ values. We note that ν is closed and not exact in $E^1(X_n/X_{-2g})$ as a chain complex computing E^2. It is closed since the only non-zero portion of $\partial_{Kh}\nu$ is in X_{-2g}. It is not exact since it would need to be the differential of something with higher I–filtration, and for those elements the differential, which is given by the Khovanov differential, is the same as in $E^1(X_n)$; however in $E^1(X_n)$, ν is not closed and hence is not exact. Thus $[\nu]$ will be non-zero in $E^2(X_n/X_{-2g})$.

Consider C_i to be the sub-complex of X_n with I–filtration greater than or equal to i. We have the commutative diagram represented in Figure 6, to which the remaining argument refers. Here $\mathbb{F}$ is the homology of X_{-2g}, Q_c is the quotient complex of X_n by $C_{I_\psi+1}$ and Q is the quotient complex by $\{\Psi \leq -g\} \cup \{I \geq I_\psi + 1\}$. The 0 in the upper left comes from the observation that there are no generators in X with $\Psi \leq -2g$ and $I \geq I_\psi + 1$. The 0 on the map in the upper right indicates that it will generate the trivial map in homology due to ν. From now on we let $X' = X_n/X_{-2g}$.

An element of $H_*(Q)$ in filtration level $I_\psi - 1$ must have non-trivial representative in $E^2(X_n/X_{-2g})$. Furthermore, the argument above shows that $[\nu] \neq 0$ in $H_*(Q)$. This is certainly true in Q_c since ν has a non-trivial differential. However, if in Q there is an element with differential equal to ν, the only other possibility is that in Q_c this element has differential equal to ν plus something in X_{-2g}. But then $\partial^2 \neq 0$ on this element.

Suppose $[\nu]$ has non-zero image, $[\omega]$, under the map $H_*(Q) \to H_*(C_{I_\psi+1})$. If $[\omega]$ has non-zero image in $H_*(X_n)$, from the middle row, then it too must have a non-zero representative in $E^2(X_n/X_{-2g})$, since $C_{I_\psi+1}$ has no representatives with Ψ–filtration $-2g$. But then the induced differential from the long exact sequence implies that $\partial[\nu] = [\omega]$ in X'. Hence, the rank of $H_*(X')$ is strictly less than that at E^2, ie there is a non-trivial differential beyond E^2.

[2] In an earlier version of this paper, the author incorrectly asserted that the vanishing of ψ is enough to conclude that $c(\xi)$ also vanishes. John Baldwin [2] pointed out the error and has since discovered examples where $c(\xi)$ is non-zero despite ψ vanishing in the reduced Khovanov homology.

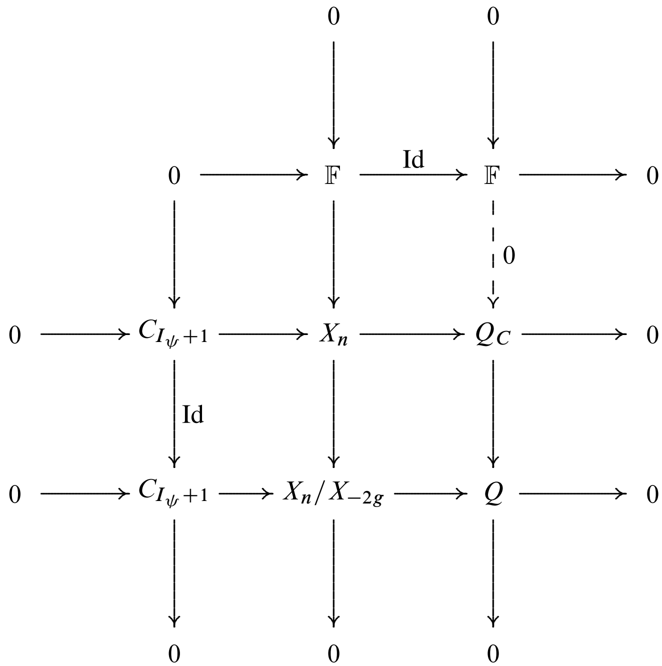


Figure 6

Given the assumptions, we must have $[\omega] = 0$ in $H_*(X_n)$. Then it is the image of some non-zero element, $[\eta]$, of $H_*(Q_c)$. This element injects into $H_*(Q)$ so that $[\nu]$ minus the image of $[\eta]$ is the image of some non-zero element of $H_*(X')$. Furthermore, since the image of $[\nu]$ under $H_*(Q) \to \mathbb{F}$ is non-zero, then the map $H_*(X') \to \mathbb{F}$ is surjective. As a result, the map $\mathbb{F} \to H_*(X_n)$ is zero, but this implies that $H_*(X_{-2g}) \to H_*(X)$ is zero.

The filtered quasi-isomorphism from X to $\widehat{CFK}$ induces a commutative diagram

$$\begin{array}{ccc} H_*(X_{-2g}) \cong \mathbb{F} & \xrightarrow{\cdot 0} & H_*(X) \\ \downarrow \cong & & \downarrow \cong \\ HFK(\widetilde{B}, -g) \cong \mathbb{F} & \longrightarrow & HF(-\Sigma(\mathbb{L})) \end{array}$$

thereby showing that the contact invariant vanishes. □

Note 1 The purpose of the $\mathbb{F}_2$–coefficients is to connect with the extant versions of Khovanov homology. In the end, the crucial observation is that Plamenevskaya's element uniquely defines the lowest filtered portion of both the skein and knot Floer homologies. As long as this remains true and there is an analogous reduced Khovanov homology, the same argument will work with other coefficients. In particular, just changing the sign conventions will not change the conclusion, but there should be some sign convention lifted from the Heegaard Floer world that will allow $\mathbb{Z}$–coefficients.

Note 2 For a braid, $\mathbb{L}$, we can lift a negative crossing or a positive crossing to negative/positive Dehn twists along homologically non-trivial curves in the fiber of the open book. These, in turn, fit into the long exact sequences of Heegaard Floer and knot Floer homology. One sign fits into the $\infty, 0, +1$–sequence for the fiber framing, while the other fits into the $-1, 0, \infty$–sequence. Doing all the surgeries at the same time yields a spectral sequence as in the previous section, with the maps in the E^1–page coming either from the maps $\infty \to 0$ or $0 \to \infty$ from the respective long exact sequence and converging to the appropriate homology of the fibered knot. This is the same sequence as that constructed above, only the basis for the framings has been altered. Namely, if the framing from the crossing is declared ∞ and the crossing is negative, then the 0–framing is ∞ in the fiber framing, and $+1$ is 0 in the fiber framing. The knot for the surgery is the same, a lift of an arc between two branched points.

