On Korn's First Inequality with Nonconstant Coefficients

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Abstract

In this paper we prove a Korn-type inequality with nonconstant coefficients which arises from applications in elasto-plasticity at large deformations. More precisely let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with nonvanishing 2-dimensional Lebesgue measure. Define $H^{1,2}_{\circ}(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|_\Gamma} = 0\}$ and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$ be given with $\det F_p(x) \geq \mu^+ > 0$. Moreover suppose that $Rot F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\begin{aligned}&\exists\, c^+ > 0 \quad \forall\, \phi \in H^{1,2}_{\circ}(\Omega, \Gamma) : \\ &\qquad \|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+ \ \|\phi\|^2_{H^{1,2}(\Omega)} \,.\end{aligned}$$

Clearly this result generalizes the classical Korn's first inequality

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^{1,2}_{\circ}(\Omega, \Gamma) : \quad \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+ \ \|\phi\|^2_{H^{1,2}(\Omega)}$$

which is just our result with $F_p = 1\!\!1$. With slight modifications we are able to treat as well forms of the type

$$\|F_p(x) \cdot \nabla\phi \cdot G(x) + G(x)^T \cdot \nabla\phi^T \cdot F_p^T(x)\|^p, \quad 1 < p < \infty \,.$$

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1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with nonvanishing 2-dimensional Lebesgue measure. For $a, b \in \mathbb{R}^3$ we let (a, b) denote the scalar product on $\mathbb{R}^3$. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 matrices and by $skew(\mathbb{M}^{3\times3})$ the skew-symmetric real 3×3 matrices. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle A,B\rangle = tr(A \cdot B^T)$ and subsequently we have $\|A\|^2 = \langle A,A\rangle$. With $\mathrm{Adj}A$ we denote the matrix of transposed cofactors $Cof(A)$ such that $\mathrm{Adj}A = \det A \cdot A^{-1} = Cof(A)^T$ if $A \in GL(3,\mathbb{R})$. The identity matrix on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $tr(A) = \langle A, 1\!\!1\rangle$. In general we work in the context of nonlinear elasticity. For $u \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $\nabla u \in C(\overline{\Omega}, \mathbb{M}^{3\times3})$. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_{\circ}(\Omega)$ which we use indifferently for scalar-valued functions as well as for vector-valued functions. We define $H^{1,2}_{\circ}(\Omega,\Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\}$ where $\phi_{|\Gamma} = 0$ is to be understood in the sense of traces and by $C_0^\infty(\Omega)$ we denote infinitely differentiable functions with compact support in Ω.

2 Motivation

In the nonlinear theory of elasto-viscoplasticity at large deformation gradients it is often assumed that the deformation gradient $F = \nabla u$ splits multiplicatively into an elastic and plastic part

$$\nabla u(x) = F(x) = F_e(x) \cdot F_p(x), \quad F_e, F_p \in GL(3,\mathbb{R}) \tag{1}$$

where F_e, F_p are explicitly understood to be incompatible configurations, i.e $F_e, F_p \neq \nabla\Psi$ for any $\Psi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$. In our context we assume that this decomposition is uniquely defined up to a rigid rotation. In addition one sometimes imposes the so called plastic incompressibility constraint, $\det F_p(x) = 1$. This multiplicative split, which has gained more or less permanent status in the literature, is micromechanically motivated by the kinematics of single crystals where dislocations move along fixed slip systems through the crystal lattice. The source for the incompatibility are those dislocations which did not completely transverse the crystal and consequently give rise to an inhomogeneous plastic deformation. Therefore it seems reasonable to introduce the deviation of the plastic intermediate configuration F_p from compatibility as a kind of plastic **dislocation density**. This deviation should be related somehow to the quantity $RotF_p$ and indeed later on we see the important role which is played by $RotF_p$, see [5, 16, 19, **?**, 21, 28] for more on this subject and for applications of this theory in the engineering field look e.g at [23, 26, 27]. The above split contrasts the additive decomposition

$$\frac{1}{2}(\nabla\tilde{u} + \nabla\tilde{u}^T) = \epsilon(\tilde{u}(x)) = \epsilon_e(x) + \epsilon_p(x)$$

where we have set $F = 1\!\!1 + \nabla\tilde{u}$ with $\tilde{u}$ the displacement vector and where subsequently $\epsilon(\tilde{u}(x))$ denotes the infinitesimal strain tensor. This decomposition is appropriate only for infinitesimal small values of $\|\nabla\tilde{u}\|$, see e.g. [2, 12, 15] and references therein. Nevertheless, the additive decomposition can be seen as a first order approximation of (1).
Generally one is then led to define an elastic energy

$$\hat{W} = \hat{W}(F_e) = \hat{W}(\nabla u \cdot F_p^{-1})\,.$$

This constitutive relation is subject to material frame indifference, i.e must remain invariant under superimposed rigid body motions. Together with isotropy of $\hat{W}$ for $F_p = 1\!\!1$ and the requirement, that $D\hat{W}(1\!\!1) = 0$ it can be shown [6, p.156] that there exist the so called Lamé constants $\lambda, \mu > 0$ such that

$$\hat{W} = \hat{W}(F_e) = \lambda\,\|F_e^TF_e - 1\!\!1\|^2 + \mu\,tr(F_e^TF_e - 1\!\!1)^2 + o(\|F_e^TF_e - 1\!\!1\|^2)$$

near a natural state.

2.1 No elastic rotations

In metal-plasticity one observes that the quantity $\|F_e^TF_e - 1\!\!1\|$ remains pointwise small. If we incorporate this experimental fact directly into the form of the elastic energy and disregard elastic rotations, i.e postulate in addition that $\|F_e - 1\!\!1\|$ is small, we are led to consider elastic energies of the kind

$$\begin{aligned} W = W(\nabla u \cdot F_p^{-1}) = W(F_e) &= 4\lambda\,\|\frac{F_e^T + F_e}{2} - 1\!\!1\|^2 + 4\mu\,tr(\frac{F_e^T + F_e}{2} - 1\!\!1)^2 \\ &= 4\lambda\,\|\frac{\nabla u \cdot F_p^{-1} + F_p^{-T} \cdot \nabla u^T}{2} - 1\!\!1\|^2 + 4\mu\,tr(\frac{\nabla u \cdot F_p^{-1} + F_p^{-T} \cdot \nabla u^T}{2} - 1\!\!1)^2 \end{aligned}$$

where we have used that $F_e = 1\!\!1 + (F_e - 1\!\!1)$ and eliminated terms which are quadratic in $(F_e - 1\!\!1)$.
If we define the corresponding functional $I : H_\circ^{1,2}(\Omega, \Gamma) \times C^2(\overline{\Omega}, GL(3,\mathbb{R})) \mapsto \mathbb{R}$

$$I(u, F_p^{-1}) := \int_\Omega W(\nabla u \cdot F_p^{-1})\,dx$$

and compute the second derivative with respect to u we see that

$$\begin{aligned} D_u^2 I(u,F_p^{-1}).(\phi,\phi) &= \int_\Omega D^2W(\nabla u \cdot F_p^{-1}).(\nabla\phi,\nabla\phi)\,dx \\ &= \int_\Omega 4\lambda\,\|\nabla\phi\cdot F_p^{-1} + F_p^{-T}\cdot\nabla\phi^T\|^2 + 4\mu\,tr(\nabla\phi\cdot F_p^{-1} + F_p^{-T}\cdot\nabla\phi^T)^2\,dx \\ &\geq 4\lambda\,\|\nabla\phi\cdot F_p^{-1} + F_p^{-T}\cdot\nabla\phi^T\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Here $D^2W(\nabla u \cdot F_p^{-1})$ is the corresponding elasticity tensor, which is not independent of the plastic evolution. Observe however that $D_u^2 I(u,F_p^{-1}).(\phi,\phi)$is independent of the deformation u itself.