We now collect some results for quasi-positive braids. We note that for a quasi-positive braid, the lifted contact structure is Stein fillable. We can use the above argument to reprove that the induced contact element is non-vanishing [12]. Let the braid be given by $w_1\sigma_{i_1}w_1^{-1}\cdots\ w_k\sigma_{i_k}w_k^{-1}$. We resolve only those crossings corresponding to the σ_{i_k} terms. For the $00\dots0$–resolution, the result will be b non-trivial circles. Any 1–resolutions make the situation more difficult, but all the non-zero terms occur in higher filtration levels. Plamenevskaya's element is then in the lowest level of the $00\dots0$–resolution. There is no possibility in the spectral sequences of a higher differential landing at this spot as they must all map to enhanced states with at least one 1 in their code. Thus the element survives in this spectral sequence. We now note that when there is a 1 in the code, and the resulting resolution does not consist of unlinked circles, that the Heegaard Floer homology of its double cover is, as a filtered group, the limit of a spectral sequence. Combining all of these shows that Plamenevskaya's element survives in the spectral sequence and thus gives the non-triviality of the contact element in the double branched cover.

8 Heegaard Floer homology for double covers branched over alternating branch loci

In this section we aim to prove the following result:

Lemma 8.1 *Let D be a (possibly disconnected) alternating diagram in $\mathbb{R}^2$ for a link $\mathbb{L} \subset S^3$ with C crossings. In the complex $X(D)$, let $\mathbf{x}$ be a generator for the torsion* Spin^c *structure on the all 0–resolution summand, and $\mathbf{y}$ be a generator for the torsion* Spin^c *structure in the all 1–resolution summand. Any homotopy class of polygons, $\psi \in \pi_2(\mathbf{x}, \Theta, \dots, \Theta, \mathbf{y})$, with $\mu(\psi) = 0$ and $n_z(\psi) = 0$ has*

$$\mathrm{gr}_{\mathbb{Q}}(\mathbf{y}) - \mathrm{gr}_{\mathbb{Q}}(\mathbf{x}) = -1 + \frac{C}{2}$$

where $\mathrm{gr}_{\mathbb{Q}}$ is the absolute rational grading of [10].

Comment This lemma is more general than it first appears. In particular, if D' is any alternating diagram with crossings C', pick any subset of crossings $C'' \subset C'$ and resolve them arbitrarily using 0 and 1–resolutions. The resulting diagram D is still alternating, so the lemma applies with C being the unresolved crossings. By choosing C'' appropriately we can apply the lemma to any homotopy class of polygons that contributes to the differential for $X(D')$. Consequently, this lemma governs every non-zero term in the differential for $X(D')$ with implications explored in the next section.

Proof Let $\mathbb{L} \subset \mathbb{R}^3$ be a link with a (potentially disconnected) alternating diagram D possessing C crossings. Decompose D as $D_1, \dots, D_k$ where D_i is the diagram for the i^{th} connected component of D, thought of as a 4–valent graph. Let C_i be the subset of crossings in D_i. Resolving D_i using all 0–resolutions results in a collection of disjoint unknots, D_i^0, sitting in the plane $\mathbb{R}^2$. To obtain the all 1–resolution diagram of D_i, we attach two-dimensional 1–handles to these unknots, with each handle corresponding to each crossing. Thought of in $\mathbb{R}^2 \times I$, these handles construct a surface, S_i, which is homeomorphic to a sphere with discs removed, since D_i is alternating, according to [5]. If we do this for all the diagrams D_i, we obtain k planar surfaces in $\mathbb{R}^2 \times I$. Take $\mathbb{R}^2$ as the xy–plane in $\mathbb{R}^3$. Using the one point compactifications of $\mathbb{R}^2$ and $\mathbb{R}^3$, and crossing with I, we obtain embeddings of these k surfaces in $S^3 \times I$. Each surface S_i bounds a handlebody in $\mathbb{R}^2 \times I$, so each S_i is unknotted in $S^2 \times I$. Call the union of these connected surfaces $S \subset S^3 \times I$.

We will think of S as the disjoint union of planar surfaces, which provides an orientable cobordism from an unlink in $S^3 \times \{0\}$ to an unlink in $S^3 \times \{1\}$.

We will now describe the structure of the four-manifold obtained by taking the double branched cover of $S^3 \times I$, branched over S. First, since the handles defining S come from the resolutions of the diagram, they occur independently over A. In particular, we can add them in any order. Thus by isotoping each component of the surface S in $S^2 \times I$, we can arrange that S in $S^3 \times [0, \frac{1}{2}]$ consists of a union of planar surfaces with ≥ 1 boundary components on $S^3 \times \{0\}$ and exactly 1 boundary component in $S^3 \times \{\frac{1}{2}\}$. Taking the double branched cover of this gives a four-manifold cobordism from

$$\#^{l-1} S^1 \times S^2 \longrightarrow \#^{k-1} S^1 \times S^2,$$

where l is the number of boundary components in $S^3 \times \{0\}$ and k is the number of components of S in $S^3 \times [0, \frac{1}{2}]$. Assuming that this isotopy has been performed, let $S_i' = S_i \cap S^3 \times \{\frac{1}{2}\}$, then each handle in S_i' joins two distinct circles and thus, in the double branched cover, corresponds to a $4D$ 2–handle addition, which cancels one $S^1 \times S^2$ factor. The surface $S \cap S^3 \times [\frac{1}{2}, 1]$ has the reverse structure in each component: each handle divides a circle, and thus in its branched double cover each handle corresponds to a two handle addition which gives rise to a new $S^1 \times S^2$–summand.

Consequently, $H_2(\partial\Sigma(S)) \longrightarrow H_2(\Sigma(S))$ is surjective, and the unique torsion Spin^c structures on the boundaries have unique extensions to $\Sigma(S)$, which are also torsion. We can then compute the change in grading for the cobordism map for the torsion Spin^c structure and the cobordism $\Sigma(S)$

$$\frac{c_1(\mathfrak{s}_0)^2 - \big(2\,\chi(\Sigma(S)) + 3\sigma(\Sigma(S))\big)}{4} = \frac{0 - \big(2\,C + 3\cdot 0\big)}{4} = -\frac{C}{2} \tag{1}$$

since we have added C 2–handles, since there is one change of resolution per crossing.

$\Sigma(S)$ also occurs in the link surgery spectral sequence approach to computing the Heegaard Floer homology of the double branched cover. The link surgery spectral sequence arises from looking at disjoint balls around each crossing intersecting the link in two unknotted arcs. These balls lift to solid tori in $\Sigma(\mathbb{L})$. Taking the double branched cover of D resolved at a crossing corresponds to removing this torus and gluing it back with a framing determined by the resolution. Going from the 0–resolution to the 1–resolution comes from removing this solid torus again and gluing it back in, but using the framing coming from the double branched cover of the 1–resolution. This surgery is effected by a four-manifold 2–handle addition that corresponds to the branched double cover of the surface locally constructed by taking the arcs of the 0–resolution and adding a handle joining the components to obtain the 1–resolution.

To see this, take $S^3 \times I$, and consider in it $D^3 \times I$, where D^3 is the small ball near the 0–resolved crossing, containing the 2 arcs used in the resolution change. In $D^3 \times I$, we take a disc whose boundary is a union of three parts: (1) in $D^3 \times \{0\}$ take the two arcs, (2) in $\partial D^3 \times I$, take the segments formed from the four endpoints of these arcs crossed with I, and (3) in $D^3 \times \{1\}$ use the pair of arcs for the 1–resolution. The resulting boundary looks like the seam on a tennis ball and bounds a disc in $D^3 \times I$.

This disc comes from the two-dimensional 1–handle addition which changes between the resolutions. If we take the double branched cover of $D^3 \times I$ over this disc, we obtain a solid torus over $D^3 \times \{0\}$, which is the torus we remove in the link surgery spectral sequence, $T^2 \times I$ over $\partial D^3 \times I$, and another solid torus over $D^3 \times \{1\}$, which is the torus we add. It is straightforward to see that, in this latter solid torus, the meridional discs have boundary the same as the framing in the surgery spectral sequence. Furthermore, the decomposition of the boundary of $D^3 \times I$ extends to $D^3 \times I$, so that the double branched cover of $D^3 \times I$ over the disc, which is homeomorphic to D^4, can be thought of instead as $D^2 \times D^2$, with $S^1 \times D^2$ the torus we remove, and $D^2 \times S^1$ the torus we glue back. This provides a four-dimensional 2–handle addition effecting the desired Dehn surgery.

In summary, there is a surface S in $D^3 \times I$ that is built from the resolution changes in D and whose branched double cover can be constructed from the 2–handle additions defining the simplest maps in the link surgery spectral sequence.

We can use these descriptions of $\Sigma(S)$ to understand the rational grading change for any homotopy class of polygons occurring the spectral sequence $X(D)$. First, taking all the four-dimensional 2–handles together, we can describe the cobordism map for $\Sigma(S)$ using a Heegaard *triple* $(\Sigma, \{\alpha_i\}_{i=1}^g, \{\beta_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g)$ [10] where:

(1) $\beta_1, \dots, \beta_C$ are framings for solid tori corresponding to those from the 0–resolution double branched cover.

(2) $\gamma_1, \dots, \gamma_C$ are the framings of the tori that correspond to the 1–resolution double branched cover.

(3) $\beta_i \simeq \gamma_i$ for all $i \geq C+1$.

(4) $(\Sigma, \{\beta_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g) \cong \#^n S^1 \times S^2$, which in $\widehat{CF}$ has a unique generator, corresponding to a single intersection point in the Floer chain complex, which is closed and will generate $\widehat{HF}$ as an $\bigwedge H_1$–module. This generator in $\widehat{CF}$ will be denoted Θ^+.

We will only be concerned with the case where

$$(\Sigma, \{\alpha_i\}_{i=1}^g, \{\beta_i\}_{i=1}^g) \quad \text{and} \quad (\Sigma, \{\alpha_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g)$$

also represent connect sums of $S^1 \times S^2$'s. Given any generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ representing the torsion structures, there is a homotopy class of triangles ψ in $\psi_2(\mathbf{x}, \Theta^+, \mathbf{y})$ with $n_z(\psi) = 0$ due to the topological description of $\Sigma(S)$ above. Furthermore, any two such classes differ only by doubly periodic domains. Using the grading change formula in [10], restricted to the $\widehat{\ }$–theory, and the calculation in equation (1) we have:

$$\mathrm{gr}_{\mathbb{Q}}(\mathbf{y}) - \mathrm{gr}_{\mathbb{Q}}(\mathbf{x}) = -\mu(\psi) - \frac{C}{2}$$

for any such ψ.

We can use these homotopy classes to construct homotopy classes of higher polygons. From ψ, we can create a homotopy class of quadrilaterals on

$$\left(\Sigma, \{\alpha_i\}_{i=1}^g, \{\beta_i\}_{i=1}^g, \{\eta_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g\right)$$

where η_i is a small Hamiltonian isotope of β_i, except when $i = 1$, where η_1 is a small Hamiltonian isotope of γ_1. In effect, we will do the first surgery alone, and then all the others simultaneously. There is a homotopy class of triangles in $\left(\Sigma, \{\beta_i\}_{i=1}^g, \{\eta_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g\right)$ that represents a class $\pi_2(\Theta^+_{\beta\eta}, \Theta^+_{\eta\gamma}, \Theta^+_{\beta\gamma})$ since the corresponding cobordism map

$$\hat{F}\colon \widehat{HF}\left(\{\beta_i\}_{i=1}^g, \{\eta_i\}_{i=1}^g\right) \to \widehat{HF}\left(\{\eta_i\}_{i=1}^g, \{\gamma_i\}_{i=1}^g\right)$$

is non-trivial (see [10, section 4]). Call this homotopy class ψ_1. Then $\psi * \psi_1$ is a homotopy class of quadrilaterals in $\pi_2(\mathbf{x}, \Theta^+_{\beta\eta}, \Theta^+_{\eta\gamma}, \mathbf{y})$ and

$$\mu(\psi * \psi_1) = \mu(\psi) + \mu(\psi_1) + 1 = \mu(\psi) + 1.$$

We can repeat process for $\gamma_2, \ldots, \gamma_C$ to build a homotopy polygon of the type considered in the link surgery spectral sequence. Each time we glue in a homotopy class of triangles from a non-zero cobordism map, and thus construct a homotopy polygon with one more edge and μ incremented by one. Consequently, when we perform all C–surgeries we will obtain a homotopy $C+2$–gon in

$$\pi_2(\mathbf{x}, \Theta^+_{\beta\eta^1}, \ldots, \Theta^+_{\eta^{C-1}\gamma}, \mathbf{y})$$

with $\mu = \mu(\psi) + C - 1$.

In the surgery spectral sequence, we care only about such polygons with $\mu = 0$. To obtain such a polygon from our construction we need $\mu(\psi) = 1 - C$. Consequently, for any pair of generators joined by a $\mu = 0$ homotopy class of higher polygons, as occurs for non-zero higher differentials in $X(D)$, we have:

$$\mathrm{gr}_{\mathbb{Q}}(\mathbf{y}) - \mathrm{gr}_{\mathbb{Q}}(\mathbf{x}) = -\mu(\psi) - \frac{C}{2} = -1 + \frac{C}{2}$$

However, given the special topology of $\Sigma(S)$ we knot that all other polygons representing the same Spin^c structure, and joining the same endpoints in the chain complex, differ only by doubly periodic domains. The doubly periodic domains in any boundary each have $\mu = 0$, since we have torsion Spin^c structures on each boundary. Consequently, every homotopy n–gon with $\mu = 0$ will differ from the one constructed above by such doubly periodic domains, and thus still have the same change in grading. Finally, the only Spin^c structure on $\Sigma(S)$ restricting to the torsion structures on the boundary is the torsion one, so every homotopy class of polygon used in the complex $X(D)$ arises from the construction above, after adding doubly periodic domains. □

Put another way, the calculation above is a homotopy-based calculation, so for any $\mathbf{x}$ and $\mathbf{y}$ joined by a homotopy n–gon, we can add doubly periodic domains to obtain an n–gon homotopic to one constructed by the gluing procedure. We can then conclude that the grading difference is as stated. Note that we have relied heavily on the fact that $H_2(X;\mathbb{Z})$ is generated by boundary classes, otherwise we would need to include more Spin^c structures for which c_1^2 might not be zero. These additional Spin^c structured do occur for diagrams of non-alternating links.