2.2 The case with elastic rotations

We can adapt the above framework so as to incorporate elastic rotations. Thus we assume only that $\|F_e^T F_e - 1\!\!1\|$ remains small. An application of the polar decomposition theorem then shows that $\|F_e - R_e\|$ has to be small as well for a uniquely defined $R_e \in O(3)$. If we repeat the above procedure with R_e instead of $1\!\!1$ we get

$$\begin{aligned} W = W(F_e) &= 4\lambda\,\|\frac{F_e^T\cdot R_e + R_e^T\cdot F_e}{2} - 1\!\!1\|^2 + 4\mu\,tr(\frac{F_e^T\cdot R_e + R_e^T\cdot F_e}{2} - 1\!\!1)^2 \\ &= 4\lambda\,\|\frac{R_e^T\nabla u F_p^{-1} + F_p^{-T}\nabla u^T R_e}{2} - 1\!\!1\|^2 + 4\mu\,tr(\frac{R_e^T\nabla u F_p^{-1} + F_p^{-T}\nabla u^T R_e}{2} - 1\!\!1)^2 \end{aligned}$$

where we have used that $F_e = R_e + (F_e - R_e)$ and eliminated terms which are quadratic in $(F_e - R_e)$.Both quantities R_e and F_p induce at the same time inhomogeneites and anisotropy.
The second derivative of the corresponding functional at a given rotation R_e can be estimated by

$$D_u^2 I(u,F_p^{-1}).(\phi,\phi) \geq 4\lambda\,\|R_e^T\cdot\nabla\phi\cdot F_p^{-1} + F_p^{-T}\cdot\nabla\phi^T\cdot R_e\|^2_{L^2(\Omega)}\,.$$

In the quasistatic viscoplastic setting without body forces we then have to solve in both cases the following system of coupled partial differential and evolution equations for $u : [0,T]\times\overline{\Omega} \mapsto \mathbb{R}^3$ and $F_p : [0,T]\times\overline{\Omega} \mapsto GL(3,\mathbb{R})$

$$\begin{aligned} div\,DW(\nabla u(t,x)\cdot F_p^{-1}(t,x)) &= 0 \quad x\in\Omega \\ \frac{d}{dt}F_p^{-1}(t,x) &= f(\nabla u(t,x), F_p^{-1}(t,x)) \\ u_\Gamma(t,x) &= g(t,x) \quad x\in\Gamma \\ F_p^{-1}(0,x) &= F_{p0}^{-1} \end{aligned}$$

with a nonlinear flow function $f : \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{3\times3}$ which governs the visco-plastic evolution and is motivated by thermodynamical considerations. Here $g(t,x)$ represents the time dependent inhomogeneous Dirichlet boundary data and $F_{p_0}^{-1}$ the initial condition for the plastic evolution. This system is formally equivalent to

$$
\begin{aligned}
\forall\, t \in [0,T]: \quad I(u(t), F_p^{-1}(t)) &\mapsto min, \quad u(t) \in g(t) + H_\circ^{1,2}(\Omega,\Gamma) \\
\frac{d}{dt} F_p^{-1}(t,x) &= f(\nabla u(t,x), F_p^{-1}(t,x)) \\
F_p^{-1}(0,x) &= F_{p_0}^{-1}\,.
\end{aligned}
$$

We have to remark that the above procedure leads to a **linear elliptic system** in u for fixed F_p with **nonconstant coefficients** which are determined by F_p which remains valid (at least from a modeling point of view) for both large plastic deformations F_p and large deformation gradients ∇u. Note however that the solution u depends nonlinear on F_p.

In the small strain case, where ϵ, ϵ_p is used the corresponding equilibrium equation form a **linear elliptic system** in $\tilde{u}$ for fixed ϵ_p with **constant coefficients** and the solution depends linear on ϵ_p.

Our main Theorem 3 in conjunction with the direct methods of the calculus of variations then tells us that for given smooth invertible F_p the problem

$$
\begin{aligned}
div\; DW(\nabla u(t,x)\cdot F_p^{-1}(t,x)) &= 0 \quad x \in \Omega \\
u_\Gamma(t,x) &= g(t,x) \quad x \in \Gamma
\end{aligned}
$$

has a unique solution. This will be the first step in an existence proof of the evolution problem.
In the presence of elastic rotations the above system has to be complemented by either an evolution equation for R_e or some incremental device, which determines the rotation R_e uniquely at every timestep, e.g. we could set $R_e^{n+1} = R_e(F_e^n)$ where $R_e(F_e^n)$ denotes the rotation associated with F_e^n.
If we set out to formulate a linear problem for the deformation u it seems impossible to use energies of the type $W = W(C, C_p)$ together with evolution equations for C_p. Even in the physically linear setting $W(C,C_p) = \langle D(x).(C - C_p), C - C_p\rangle$ where D denotes a fourth order tensor and the assumption that $C - C_p$ remains small the problem for u will be nonlinear. This underlines again the importance of a formulation where rotations R_e are explicitly involved.
The fully nonlinear case, where $\hat{W} = \hat{W}(F)$ is only required to be **polyconvex** has been investigated by the author in [**?**]. There one can find a local in time existence theorem of a suitably regularized coupled visco-plastic problem.

The theory of coercive forms has a long dating history and we dare not trace its origins. One refers usually to [18] for a first version of Korn's inequality. By the classical Korn's first inequality we mean

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega,\Gamma) : \\ \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+ \; \|\phi\|^2_{H^{1,2}(\Omega)} \, .$$

and we say that the classical Korn's second inequality holds, if

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega,\Gamma) : \\ \|\nabla\phi + \nabla\phi^T\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geq c^+ \; \|\phi\|^2_{H^{1,2}(\Omega)} \, .$$

Friedrichs furnished a modern proof [9] of the above inequalities. See [25, 9, 13, 4, 17] for more on this subject. The widespread popularity of Korn's inequalities may be explained by their applicability to the linearized systems of elasticity. In this case they yield existence, uniqueness and continuous dependence upon data. Recently, Weck [29] has shown how to circumvent Korn's second inequality in case of irregular domains and if only questions of existence are to be settled.
Ciarlet has shown [8, 7] how to extend Korn's inequalities to curvilinear coordinates which has applications in shell theory. The main contribution of this article is to extend Korn's first inequality to nonconstant coefficients which cannot be realized as metric of an underlying deformation . We rely on a theorem on coerciveness of [13] which was subsequently generalized by [4]. This theorem generalizes the Korn's second inequality to nonconstant coefficients. We then proceed to show that the nullspace of our form is trivial. A compactness argument then gives the generalized Korn's first inequality. As a special case we recover in different terms the situation of [8, p.44].