9 Knot Floer results for alternating branch loci

We now explore the implications of the previous sections for knot Floer homology. First, we define some notation. Let $\mathbb{L}$ be a link in $A \times I$ admitting a connected, alternating projection to A. According to [13], the Heegaard Floer homology of $\Sigma(\mathbb{L}, \mathfrak{s})$ is congruent to $\mathbb{F}$ for each of the Spin^c structures on $\Sigma(\mathbb{L})$. For a Spin^c structure $\mathfrak{s}$ and a null-homologous knot $K \subset \Sigma(\mathbb{L})$, define

$$\tau(K, \mathfrak{s}) = \min_{s \in \mathbb{Z}} \left\{ s \colon\ \widehat{HF}(\mathcal{F}_s, \mathfrak{s}) \xrightarrow{i_*} \widehat{HF}(\Sigma(\mathbb{L}), \mathfrak{s}) \text{ is nontrivial} \right\}$$

where $\mathcal{F}_s$ is the sub-complex of generators with filtration index less than or equal to s. Using the results of the previous sections and Lemma 5.4 we can prove Theorem 1.2:

Theorem 9.1 *Let $\mathbb{L}$ be a non-split alternating link in $A \times I$ intersecting the spanning disc for B in an odd number of points. Then for each k there is an isomorphism*

$$\widehat{HFK}\left(-\Sigma(\mathbb{L}) \#^2 \left(S^1 \times S^2\right), \widetilde{B} \# B(0,0), k\right) \cong \bigoplus_{i,j \in \mathbb{Z}} H^{i;j,2k}(\mathbb{L})$$

where, for each Spin^c structure, the elements on the right side all have the same absolute $\mathbb{Z}/2\mathbb{Z}$–grading. Together these isomorphisms induce a filtered quasi-isomorphism from the E^2–page of the knot Floer homology spectral sequence to that of the skein homology spectral sequence. Thus the knot Floer spectral sequence collapses after two steps. Furthermore, for any $\mathfrak{s} \in \mathrm{Spin}^c(\Sigma(\mathbb{L}))$ we have that

$$\tau(\widetilde{B}, \mathfrak{s}) = 0$$

where $\widetilde{B}$ is considered in $\Sigma(\mathbb{L})$.

The content of this theorem is that all the knots $\widetilde{B}$ have the same knot Floer properties as alternating knots in S^3, and their knot Floer homology (over all Spin^c structures) is determined by the skein homology, and the Göritz matrix of $\mathbb{L}$, when applicable. The last is used to calculate the signature, through a formula of C Gordon and R Litherland, and the Heegaard Floer invariants, $d(\mathfrak{s})$, for $\mathfrak{s} \in \mathrm{Spin}^c(-\Sigma(\mathbb{L}))$ [13], which determine the precise absolute grading for the homology groups. However, it seems difficult to recover data about individual Spin^c structures from the Khovanov formalism.

Proof of Theorem 1.2 We have established that there is a spectral sequence starting at the right side of the isomorphism and converging to the left side. The right side is the E^2–page of this spectral sequence for the homological filtration. The higher pages are computed using maps between resolutions, R_1 and R_2 differing in at least two positions. Thus the maps will necessarily increase I–grading by the number of crossings where the resolutions differ, N. However, the pair R_1 and R_2 determine an alternating diagram (by leaving the crossings where they differ alone) to which the results of the previous section can be applied. By the absolute rational grading calculation in the preceding section, the homotopy classes of polygons with $\mu = 0$, and thus potentially giving rise to higher differentials, change the q–grading by $2(-1 + \frac{N}{2}) + N$, where (1) we multiply by two to change from absolute grading on $\widehat{HF}(Y_I)$ into q–gradings and (2) perform the obligatory shift of the q–grading by N since the I–grading has also increased. As a result, for any polygon used in the differential, the grading change is

$$I \longrightarrow I + N$$
$$J \longrightarrow J + 2N - 2$$

where N is the number of crossings where the corresponding resolutions are different.

Let $\delta = J - 2I$. Then δ changes by -2 for all relevant homotopy classes. Note that the Khovanov differential, which corresponds to $N = 1$ as described below, also alters δ by -2.

We can compute the spectral sequence from the reduced Khovanov homology to $\widehat{HF}(\Sigma(\mathbb{L}))$ in the following manner. First, compute the E^1–page in the spectral sequence by reducing those differentials with (I, J) change of $(0, 0)$. This yields the standard Khovanov complex on the E^1–page, and the higher differentials retain the grading shifts described above. The differential on the E^1–page consists only of the Khovanov $(1, 0)$–differentials (corresponding to $N = 1$) and reducing them yields the Khovanov homology at E^2. Furthermore, when we cancel the $(1, 0)$–differentials, we cancel differentials with δ–shift of -2, and thus all potential higher differentials continue to change the δ–grading by -2. However, by Lee's theorem [8], in the reduced Khovanov complex for an alternating knot or link, there is a constant C for which all the generators lie on diagonals $J-2I=C$. Thus the E^2–page is supported in a single grading. Together these results imply that all the higher differentials are trivial, and the spectral sequence collapses at E^2.

If we consider the spectral sequence for a projection into the annulus, we have E^2–page equal to the Khovanov skein homology. For the Khovanov skein homology we have a relationship of the form $K-J+2I=C$, even after adding several non-trivial unlinked circles to the projection. For now we will write the gradings on the complex $X(D)$ as (k,δ), where the second entry is the δ–grading above. Each of the differentials in the complex $X(D)$ used to compute $\widehat{HF}(\Sigma(\mathbb{L}))$ will change this pair by $(\Delta k,-2)$, from the argument above. Furthermore, $\Delta k \leq 0$ since the higher differentials are filtered by $\widetilde{B}$. Consequently, since at E^2 we have $K-\delta=C$ for some constant, after a higher differential we will have a generator for which $K-\delta=C+\Delta k-2<C$. Thus, the spectral sequence for each Alexander grading collapses at the E^2–page, and the only differentials showing up in the spectral sequence from $\bigoplus \widehat{HFK}(\Sigma(\mathbb{L}),\widetilde{B},k) \Rightarrow \widehat{HF}(\Sigma(\mathbb{L}))$ must occur before the E^2–page. With the preceding paragraph this implies that the spectral sequence is modeled on the spectral sequence from the reduced skein homology to the reduced Khovanov homology.