3 Preliminaries

In the sequel we need the following operations between $\mathbb{M}^{3\times3}$ and the Euclidean real vector space $\mathbb{R}^9$:

Definition 1 (Identification of $\mathbb{R}^9$ and $\mathbb{M}^{3\times3}$)
We define the following operator $matrix : \mathbb{R}^9 \mapsto \mathbb{M}^{3\times3}$.

$$matrix \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} .$$

Definition 2
We define the following operator $vec : \mathbb{M}^{3\times3} \mapsto \mathbb{R}^9$.

$$vec \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{pmatrix}^T .$$

Of course, both operations are just the canonical identifications of $\mathbb{R}^9$ and $\mathbb{M}^{3\times3}$. We need as well the following identification of $skew(\mathbb{M}^{3\times3})$ and $\mathbb{R}^3$:

Lemma 1
Let $A \in \mathbb{M}^{3\times3}$ be skew symmetric, i.e $A = -A^T$. If $A \neq 0$ then $rank(A) = 2$. In addition there is a vector $\omega \in \mathbb{R}^3$ such that

$$A = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}$$

.

Lemma 2
Let $A \in \mathbb{M}^{3\times3}$ be skew symmetric and $B \in GL(3,\mathbb{R})$. If $rank(A \cdot B) \leq 1$ then $A = 0$.

Proof. If $rank(A \cdot B) \leq 1$ then we can find two linear independent vectors $\tau_1, \tau_2 \in \mathbb{R}^3$ such that $(A \cdot B).\tau_1 = (A \cdot B).\tau_2 = 0$. But B is invertible and we see that $dim(ker(A)) \geq 2$ which is only possible for $A = 0$ because of Lemma 1. ■

Corollary 1
$skew(\mathbb{M}^{3\times3})$ and $\mathbb{R}^3$ can be identified via

$$\omega : \mathbb{R}^3 \mapsto skew(M^{3\times3})$$
$$\omega \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 \\ -\zeta_1 & 0 & \zeta_3 \\ -\zeta_2 & -\zeta_3 & 0 \end{pmatrix}$$

and ω is bijective onto its range.

Proof. Obvious. ■

Definition 3 (Rot)
We define the operator $Rot : C^1(\overline{\Omega}, \mathbb{M}^{3\times3}) \mapsto C(\overline{\Omega}, \mathbb{M}^{3\times3})$ such that we take the operator $rot : C^1(\overline{\Omega}, \mathbb{R}^3) \mapsto C(\overline{\Omega}, \mathbb{R}^3)$ rowwise; for example let $Y \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ then

$$Rot(Y) = \begin{pmatrix} rot[Y_{11}(x,y,z), Y_{12}(x,y,z), Y_{13}(x,y,z)] \\ rot[Y_{21}(x,y,z), Y_{22}(x,y,z), Y_{23}(x,y,z)] \\ rot[Y_{31}(x,y,z), Y_{32}(x,y,z), Y_{33}(x,y,z)] \end{pmatrix}$$

Lemma 3
For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ with $A = -A^T$ and $B \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ we have

$$Rot(A \cdot B) = matrix[L_B.vec[\nabla(\omega^{-1}(A))]] + A \cdot Rot(B)$$

with a linear map $L_B : \mathbb{R}^9 \mapsto \mathbb{R}^9$

$$L_B = \begin{pmatrix} 0 & b_{23} & -b_{22} & 0 & b_{33} & -b_{32} & 0 & 0 & 0 \\ -b_{23} & 0 & b_{21} & -b_{33} & 0 & b_{31} & 0 & 0 & 0 \\ b_{22} & -b_{21} & 0 & b_{32} & -b_{31} & 0 & 0 & 0 & 0 \\ 0 & -b_{13} & b_{12} & 0 & 0 & 0 & 0 & b_{33} & -b_{32} \\ b_{13} & 0 & -b_{11} & 0 & 0 & 0 & -b_{33} & 0 & b_{31} \\ -b_{12} & b_{11} & 0 & 0 & 0 & 0 & b_{32} & -b_{31} & 0 \\ 0 & 0 & 0 & 0 & -b_{13} & b_{12} & 0 & -b_{23} & b_{22} \\ 0 & 0 & 0 & b_{13} & 0 & -b_{11} & b_{23} & 0 & -b_{21} \\ 0 & 0 & 0 & -b_{12} & b_{11} & 0 & -b_{22} & b_{21} & 0 \end{pmatrix}.$$

Moreover $L_B \in \mathbb{M}^{9\times 9}$ *is bijective if* B *is bijective with*

$$\det(L_B) = 2 \cdot \det(B)^3$$

and the map $B \mapsto L_B \in \mathbb{M}^{9\times 9}$ *is linear.*

Proof. The proof consists of simple but long and tedious calculations. Because this formula is the heart of the argument we give it anyhow. First of all we evaluate the expression $Rot(A \cdot B)$ for all $A, B \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. We write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{pmatrix}$$

with $\bar{a}_i,\ i = 1, 2, 3$ the rows of A and

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} |b_1 & |b_2 & |b_3 \end{pmatrix}$$

with $|b_i,\ i = 1, 2, 3$ the columns of B. Then we have, of course,

$$\begin{aligned} A \cdot B &= \begin{pmatrix} (\bar{a}_1, |b_1) & (\bar{a}_1, |b_2) & (\bar{a}_1, |b_3) \\ (\bar{a}_2, |b_1) & (\bar{a}_2, |b_2) & (\bar{a}_2, |b_3) \\ (\bar{a}_3, |b_1) & (\bar{a}_3, |b_2) & (\bar{a}_3, |b_3) \end{pmatrix} \\ Rot(A \cdot B) &= \begin{pmatrix} rot[(\bar{a}_1, |b_1) & (\bar{a}_1, |b_2) & (\bar{a}_1, |b_3)] \\ rot[(\bar{a}_2, |b_1) & (\bar{a}_2, |b_2) & (\bar{a}_2, |b_3)] \\ rot[(\bar{a}_3, |b_1) & (\bar{a}_3, |b_2) & (\bar{a}_3, |b_3)] \end{pmatrix} \\ &= \begin{pmatrix} \partial_y(\bar{a}_1, |b_3) - \partial_z(\bar{a}_1, |b_2) & -[\partial_x(\bar{a}_1, |b_3) - \partial_z(\bar{a}_1, |b_1)] & \partial_x(\bar{a}_1, |b_2) - \partial_y(\bar{a}_1, |b_1) \\ \partial_y(\bar{a}_2, |b_3) - \partial_z(\bar{a}_2, |b_2) & -[\partial_x(\bar{a}_2, |b_3) - \partial_z(\bar{a}_2, |b_1)] & \partial_x(\bar{a}_2, |b_2) - \partial_y(\bar{a}_2, |b_1) \\ \partial_y(\bar{a}_3, |b_3) - \partial_z(\bar{a}_3, |b_2) & -[\partial_x(\bar{a}_3, |b_3) - \partial_z(\bar{a}_3, |b_1)] & \partial_x(\bar{a}_3, |b_2) - \partial_y(\bar{a}_3, |b_1) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} (\bar{a}_{1_y}, |b_3) + (\bar{a}_1, |b_{3_y}) - (\bar{a}_{1_z}, |b_2) - (\bar{a}_1, |b_{2_z}) & 0 & 0 \\ (\bar{a}_{2_y}, |b_3) + (\bar{a}_2, |b_{3_y}) - (\bar{a}_{2_z}, |b_2) - (\bar{a}_2, |b_{2_z}) & 0 & 0 \\ (\bar{a}_{3_y}, |b_3) + (\bar{a}_3, |b_{3_y}) - (\bar{a}_{3_z}, |b_2) - (\bar{a}_3, |b_{2_z}) & 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -(\bar{a}_{1_x}, |b_3) - (\bar{a}_1, |b_{3_x}) + (\bar{a}_{1_z}, |b_1) + (\bar{a}_1, |b_{1_z}) & 0 \\ 0 & -(\bar{a}_{2_x}, |b_3) - (\bar{a}_2, |b_{3_x}) + (\bar{a}_{2_z}, |b_1) + (\bar{a}_2, |b_{1_z}) & 0 \\ 0 & -(\bar{a}_{3_x}, |b_3) - (\bar{a}_3, |b_{3_x}) + (\bar{a}_{3_z}, |b_1) + (\bar{a}_3, |b_{1_z}) & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & (\bar{a}_{1_x}, |b_2) + (\bar{a}_1, |b_{2_x}) - (\bar{a}_{1_y}, |b_1) - (\bar{a}_1, |b_{1_y}) \\ 0 & 0 & (\bar{a}_{2_x}, |b_2) + (\bar{a}_2, |b_{2_x}) - (\bar{a}_{2_y}, |b_1) - (\bar{a}_2, |b_{1_y}) \\ 0 & 0 & (\bar{a}_{3_x}, |b_2) + (\bar{a}_3, |b_{2_x}) - (\bar{a}_{3_y}, |b_1) - (\bar{a}_3, |b_{1_y}) \end{pmatrix}$$
$$= \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix}$$
$$+ \begin{pmatrix} (\bar{a}_1, |b_{3_y} - |b_{2_z}) & (\bar{a}_1, |b_{1_z} - |b_{3_x}) & (\bar{a}_1, |b_{2_x} - |b_{1_y}) \\ (\bar{a}_2, |b_{3_y} - |b_{2_z}) & (\bar{a}_2, |b_{1_z} - |b_{3_x}) & (\bar{a}_2, |b_{2_x} - |b_{1_y}) \\ (\bar{a}_3, |b_{3_y} - |b_{2_z}) & (\bar{a}_3, |b_{1_z} - |b_{3_x}) & (\bar{a}_3, |b_{2_x} - |b_{1_y}) \end{pmatrix}$$
$$= \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix}$$
$$+ \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{pmatrix} \cdot \begin{pmatrix} |b_{3_y} - |b_{2_z} & |b_{1_z} - |b_{3_x} & |b_{2_x} - |b_{1_y} \end{pmatrix}$$
$$= \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix}$$
$$+ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{13y} - b_{12z} & b_{11z} - b_{13x} & b_{12x} - b_{11y} \\ b_{23y} - b_{22z} & b_{21z} - b_{23x} & b_{22x} - b_{21y} \\ b_{33y} - b_{32z} & b_{31z} - b_{33x} & b_{32x} - b_{31y} \end{pmatrix}$$
$$= \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix} + A \cdot Rot(B)\,.$$

Let us now use the assumption that $A = -A^T$ and set $\zeta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. We may put $A = \omega(\zeta)$. Thus

$$\nabla \zeta = \begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} .$$

This yields

$$\begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix}$$
$$= \begin{pmatrix} ((0, \alpha_y, \beta_y), |b_3) - ((0, \alpha_z, \beta_z), |b_2) & 0 & 0 \\ ((-\alpha_y, 0, \gamma_y), b_3) - (-\alpha_z, 0, \gamma_z), |b_2) & 0 & 0 \\ ((-\beta_y, -\gamma_y, 0), |b_3) - ((-\beta_z, -\gamma_z, 0), |b_2) & 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -((0, \alpha_x, \beta_x), |b_3) + ((0, \alpha_z, \beta_z), |b_1) & 0 \\ 0 & -((-\alpha_x, 0, \gamma_x), |b_3) + ((-\alpha_z, 0, \gamma_z), |b_1) & 0 \\ 0 & -((-\beta_x, -\gamma_x, 0), |b_3) + ((-\beta_z, -\gamma_z, 0), |b_1) & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & ((0, \alpha_x, \beta_x), |b_2) - ((0, \alpha_y, \beta_y), |b_1) \\ 0 & 0 & ((-\alpha_x, 0, \gamma_x), |b_2) - ((-\alpha_y, 0, \gamma_y), |b_1) \\ 0 & 0 & ((-\beta_x, -\gamma_x, 0), |b_2) - ((-\beta_y, -\gamma_y, 0), |b_1) \end{pmatrix} .$$