We now consider the spectral sequence from the knot Floer homology to the Heegaard Floer homology. The spectral sequence on the reduced skein homology induced by *the Alexander filtration* also collapses at its E^2–term. Following the above, after reducing the Khovanov skein homology differentials, the remaining portion of the Khovanov differential preserves j, but increases i by 1. Since $k-j+2i=\sigma(\mathbb{L})$ for any non-zero summand of the Khovanov skein homology, any non-trivial entry in these higher differentials changes k by -2. Canceling these yields the Khovanov homology. At that stage we recover $\widehat{HF}(-\Sigma(\mathbb{L}))$, as the reduced Khovanov homology has total rank given by $\det(\mathbb{L})$. By Lemma 5.4, the spectral sequence on the Khovanov skein homology is quasi-isomorphic to that on the knot Floer homology of $\widetilde{B}\,\#\,B(0,0)$ in $-\Sigma(\mathbb{L})$. This allows us to draw the conclusion concerning τ. Namely, the Heegaard Floer homology of $-\Sigma(\mathbb{L})\,\#^2\,(S^1\times S^2)$ will have the form $\widetilde{H}\otimes V^{\otimes 2}$ under the isomorphism to Khovanov homology, and will lie on the four lines $j-2i=-\sigma(\mathbb{L})$ (with multiplicity 2) and $j-2i=-\sigma(\mathbb{L})\pm 2$ due to our grading convention for the marked circle. When we factor out the $V^{\otimes 2}$, we have the reduced homology lying on $k=\sigma(\mathbb{L})+j-2i=0$. Since there is only one grading in each filtration level in the knot Floer homology, this implies that

$$\tau(\overline{\widetilde{B}})=0. \qquad \square$$

We can also derive some information about $\widetilde{B}$ for the branch loci depicted in Figure 1, regardless of whether $\mathbb{L}$ is alternating. In particular, $\widetilde{B} \subset S^3$ in these cases and:

Lemma 9.2 *For $\widetilde{B}$ coming from the branch loci depicted in Figure 1,*

$$\tau(\widetilde{B}) = -\tfrac{1}{2}T(\mathbb{L}).$$

Proof Add two non-trivial, non-interacting unknots to $\mathbb{L}$ and mark one of these. There is then a spectral sequence converging to the knot Floer homology of $\widetilde{B}\,\#^2\,B(0,0)$ from $H^{*;**}(\overline{\mathbb{L}}) \otimes V$. Consider an element in the sub-complex corresponding to knot filtrations less than or equal to $\tau(\widetilde{B})-1$ that maps to $\Theta^{--} \in \widehat{HF}(\#^2\,S^1 \times S^2)$ under inclusion of the sub-complex. Then there is a element with k–gradings less than or equal to $2\tau(\widetilde{B})-2$ that survives the spectral sequence to the knot Floer homology. However, since $\mathbb{L}$ is an unknot, Θ^{--} is the element $\mathbf{u}_{-1} \otimes v_-$. Therefore, this same element will survive the spectral sequence from the skein homology to the Khovanov homology. Hence $T(\overline{\mathbb{L}})-2 \leq 2\tau(\widetilde{B})-2$ and $-\frac{1}{2}T(\mathbb{L}) \leq \tau(\widetilde{B})$. This is also true for $\overline{B}$ whence

$$-\tfrac{1}{2}T(\overline{\mathbb{L}}) \leq \tau(\overline{\widetilde{B}}).$$

Therefore, $-\frac{1}{2}T(\mathbb{L}) \geq \tau(\widetilde{B})$ as well. □

These results hold in slightly greater generality. In the sequel to this paper an argument is given, which holds for a broader class of links, similar to the quasi-alternating links of [13]. This is the *smallest* subset of links in $A \times I$, denoted $\mathcal{Q}'$, with the property that:

(1) The alternating, twisted unknots, linking B an odd number of times, are in $\mathcal{Q}'$.

(2) If $L \subset A \times I$ is a link admitting a connected projection to A, with a crossing such that
  - the two resolutions of this crossing, L_0 and L_1, are in $\mathcal{Q}'$ and are connected in A, and
  - $\det(L) = \det(L_0) + \det(L_1)$,

  then L is in $\mathcal{Q}'$

The alternating L used above are in $\mathcal{Q}'$, and the elements of $\mathcal{Q}'$ when considered in S^3 are all quasi-alternating as in [13]. For this class of links Wehrli's algorithm terminates at the base cases of our induction, from which the conclusion in the theorem can be drawn. For braids in $\mathcal{Q}'$ we can be more precise about Plamenevskaya's element:

Corollary 9.3 *Let $\mathbb{L}$ be in $\mathcal{Q}'$. If the element $\widetilde{\psi}$ vanishes in the reduced Khovanov homology, then $c(\xi) = 0$.*

Proof This corollary follows from the non-vanishing result in Section 8 since the spectral sequence for X_{2-2g}/X_{-2g} collapses according to Theorem 1.2. However we need to verify that $\widetilde{\psi}$ is zero in Kh_{1-g}. The only difficulty arises if there is a ν whose Khovanov differential is $\widetilde{\psi}$ and $\nu = \sum \nu_i$ where ν_i is in Ψ–filtration level $2i$. Since the Khovanov differential reduces Ψ by at most 2, this requires the i indices to range from $1-g$ to l, and for there to be a summand for each index in the range. As we collapse the complex along differentials preserving the Ψ–filtration level, the complex stabilizes at E^2, and the structure described above yields a differential from ν_l to $\widetilde{\psi}$. However, we know that at E^2, $k - j + 2i = \sigma(\mathbb{L})$, and ν_l and $\widetilde{\psi}$ must have the same j value since they are linked by Khovanov differentials. In addition, the change in i is an increase of 1 from ν_l to $\widetilde{\psi}$. This implies that k must decrease by 2, and thus $l = 1-g$ as required. □

10 Examples

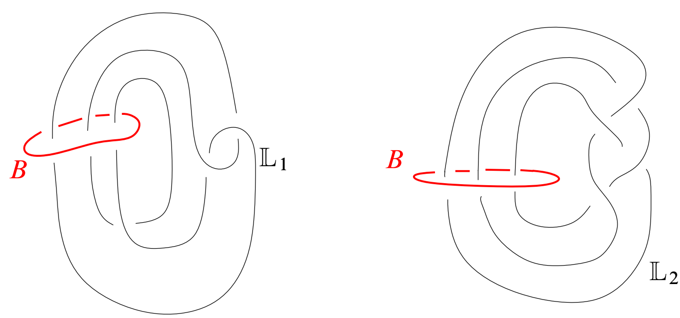
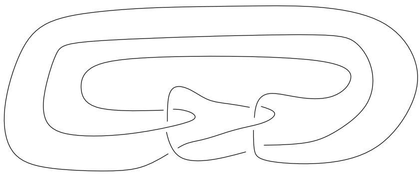


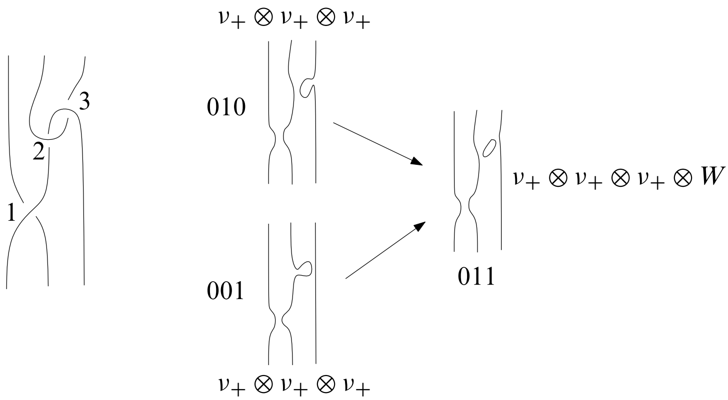
Figure 7: The diagram for Example 1 is on the left; that for Example 2 is on the right.