Thus we arrive at

$$vec \begin{pmatrix} (\bar{a}_{1_y}, |b_3) - (\bar{a}_{1_z}, |b_2) & -(\bar{a}_{1_x}, |b_3) + (\bar{a}_{1_z}, |b_1) & (\bar{a}_{1_x}, |b_2) - (\bar{a}_{1_y}, |b_1) \\ (\bar{a}_{2_y}, |b_3) - (\bar{a}_{2_z}, |b_2) & -(\bar{a}_{2_x}, |b_3) + (\bar{a}_{2_z}, |b_1) & (\bar{a}_{2_x}, |b_2) - (\bar{a}_{2_y}, |b_1) \\ (\bar{a}_{3_y}, |b_3) - (\bar{a}_{3_z}, |b_2) & -(\bar{a}_{3_x}, |b_3) + (\bar{a}_{3_z}, |b_1) & (\bar{a}_{3_x}, |b_2) - (\bar{a}_{3_y}, |b_1) \end{pmatrix} =$$
$$\begin{pmatrix} ((0, \alpha_y, \beta_y), |b_3) - ((0, \alpha_z, \beta_z), |b_2) \\ -((0, \alpha_x, \beta_x), |b_3) + ((0, \alpha_z, \beta_z), |b_1) \\ ((0, \alpha_x, \beta_x), |b_2) - ((0, \alpha_y, \beta_y), |b_1) \\ ((-\alpha_y, 0, \gamma_y), |b_3) - (-\alpha_z, 0, \gamma_z), |b_2) \\ -((-\alpha_x, 0, \gamma_x), |b_3) + ((-\alpha_z, 0, \gamma_z), |b_1) \\ ((-\alpha_x, 0, \gamma_x), |b_2) - ((-\alpha_y, 0, \gamma_y), |b_1) \\ ((-\beta_y, -\gamma_y, 0), |b_3) - ((-\beta_z, -\gamma_z, 0), |b_2) \\ -((-\beta_x, -\gamma_x, 0), |b_3) + ((-\beta_z, -\gamma_z, 0), |b_1) \\ ((-\beta_x, -\gamma_x, 0), |b_2) - ((-\beta_y, -\gamma_y, 0), |b_1) \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & b_{23} & -b_{22} & 0 & b_{33} & -b_{32} & 0 & 0 & 0 \\ -b_{23} & 0 & b_{21} & -b_{33} & 0 & b_{31} & 0 & 0 & 0 \\ b_{22} & -b_{21} & 0 & b_{32} & -b_{31} & 0 & 0 & 0 & 0 \\ 0 & -b_{13} & b_{12} & 0 & 0 & 0 & 0 & b_{33} & -b_{32} \\ b_{13} & 0 & -b_{11} & 0 & 0 & 0 & -b_{33} & 0 & b_{31} \\ -b_{12} & b_{11} & 0 & 0 & 0 & 0 & b_{32} & -b_{31} & 0 \\ 0 & 0 & 0 & 0 & -b_{13} & b_{12} & 0 & -b_{23} & b_{22} \\ 0 & 0 & 0 & b_{13} & 0 & -b_{11} & b_{23} & 0 & -b_{21} \\ 0 & 0 & 0 & -b_{12} & b_{11} & 0 & -b_{22} & b_{21} & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \\ \beta_x \\ \beta_y \\ \beta_z \\ \gamma_x \\ \gamma_y \\ \gamma_z \end{pmatrix}
$$
$$
= L_B.vec(\nabla\zeta)\,.
$$

Therefore

$$
vec(Rot(A \cdot B)) = L_B.vec(\nabla\zeta) + vec(A \cdot RotB)
$$

and we get the conclusion that

$$
Rot(A \cdot B)) = matrix(L_B.vec(\nabla\omega^{-1}(A)) + (A \cdot RotB)
$$

which is the first part of the lemma.

To find a simple direct proof of

$$
\det L_B = 2 \cdot (\det B)^3
$$

which shows in a few lines the above assertion, has so far eluded the efforts of the author. Instead one has to do all the computation by hand but I hesitate to confront the reader with them. ■

Lemma 4
For $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ with $A = -A^T$ and $B \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we have

$$
Rot(B \cdot A) = \hat{L}_A.DB + B \cdot Rot(A)
$$

where for fixed A the map $\hat{L}_A : \mathbb{R}^{27} \mapsto \mathbb{M}^{3\times 3}$ is linear and the application $A \mapsto \hat{L}_A$ is also linear.(Here DB denotes all partial derivatives of B with respect to (x_1, x_2, x_3).)

Proof. Is obvious from the foregoing analysis. ■
Let us quickly see what happens in the standard case $B = 1\!\!1$ which is usually involved in proving Korn's first inequality:

Corollary 2
Assume that $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ with $A = -A^T$. Then

$$
Rot(A) = 0 \Longrightarrow A = const.
$$

Proof. Retaining the same notations as in Lemma 3, we have for

$$A = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}$$

that

$$Rot(A) = \begin{pmatrix} \beta_y - \alpha_z & -\beta_x & \alpha_x \\ \gamma_y & -\gamma_x - \alpha_z & \alpha_y \\ -\gamma_z & -\beta_z & -\gamma_x + \beta_y \end{pmatrix}$$
$$vec(Rot(A)) = L_{1\!\!1}.vec(\nabla\zeta)\,.$$

Now if $Rot(A) = 0$ then this implies that $\alpha_x, \alpha_y, \beta_x, \beta_z, \gamma_y, \gamma_z = 0$ and

$$\begin{array}{r} \beta_y - \alpha_z = 0 \\ -\gamma_x - \alpha_z = 0 \\ -\gamma_x + \beta_y = 0 \end{array} \Leftrightarrow \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}}_{\text{invertible}} \cdot \begin{pmatrix} \alpha_z \\ \beta_y \\ \gamma_x \end{pmatrix} = 0$$

which yields $\alpha_z, \beta_y, \gamma_x = 0$. Hence $\alpha, \beta, \gamma = const.$
This is equivalent to $A = const.$
Note that we have implicitly also shown that $L_{1\!\!1} : \mathbb{R}^9 \mapsto \mathbb{R}^9$ is invertible. ■

Corollary 3
Assume that $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3})$ with $A = -A^T$ and either $B \in GL(3,\mathbb{R}),\ B = const.$ or $B \in C^1(\overline{\Omega}, \mathbb{M}^{3\times3}),\ B = \nabla\psi$. Then if $Rot(A \cdot B) = 0$ we have $A = const.$

Proof. From Lemma 3 we know that $Rot(A \cdot B) = 0$ implies

$$0 = matrix(L_B.vec(\nabla\omega^{-1}(A))) + (A \cdot RotB)\,.$$

Because B is invertible so is L_B by way of the second part of Lemma 3 and we can write

$$vec(\nabla\omega^{-1}(A)) = L_B^{-1}.vec(A \cdot RotB)\,.$$

But in both cases for B we have $RotB = 0$ and if we use the assumption that $A = -A^T$ and put $\zeta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ and $A = \omega(\zeta)$ then we can write in terms of ζ equivalently

$$\nabla\zeta = 0$$

Hence the conclusion. ■

Lemma 5
Assume that $A \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ *with* $A = -A^T$ *and* $B \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ *and that* $Rot(A \cdot B) = 0$ *and* $\det B \geq c^+ > 0$. *If furthermore there is an* $x_0 \in \overline{\Omega}$ *with* $A(x_0) = 0$ *then* $A = 0$ *everywhere.*

Proof. From Lemma 3 we know that $Rot(A \cdot B) = 0$ implies

$$0 = matrix(L_B.vec(\nabla\omega^{-1}(A))) + (A \cdot RotB)\,.$$

Because B is invertible so is L_B by way of the second part of Lemma 3 and we can write

$$vec(\nabla\omega^{-1}(A)) = L_B^{-1}.vec(A \cdot RotB)\,.$$

Let us now use once more the assumption that $A = -A^T$ and put $\zeta = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ and $A = \omega(\zeta)$. This gives in terms of ζ equivalently

$$\nabla\zeta = matrix(L_B^{-1}.vec(\omega(\zeta)) \cdot RotB)\,.$$

Consider now a smooth curve $x : [0, T] \mapsto x(t) \in \overline{\Omega}$ starting at x_0 i.e $x(0) = x_0$. With such smooth curves we can reach every point $x \in \overline{\Omega}$. We are interested in the behaviour of ζ along these curves. We differentiate the function $t \mapsto \eta(t) := \zeta(x(t))$ to get

$$\begin{aligned}\frac{d}{dt}\eta(t) = \frac{d}{dt}\zeta(x(t)) &= \nabla\zeta(x(t)).\dot{x(t)}\\ &= matrix(L_{B(x(t))}^{-1}.vec(\omega(\zeta(x(t))) \cdot RotB(x(t))).\dot{x(t)}\\ &= matrix(L_{B(x(t))}^{-1}.vec(\omega(\eta(t)) \cdot RotB(x(t))).\dot{x(t)}\end{aligned}$$

Together with $\eta(0) = \zeta(x(0)) = \zeta(x_0) = \omega^{-1}(A(x_0)) = \omega^{-1}(0) = 0$ this gives the following linear system of ordinary differential equations for η along $x(t)$

$$\begin{aligned}\frac{d}{dt}\eta(t) &= matrix(L_B(x(t))^{-1}.vec(\omega(\eta(t)) \cdot RotB(x(t))).\dot{x(t)}\\ \eta(0) &= 0\,.\end{aligned}$$

Because this system has a unique solution and $\eta = 0$ is a solution we must have $\zeta(x(t))$ identically 0. With the arbitrariness of $x(t)$ we see that $\zeta(x)$ is zero everywhere in $\overline{\Omega}$. But $A = \omega(\zeta)$ and we conclude $A = 0$ everywhere in $\overline{\Omega}$. ■

4 Korn-type inequalities with nonconstant coefficients

Lemma 6 (Ad Hoc Higher Regularity)
Assume that $\phi \in H^{1,2}(\Omega)$ *and* $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$. *Furthermore suppose that* $RotF_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. *If*

$$\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T = 0 \quad x \in \Omega$$

then $\phi \in C^2(\overline{\Omega}, \mathbb{R}^3)$ *and* $A := \nabla\phi \cdot F_p^{-1} \in C^{1,\frac{1}{2}}(\overline{\Omega}, \mathbb{M}^{3\times 3})$.

Proof. Put $A = \nabla\phi \cdot F_p^{-1}(x)$. Then $A = -A^T$ and $A \in L^2(\Omega)$ because of $\phi \in H^{1,2}(\Omega)$ and $F_p^{-1} \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$. We can solve for $\nabla\phi$ because F_p is invertible which gives $\nabla\phi = A \cdot F_p$. Taking the operator *Rot* on both sides in the sense of distributions we have

$$0 = Rot(\nabla\phi) = Rot(A \cdot F_p)\,.$$

Now we use our formula for $Rot(A \cdot F_p)$ which gives

$$0 = matrix[L_{F_p}.vec[\nabla(\omega^{-1}(A))]] + A \cdot Rot(F_p)\,.$$

Taking *vec* on both sides we get

$$0 = L_{F_p}.vec[\nabla(\omega^{-1}(A))] + vec(A \cdot Rot(F_p))\,.$$

By assumption, F_p is everywhere invertible and so is then L_{F_p}. Thus we can write this equivalently as

$$\begin{aligned} vec[\nabla(\omega^{-1}(A))] &= -L_{F_p}^{-1}.vec(A \cdot Rot(F_p)) \\ \nabla(\omega^{-1}(A)) &= -matrix[L_{F_p}^{-1}.vec(A \cdot Rot(F_p))]\,. \end{aligned} \tag{2}$$

Because $A \in L^2(\Omega)$, $F_p \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$ and $RotF_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ we read from this formula that $\nabla(\omega^{-1}(A)) \in L^2(\Omega)$. But $\nabla(\omega^{-1}(A))$ controls all first derivatives of A which means $A \in H^{1,2}(\Omega)$. Differentiating the above expression 2 on both sides once more we get that $A \in H^{2,2}(\Omega)$ since $F_p, RotF_p$ are continuously differentiable. Hence the Sobolev embeddding theorem [1] yields $A \in C^{0,\frac{1}{2}}(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Looking again at 2 we see that indeed $A \in C^{1,\frac{1}{2}}(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Together with $\nabla\phi = A \cdot F_p$ we see that $\nabla\phi \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Thus evidently $\phi \in C^2(\overline{\Omega}, \mathbb{R}^3)$. ■

Lemma 7
Assume that $\phi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ *and* $\phi_{|\Gamma} = 0$. *Moreover let* $\Gamma \subset \partial\Omega$ *be a two-dimensional smooth surface. Then there are two linear independent tangential directions* τ_1, τ_2 *on* Γ *such that*

$$\nabla\phi(x).\tau_1(x) = 0, \quad \nabla\phi(x).\tau_2(x) = 0\,.$$

Hence

$$rank(\nabla\phi(x)) \leq 1 \quad x \in \Gamma\,.$$

Proof. Look at curves $s(t)$ on the surface Γ starting in $x \in \Gamma$. Then $\phi(s(t)) = 0$. Differentiating yields $\nabla\phi(s(t)).\dot{s}(t) = 0$. Because Γ is a two-dimensional smooth surface, there are 2 linear independent tangential directions in every point $x \in \Gamma$. If we choose the curves such that $\dot{s}(0) = \tau_{1,2}$ we see the first part of the lemma. Because then $dim(ker(\nabla\phi(x))) = 2$ we see the second part as well. ■

Theorem 1 (Trivial Nullspace)
Assume that $\phi \in H_\circ^{1,2}(\Omega,\Gamma)$ *and* $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$. *Furthermore suppose that* $RotF_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. *Then*

$$\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} = 0 \quad \Longrightarrow \phi \equiv 0\,.$$

Proof. Because of $\phi \in H_\circ^{1,2}(\Omega,\Gamma)$ and the smoothness assumptions on F_p we know by virtue of Lemma 6 that $\phi \in C^1(\overline{\Omega}, \mathbb{R}^3)$. Therefore we can apply Lemma 7 to get that $rank(\nabla\phi) \leq 1$ for $x \in \Gamma$. Now set $\nabla\phi \cdot F_p^{-1} = A(x)$. In Lemma 6 we showed also that $A \in C^{1,\frac{1}{2}}(\overline{\Omega}, \mathbb{M}^{3\times 3})$ and of course A is skewsymmetric. We see with Lemma 2 that $A_{|_\Gamma} = 0$. If we solve for $\nabla\phi$ we arrive at

$$\nabla\phi = A \cdot F_p\,.$$

Taking now Rot on both sides in the strong sense yields $Rot(A \cdot F_p) = 0$ and we are in the position to take Lemma 5 into account. Thus we conclude that $A = 0$ everywhere. Whence also $\nabla\phi = 0$ everywhere. From $\phi \in H_\circ^{1,2}(\Omega,\Gamma)$ together with Poincare's inequality [6, p.281] we conclude that indeed $\phi = 0$. ■