Example 0 Let $\mathbb{L}$ be a non-split alternating link and suppose B is a meridian of one of the components. Then $\widetilde{B}$ is an unknot in $-\Sigma(\mathbb{L})$ since the spanning disc lifts to a disc. Mark the link as above, then the reduced skein homology after the final shifting agrees with the reduced Khovanov homology. On the other hand, the knot Floer homology of this unknot is just $\widehat{HF}(-\Sigma(\mathbb{L}))$ in filtration level 0. The equivalence of these two groups is a consequence of [13]. In this sense, Theorem 1.2 is a generalization of the result in [13].

Example 1 See Figure 7 for the diagram. Here $\mathbb{L}$ is an unknot in S^3, so $\widetilde{B}$ is a knot in S^3 as well. Untwisting and taking the branched double cover (or using symmetry between the two components) shows that $\widetilde{B}$ is the knot:



This is the alternating knot, 6_1, with signature equal to 0. The main result in [9] now verifies the knot Floer conclusions of Theorem 1.2. Furthermore, the Alexander polynomial is $-2\,T^{-1}+5-2\,T$. We content ourselves with a direct verification of the rank of the highest filtration level. Only resolutions with three non-trivial circles contribute to this level. These resolutions and the associated generators are:



The maps from 010 and 001 to 011 both take $v_+\otimes v_+\otimes v_+$ to $v_+\otimes v_+\otimes v_+\otimes w_-$, and thus their sum is closed, as is $v_+\otimes v_+\otimes v_+\otimes w_+$. The latter is two q–gradings above the former closed element, but it also has one more 1–resolution. Shifting q down by 2 decreases the homological grading by 1 when identifying with knot Floer homology. Thus, these generators are in the same grading in the knot Floer complex. This confirms Theorem 1.2 for the highest filtration level (modulo some shifting).

Example 2 See Figure 7 for the diagram. Here $\mathbb{L}$ is the figure 8 knot, 4_1, whose branched double cover is $L(5,2)$. In this arrangement, $\widetilde{B}$ is a genus 1 fibered knot in $L(5,2)$. The possibilities for such a knot are strictly limited, since there is only a $\mathbb{Z}$ in filtration levels ± 1. The real content of the theorem here is that $\tau(\widetilde{B})=0$, as this implies that there is one Spin^c structure where the knot Floer homology is that of 4_1. We give a non genus 1 example later.

The monodromy for this knot is $(\gamma_1\gamma_2^{-1})^2$ where γ_i is a positive Dehn twist around a standard symplectic basis element for $H_1(T^2-D^2)$. The monodromy action on

H_1 and the Alexander polynomial associated to the $\mathbb{Z}$–covering from the fibering are computed to be

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \qquad \Longrightarrow \qquad \det(I - tA) \doteq \Delta_{\widetilde{B}}(t) = -T^{-1} + 7 - T^{1}$$

where we have symmetrized and normalized $\det(I - tA)$. We can compute the skein homology directly, but instead we use the theory to compute it from the Euler characteristic polynomial $V(-1,q,x)$. The polynomial satisfies:

$$V(-1,q,x)/(qx + q^{-1}x^{-1}) = q^{-4} + q^{4} - 2q^{-2} - 2q^{2} + q^{2}x^{2} + 1 + q^{-2}x^{-2}$$

From our conventions, we should add two non-trivial strands, and at the end factor out $V^{\otimes 2}$ to get to the knot Floer homology. However, adding a marked non-trivial circle and a non-trivial circle amounts to multiplying $qx(qx + q^{-1}x^{-1})V(t,q,x)$, so it will equal the above after shifting and removing $V^{\otimes 2}$.

Since the signature of 4_1 is zero, we can use $k - j + 2i = 0$ and the form $x^k q^j$ in the polynomial above to compute the i–grading. This gives the following diagram, where the coordinates are on the (j,k)–axes and the subscripts are the values of i.

$+4$					
$+3$					
$+2$				$\mathbb{F}_0$	
$+1$					
0	$\mathbb{F}_{-2}$	$\mathbb{F}^2_{-1}$	$\mathbb{F}_0$	$\mathbb{F}^2_1$	$\mathbb{F}_2$
-1					
-2		$\mathbb{F}_0$			
-3					

$-3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4$

Note that if we shift the elements to $j = 0$, decreasing i by 1 each time j decreases by 2, every group in the same horizontal row shifts to the same grading. Note also that the ranks after shifting horizontally reflect the coefficients of the Alexander polynomial; and, up to a minus sign, the $\mathbb{Z}/2\mathbb{Z}$–gradings are correct. Furthermore, if we consider the Ψ–filtration, in the E^{∞}–page of the spectral sequence there will be five terms on the $k = 0$ horizontal line, corresponding to the five Spin^c structures on $L(5,2)$. All that remains is to identify which generators correspond to which Spin^c structure and then use the Göritz matrix for 4_1 to complete the absolute grading calculations. To do this we should use the more refined torsion, $\check{\tau}(Y-K)$, in our Euler characteristic computations [6]. We complete this argument in the sequel to this paper. Comparing the two will show that the $\mathbb{Z}/2\mathbb{Z}$–gradings from the knot Floer homology correspond to those from the skein homology. However, the correspondence only occurs when we add over all Spin^c structures and all q–gradings.

Appendix A: Homological algebra

All coefficients are taken in $\mathbb{F}_2$, hence the difference from the usual signs. However, everything can be adapted to work with coefficients in $\mathbb{Z}$.

Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be filtered differential modules. Let $f\colon A \to B$ be a filtered chain map. Then the mapping cone $M(f)$ inherits a filtration by declaring $\mathcal{M}_i = \mathcal{A}_i \oplus \mathcal{B}_i$. That the differential preserves this filtration follows from f being filtered. When undeclared, a filtration on a mapping cone complex will come from this construction. The definitions imply that $E^1(\mathcal{M}) \cong \mathrm{MC}(E^1(f))$.

A filtered chain map f will be a 1–quasi-isomorphism if it induces an isomorphism between the E^1–pages of the spectral sequences for the source and the target. For the morphism of spectral sequences induced by f, in which the induced maps intertwine the differentials on each page, this implies that all the higher pages, E^r, are quasi-isomorphic by the induced map, $E^r(f)$. This is probably weaker than f being a filtered chain isomorphism, but enough for spectral sequence computations.

Let $\{(A_i, \mathcal{A}_i)\}_{i=0}^{\infty}$ be a set of filtered chain complexes with each filtration $\mathcal{A}_i$ being bounded and ascending:

$$\mathcal{A}_i\colon \{0\} = A_i^{n_i} \subset \cdots \subset A_i^j \subset A_i^{j+1} \subset \cdots \subset A_i^{N_i} \cong A_i$$

Let $\{f_i\colon A_i \to A_{i+1}\}$ be a set of chain maps satisfying:

(1) f_i is a filtered map for each i.

(2) $f_{i+1} \circ f_i$ is filtered chain homotopic to 0, ie, there is a filtered map

$$H_i\colon A_i \to A_{i+2}$$

such that $f_{i+1} \circ f_i = \partial_{i+2} \circ H_i + H_i \circ \partial_i$.

(3) $f_{i+2} \circ H_i + H_{i+1} \circ f_i\colon A_i \to A_{i+3}$ is a 1–quasi-isomorphism.