Only for the convenience of the reader we give the following expression which we need in the sequel. Let $P \in C(\overline{\Omega}, M^{3\times 3})$ and $\phi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ then, as usual,

$$\nabla\phi \cdot P = \begin{pmatrix} \frac{\partial\phi^1}{\partial x_1} & \frac{\partial\phi^1}{\partial x_2} & \frac{\partial\phi^1}{\partial x_3} \\ \frac{\partial\phi^2}{\partial x_1} & \frac{\partial\phi^2}{\partial x_2} & \frac{\partial\phi^2}{\partial x_3} \\ \frac{\partial\phi^3}{\partial x_1} & \frac{\partial\phi^3}{\partial x_2} & \frac{\partial\phi^3}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial\phi^1}{\partial x_1}p_{11} + \frac{\partial\phi^1}{\partial x_2}p_{21} + \frac{\partial\phi^1}{\partial x_3}p_{31} & \frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} & \frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} \\ \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} & \frac{\partial\phi^2}{\partial x_1}p_{12} + \frac{\partial\phi^2}{\partial x_2}p_{22} + \frac{\partial\phi^2}{\partial x_3}p_{32} & \frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} \\ \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} & \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} & \frac{\partial\phi^3}{\partial x_1}p_{13} + \frac{\partial\phi^3}{\partial x_2}p_{23} + \frac{\partial\phi^3}{\partial x_3}p_{33} \end{pmatrix}$$

and we have of course

$$
\nabla\phi\cdot P + P^T\cdot\nabla\phi^T = \begin{pmatrix}
2\left(\frac{\partial\phi^1}{\partial x_1}p_{11}+\frac{\partial\phi^1}{\partial x_2}p_{21}+\frac{\partial\phi^1}{\partial x_3}p_{31}\right) & \frac{\partial\phi^1}{\partial x_1}p_{12}+\frac{\partial\phi^1}{\partial x_2}p_{22}+\frac{\partial\phi^1}{\partial x_3}p_{32}+\frac{\partial\phi^2}{\partial x_1}p_{11}+\frac{\partial\phi^2}{\partial x_2}p_{21}+\frac{\partial\phi^2}{\partial x_3}p_{31} & \frac{\partial\phi^1}{\partial x_1}p_{13}+\frac{\partial\phi^1}{\partial x_2}p_{23}+\frac{\partial\phi^1}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{11}+\frac{\partial\phi^3}{\partial x_2}p_{21}+\frac{\partial\phi^3}{\partial x_3}p_{31} \\
\frac{\partial\phi^1}{\partial x_1}p_{12}+\frac{\partial\phi^1}{\partial x_2}p_{22}+\frac{\partial\phi^1}{\partial x_3}p_{32}+\frac{\partial\phi^2}{\partial x_1}p_{11}+\frac{\partial\phi^2}{\partial x_2}p_{21}+\frac{\partial\phi^2}{\partial x_3}p_{31} & 2\left(\frac{\partial\phi^2}{\partial x_1}p_{12}+\frac{\partial\phi^2}{\partial x_2}p_{22}+\frac{\partial\phi^2}{\partial x_3}p_{32}\right) & \frac{\partial\phi^2}{\partial x_1}p_{13}+\frac{\partial\phi^2}{\partial x_2}p_{23}+\frac{\partial\phi^2}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{12}+\frac{\partial\phi^3}{\partial x_2}p_{22}+\frac{\partial\phi^3}{\partial x_3}p_{32} \\
\frac{\partial\phi^1}{\partial x_1}p_{13}+\frac{\partial\phi^1}{\partial x_2}p_{23}+\frac{\partial\phi^1}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{11}+\frac{\partial\phi^3}{\partial x_2}p_{21}+\frac{\partial\phi^3}{\partial x_3}p_{31} & \frac{\partial\phi^2}{\partial x_1}p_{13}+\frac{\partial\phi^2}{\partial x_2}p_{23}+\frac{\partial\phi^2}{\partial x_3}p_{33}+\frac{\partial\phi^3}{\partial x_1}p_{12}+\frac{\partial\phi^3}{\partial x_2}p_{22}+\frac{\partial\phi^3}{\partial x_3}p_{32} & 2\left(\frac{\partial\phi^3}{\partial x_1}p_{13}+\frac{\partial\phi^3}{\partial x_2}p_{23}+\frac{\partial\phi^3}{\partial x_3}p_{33}\right)
\end{pmatrix}.
$$

For $n = 3$ spatial dimensions we give the following

Definition 4
Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index and let a system of operators

$$N_l,\ l = 1, \dots 9 : H^{1,2}(\Omega) \mapsto L^2(\Omega)$$

be given in such a way that for $\phi = (\phi_1, \phi_2, \phi_3) \in H^{1,2}(\Omega)$

$$N_l.\phi := \sum_{s=1}^{3} \sum_{|\alpha|=1} n^l_{s\alpha}(x) \cdot D^\alpha \phi_s\,.$$

We say that this system is **weakly coercive** *with respect to $H^{1,2}(\Omega)$ if there exists $c^+ > 0$ such that*

$$\sum_{l=1}^{9} \|N_l.\phi\|^2_{2,\Omega} + \|\phi\|^2_{2,\Omega} \geq c^+ \|\phi\|^2_{1,2,\Omega}$$

for all $\phi \in H^{1,2}(\Omega)$.
For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$ we define the matrix

$$N_{l_s}(x)\xi := \sum_{|\alpha|=1} n^l_{s\alpha}(x) \cdot \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdot \xi_3^{\alpha_3}\,.$$

According to Theorem 3.2 in [13, p.310] we have the following

Theorem 2
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $n^l_{s\alpha} \in C(\overline{\Omega}, \mathbb{R})$. Then the system N_l is **weakly coercive** *if and only if*

$$\begin{aligned}
&\forall\, x \in \Omega : \forall\, \xi \in \mathbb{R}^3, \xi \neq 0 \quad \Longrightarrow rank(N_{l_s}(x)\xi) = 3 \\
&\forall\, x \in \partial\Omega : \forall\, \xi \in \mathbb{C}^3, \xi \neq 0 \quad \Longrightarrow rank(N_{l_s}(x)\xi) = 3\,.
\end{aligned}$$