In this setting we have the lemma, following [13],

Lemma A.1 *The mapping cone* $\mathrm{MC}(f_2)$ *is 1–quasi-isomorphic to* A_4*.*

Proof The hypotheses above guarantee that the maps in the proof of lemma 4.4 of [13] are filtered maps. We need only check the filtering condition for maps in and out of the mapping cone, but with the aforementioned convention these are clearly filtered. In particular the map $\psi_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i$ is a 1–quasi-isomorphism by assumption, and the same argument as in [13] implies that α_2 is a quasi-isomorphism that is also filtered. This is not quite enough to conclude, but it does ensure that α_i induces maps at each page in the spectral sequence.

The module $Gr(A_i) \cong \bigoplus_{j\in\mathbb{Z}} A_i^j / A_i^{j-1}$ inherits a differential which maps the j^{th}–graded component to itself, whose homology provides E^1. The maps f_i induce chain maps between these complexes for each grading level. Indeed each of the maps ψ_i, H_i, f_i, etc, likewise induce such maps. Compositions such as $f_{i+1} \circ H_i$ induce maps on the graded components which are the same as the compositions for the maps induced from f_{i+1} and H_i separately. Thus for each j, we have the situation in the lemma in [13] applied solely to the j^{th}–graded component. Applying the lemma in each grading guarantees that the map induced in that grading by α_2 is a quasi-isomorphism, ie, that the induced map on the E^1–page is an isomorphism of spectral sequences. Thus, α_2 induces an isomorphism from the E^1–page for A_4 to $\mathrm{MC}(E^1(f_2)) \cong E^1(\mathrm{MC}(f_2))$, which is the desired result. □

As in [13], we can reinterpret this as a result on iterated mapping cones. Let $M = \mathrm{MC}(f_1, f_2, f_3)$ be the filtered chain complex on $A_1 \oplus A_2 \oplus A_3$, filtered by

$$A_1^j \oplus A_2^j \oplus A_3^j,$$

and equipped with the differential

$$\begin{pmatrix} \partial_1 & 0 & 0 \\ f_1 & \partial_2 & 0 \\ H_1 & f_2 & \partial_3 \end{pmatrix}$$

That this is a differential is a consequence of the assumptions made before the lemma. The lemma then implies that the induced spectral sequence on the iterated mapping cone collapses at the E^1–term. This follows according to the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3^{j-1} & \longrightarrow & A_1^{j-1} \oplus A_2^{j-1} \oplus A_3^{j-1} & \longrightarrow & A_1^{j-1} \oplus A_2^{j-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3^{j} & \longrightarrow & A_1^{j-1} \oplus A_2^{j} \oplus A_3^{j} & \longrightarrow & A_1^{j} \oplus A_2^{j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3^{j}/A_3^{j-1} & \longrightarrow & M^j/M^{j-1} & \longrightarrow & \mathrm{MC}^j(f_1)/\mathrm{MC}^{j-1}(f_1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the top two rows are exact, and all the columns are exact. The nine lemma now guarantees that the bottom row is exact, and each of the maps is a chain map. In the long exact sequence from the bottom row, there is one map guaranteed to be an isomorphism by the lemma. Consequently, the groups in $E^1(M)$ are trivial.

Appendix B: The structure of twisted unknots

First let:

- $\mathbb{L}$ be a knot or link in $A \times [0,1]$ where $A = \{z | 1 < |z| < 2\}$.
- D be the a generic diagram for the projection of $\mathbb{L}$ along $[0,1]$ in $A \subset \mathbb{R}^2$.
- Γ_D be the 4–valent graph in A found by stripping the crossing data from D.

If c is a crossing for D, then $D_0(c)$, $D_1(c)$ are the $0/1$–resolutions of the diagram D at c. By an isotopy of a diagram we mean an isotopy in A of the graph Γ_D that does not change the combinatorial structure of the graph, and thus allows us to carry along the crossing data in D.

Proposition B.1 *Let D be a diagram such that:*

(1) *Γ_D is connected.*

(2) *Either $\Gamma_{D_0(c)}$ or $\Gamma_{D_1(c)}$ is disconnected for each crossing c in D.*

Then in $\mathbb{R}^2$, D can be simplified to a standard unknot diagram using only diagram isotopies and the first Reidemeister move. In A, D can be simplified using diagram isotopies and the first Reidemeister move to a diagram in A which differs from one of those depicted in Figure 1 only by second Reidemeister moves in the horizontal twisted bands.

First, we establish some conventions. Since $\Gamma_D \subset A$ we can checkerboard color $(\mathbb{R}^2, \Gamma_D)$ with the region $|z| < 1$ being black, and consider D to be in $\mathbb{R}^2$. To recover information about the embedding in A we will keep this black region fixed. Let B_D be the black Tait graph, ie, the planar graph whose vertices are the black regions with edges joining regions which abut at a crossing. The region containing $|z| < 1$ will be a root for B_L, which will not participate in any simplification due to Reidemeister moves. We will sometimes think of B_L as embedded in A with each vertex lying in the black region corresponding to it, and each edge passing through the corresponding crossing.

Given a finite planar graph, Γ, with at least one vertex, call the components of $\mathbb{R}^2 \backslash \Gamma$ the regions of Γ. Each region comes with a canonical embedding in $\mathbb{R}^2$. We can thus take the closure of a region, R, in $\mathbb{R}^2$, which we denote by $\bar{R}$. We define the set of discs of Γ to be

$$\mathcal{D}(\Gamma) = \Big\{ D \subset \mathbb{R}^2 \;\Big|\; D \cong D^2, D = \bigcup_{i \in I} \bar{R}_i \Big\}$$

where D^2 is the closed unit disc, and $\{R_i \mid i \in I\}$ is a finite subset of regions for Γ. Thus elements of $\mathcal{D}(\Gamma)$ are those discs in $\mathbb{R}^2$ formed by taking unions of closures of regions. Note that the boundary of D is a loop in Γ, without backtracking, since the regions are taken to be closed. We give a partial order to this set: $D_1 \le D_2 \iff D_1 \subset D_2$. We also note that if $\Gamma_1 \subset \Gamma_2$, as graphs, then $\mathcal{D}(\Gamma_1) \subset \mathcal{D}(\Gamma_2)$.

We can divide $\mathcal{D}(\Gamma)$ into two subsets $\mathcal{D}_1 \cup \mathcal{D}_{\ge 2}$. $\mathcal{D}_1$ consists of those discs with one or fewer vertices of Γ in its boundary. $\mathcal{D}_{\ge 2}$ consists of all the others. Note that the boundary of an element of $\mathcal{D}(\Gamma)$ is in Γ. So a disc for Γ with no vertices in its boundary would imply that Γ is either disconnected or a loop with no vertices. Since we assume Γ is connected with at least one vertex every disc in $\mathcal{D}_1$ has a boundary consisting of a self-loop, a single edge joining a vertex to itself. Furthermore, if we take a subgraph of Γ, discs may switch from $\mathcal{D}_{\ge 2}$ to $\mathcal{D}_1$, but not vice-versa.

For future use, note that two discs in $\mathcal{D}_1$ intersect if and only if one is a proper subset of the other or they intersect only at a shared vertex. We can see this by noting that the boundary of one is a circle, and so cannot be contained in the boundary of the other, unless they are equal. Since the circle bounds only one disc in $\mathbb{R}^2$, this would imply the discs are equal. However, if the discs intersect, then the boundary of one must intersect the boundary of the other, or one must be strictly contained in the other. The boundaries can only intersect if they share the same vertex.

For a rooted planar graph, Γ, define Γ' to be the graph resulting from deleting the self-loops. This is still planar, and the vertices and other edges are assumed to be unchanged. Let Γ'' be the result of removing from Γ' all the edges e such that $\Gamma' \backslash e$ is disconnected, then deleting all the non-root vertices which abut no edges. Each disc in $\mathcal{D}_{\geq 2}(\Gamma)$ corresponds to a unique disc in $\mathcal{D}(\Gamma'')$.

Proof of Proposition B.1 Let D be as in the proposition. Then B_D is connected since Γ_D is connected: for two vertices of B_D we can construct a path between them by first considering $B_D \subset A$ as above. Then we start with a path from v_1 to Γ_D lying entirely in the black region corresponding to v_1, followed by a path in Γ_D to a point in the boundary of the region corresponding to v_2, followed by a path to v_2 contained in that black region. Away from crossings, we may push this path entirely into the black regions, and passing through crossings. Reading off the regions the path goes through interleaved with the crossings the path goes through provides a path in B_D from v_1 to v_2. Reversing the process gives shows that Γ_D is connected when B_D is connected.

B_D always has a vertex, the root, and an edge, unless D is the standard unknot diagram. The assumptions in the proposition require $\mathcal{D}_{\geq 2}(B_D) = \varnothing$. If not, we may use the partial order on $\mathcal{D}_{\geq 2}$ to find a smallest such disc. The interior of this disc can only intersect B_D in trees with self-loops. The boundary of this disc contains more than one vertex, and thus two or more edges which have distinct endpoints. Each of these edges corresponds to a crossing in D. If we pick the crossing, c, corresponding to one of these edges, then $D_0(c)$ and $D_1(c)$ are both connected. One resolution corresponds to collapsing the edge in B_D and the other corresponds to deleting the edge (depending on the type of crossing), but in either case B_D remains connected since the edge is part of a cycle in B_D. Consequently, B''_D is just the root vertex, since any edge remaining in B''_D would participate in a cycle which bounded a disc.

Working backwards, B_D is a rooted tree with self-loops attached, with more than one edge. In A, we apply the following two lemmas to subsets homeomorphic to D^2 (no longer necessarily members of $\mathcal{D}(D)$). Γ is taken to be a subgraph of B_D contained in D^2 with a vertex on the boundary of D^2, for instance along a branch of the tree.

Lemma B.2 *Suppose $\Gamma \subset D^2$ with its root on ∂D^2 and otherwise contained in the interior of the disc. Suppose Γ is connected, and Γ'' is just the root. Either Γ is just the root or in D^2 there is a sub-disc, $\widetilde{D}$, whose intersection with Γ consists solely in either (1) an edge with two vertices, one in the interior of $\widetilde{D}$, the other on its boundary (possibly the root), or (2) a vertex on $\partial \widetilde{D}$ and a self-loop in the interior of $\widetilde{D}$.*

Proof Assume that there are no subdiscs of type (2). If Γ is not just the root there must be a vertex or edge in the interior of D^2. Since Γ'' is just the root, $\mathcal{D}(\Gamma) = \mathcal{D}_1(\Gamma)$. If $\mathcal{D}_1(\Gamma) \neq \varnothing$, choose an element, $\widetilde{D}'$, which is minimal in the partial order. This disc has boundary given by a vertex, v, and a self-loop. Since there are no regions of type (2), Γ must intersect the interior of this disc (or we could enlarge it slightly to obtain a region of type (2)). We now restrict to this smaller disc, taking $\Delta = \Gamma \cap (\mathrm{int}(\widetilde{D}') \cup v)$. Δ cannot have any self-loops, as these would generate elements of $\mathcal{D}_1(\Gamma)$ that are smaller in the partial order. Therefore, Δ is a tree with at least one edge. Thus Δ has a leaf edge. A small neighborhood of this edge, not passing through the terminal vertex, but passing through the other vertices, will provide a region of type (1) in the proposition. □

Lemma B.3 *If D is a diagram in $\mathbb{R}^2$ and there is a disc $\widetilde{D} \subset \mathbb{R}^2$ intersecting B_D as in the conclusion of the previous lemma, then we can reduce D in $\widetilde{D}$ using only the first Reidemeister move.*

Proof Let $\Gamma = \widetilde{D} \cap B_D$, rooted at the single vertex on the boundary of $\widetilde{D}$. The assumption is that Γ'' is just this root. In the corresponding portion of D a leaf vertex, without a self-loop, corresponds to an RI–move. We perform all such moves iteratively until we arrive at a diagram without any more. By the preceding lemma this is either just the root, or there are now self-loops. These correspond to RI–moves as well. Adding a leaf corresponds to an RI–move that extends the black regions, whereas a self-loop bounding a disc corresponds to an RI–move that extends the white regions. Thus we can continue using RI–moves to reduce the diagram. The preceding lemma continues to apply to each successive diagram and its black graph until all that remains is the root, which corresponds to the sole black region not contained in the interior of $\widetilde{D}$. □

Summarizing, the terminal state of Wehrli's algorithm gives a diagram D such that $B_D'' = \{\text{root}\}$. Inside the annulus A, any self-loop bounding a disc not containing the root corresponds to an allowable RI–move, as does any leaf, using the lemmas above. If we apply such a move to D, we obtain a new diagram D_1 with B_{D_1} equal to B_D minus a leaf edge and vertex, or a self-loop. In particular, $B_{D_1}'' = \varnothing$. We can proceed, therefore, until B_D no longer has any leaf edges, or self-loops bounding a disc in A. All self-loops must bound a disc containing the root, therefore the discs in $\mathcal{D}_1$ form a chain in the partial order, with the smallest containing $\{z \mid |z| < 1\}$. If we take two consecutive self-loops, they cobound an annulus, which B_D intersects in a contractible graph with a vertex on each boundary. Since this graph also does not have any leaf edges, it must be a linear graph, with some number of vertices in between the two boundary vertices.

Thus, B_D must have the following form: it starts at the root, is followed by $n_1 \geq 0$ vertices without self-loops, a vertex with $s_1 > 1$ self-loops, then $n_2 \geq 0$ vertices without self-loops, and a vertex with s_2 self-loops, ... This terminates at a vertex with s_k self-loops (otherwise it is a leaf). When we put in the crossing data to return to D, we obtain a diagram which is almost isotopic to one of the twisted unknots in Figure 1. However, if two consecutive edges abutting the same vertex (self-loops or not) correspond to crossings of different handedness, we will need an RII–move to obtain the monotonicity of the twisting regions. After performing some isotopies of the diagram, and a finite number of RII–moves we arrive at a diagram as in the figure. Of course, if we are considering the diagram in $\mathbb{R}^2$ we do not need the RII–moves, as we can start at the root, which is a leaf vertex, and simplify from there using only RI–moves. □

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