Proof. [13, 4]. ■

Corollary 4
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $P \in C(\overline{\Omega}, GL(3))$. Then the system

$$\{N_l\phi\}_{l=1}^9 := vec(\nabla\phi \cdot P + P^T \cdot \nabla\phi^T)$$

of operators is **weakly coercive** *over $H^{1,2}(\Omega)$.*

Proof. Obviously, the coefficients of $N_l\phi$ satisfy the continuity condition of the theorem. We check the rank condition for $\xi \in \mathbb{C}^3,\ \xi \neq 0$. We have

$$\{N_l\phi\}_{l=1}^9 := vec(\nabla\phi \cdot P + P^T \cdot \nabla\phi^T)$$
$$= \begin{pmatrix}
2\left(\frac{\partial\phi^1}{\partial x_1}p_{11} + \frac{\partial\phi^1}{\partial x_2}p_{21} + \frac{\partial\phi^1}{\partial x_3}p_{31}\right) \\
\frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} + \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} \\
\frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} \\
\frac{\partial\phi^1}{\partial x_1}p_{12} + \frac{\partial\phi^1}{\partial x_2}p_{22} + \frac{\partial\phi^1}{\partial x_3}p_{32} + \frac{\partial\phi^2}{\partial x_1}p_{11} + \frac{\partial\phi^2}{\partial x_2}p_{21} + \frac{\partial\phi^2}{\partial x_3}p_{31} \\
2\left(\frac{\partial\phi^2}{\partial x_1}p_{12} + \frac{\partial\phi^2}{\partial x_2}p_{22} + \frac{\partial\phi^2}{\partial x_3}p_{32}\right) \\
\frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} \\
\frac{\partial\phi^1}{\partial x_1}p_{13} + \frac{\partial\phi^1}{\partial x_2}p_{23} + \frac{\partial\phi^1}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{11} + \frac{\partial\phi^3}{\partial x_2}p_{21} + \frac{\partial\phi^3}{\partial x_3}p_{31} \\
\frac{\partial\phi^2}{\partial x_1}p_{13} + \frac{\partial\phi^2}{\partial x_2}p_{23} + \frac{\partial\phi^2}{\partial x_3}p_{33} + \frac{\partial\phi^3}{\partial x_1}p_{12} + \frac{\partial\phi^3}{\partial x_2}p_{22} + \frac{\partial\phi^3}{\partial x_3}p_{32} \\
2\left(\frac{\partial\phi^3}{\partial x_1}p_{13} + \frac{\partial\phi^3}{\partial x_2}p_{23} + \frac{\partial\phi^3}{\partial x_3}p_{33}\right)
\end{pmatrix}.$$

Therefore in this case the matrix $N_{l_s}\xi$ looks like

$$\left(\begin{array}{lll} 2\,(\xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31}) & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} \\ 0 & \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & 0 \\ 0 & 0 & \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} \end{array} \right.$$

$$\begin{array}{lll} \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & 0 & 0 \\ \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & 2\,(\xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32}) & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} \\ 0 & 0 & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} \end{array}$$

$$\left. \begin{array}{lll} \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} & 0 & 0 \\ 0 & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} & 0 \\ \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & 2\,(\xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33}) \end{array} \right) .$$

Now we show that $rank(N_{l_s}) \le 2$ implies $\xi = 0$ which will give the desired theorem. If $rank(N_{l_s}) \le 2$ then the matrices

$$E_1 := \begin{pmatrix} 2\,(\xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31}) & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} \\ 0 & \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & 0 \\ 0 & 0 & \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} \end{pmatrix}$$

$$E_2 := \begin{pmatrix} \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & 0 & 0 \\ \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & 2\,(\xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32}) & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} \\ 0 & 0 & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} \end{pmatrix}$$

$$E_3 := \begin{pmatrix} \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} & 0 & 0 \\ 0 & \xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33} & 0 \\ \xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31} & \xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32} & 2\,(\xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33}) \end{pmatrix}$$

must each be singular, which implies that the determinants, respectively have to vanish. But

$$\begin{aligned} 0 &= \det E_1 = 2\,(\xi_1 p_{11}+\xi_2 p_{21}+\xi_3 p_{31})^3 \\ 0 &= \det E_2 = 2\,(\xi_1 p_{12}+\xi_2 p_{22}+\xi_3 p_{32})^3 \\ 0 &= \det E_3 = 2\,(\xi_1 p_{13}+\xi_2 p_{23}+\xi_3 p_{33})^3 \ . \end{aligned}$$

This in turn implies that $P^T.\xi = 0$. But P is invertible and therefore $\xi = 0$. ■

Corollary 5 (Korn's second inequality for nonconstant coefficients) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_p^{-1} \in C(\overline{\Omega}, GL(3))$. Then*

$$\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)}$$

is a norm on $H^{1,2}(\Omega)$ equivalent to the standard norm.

Proof. As a consequence of weak coercivity we get the existence of $c^+ > 0$ such that

$$\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geq \|\phi\|^2_{H^{1,2}(\Omega)}$$

however, the continuity of F_p^{-1} implies that

$$\begin{aligned}\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} &\leq \|\phi\|^2_{L^2(\Omega)} + K^+ \cdot \|\nabla\phi\|^2_{L^2(\Omega)} \\ &\leq K^+\|\phi\|^2_{H^{1,2}(\Omega)} .\end{aligned}$$

Hence the conclusion. ■

Remark 1
This is decisively more than Garding's-inequality, which, in the case of nonconstant coefficients, together with the strict Legendre-Hadamard condition, is only valid for functions in $H^{1,2}_\circ(\Omega)$. Note that for constant coefficients we have more, namely coercivity over $H^{1,2}(\Omega)$, compare with [22, p.323]. But here we have proved a generalization of Korn's second inequality which might not have been noticed before in this special form for invertible smooth F_p.

For clarity of exposition we cite the Garding's inequality for comparison in our context.

Lemma 8 (Garding's inequality)
Let $F_p^{-1} \in C^{0,\alpha}(\overline{\Omega}, \mathbb{M}^{3\times 3})$ be given with $\det F_p(x) \geq \mu^+ > 0$. Then for all $\xi, \eta \in \mathbb{R}^3$

$$\|(\eta \otimes \xi) \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot (\eta \otimes \xi)^T\|^2 \geq c^+(\mu^+)\ \|\eta\|^2 \cdot \|\xi\|^2$$

and as a consequence

$$\begin{aligned}&\exists\, c^+ > 0\ \forall\, \phi \in H^{1,2}_\circ(\Omega) : \\ &\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)} .\end{aligned}$$

Proof. See, e.g [10, p.9]. ■
We are now in a position to prove our main

Theorem 3 (Generalized Korn's first inequality)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Gamma \subset \partial\Omega$ be a smooth part of the boundary with nonvanishing 2-dimensional Lebesgue measure. Let

$$H^{1,2}_\circ(\Omega, \Gamma) := \{\phi \in H^{1,2}(\Omega) \mid \phi_{|\Gamma} = 0\}$$

and let $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL(3, \mathbb{R}))$ be given with $\det F_p(x) \geq \mu^+ > 0$. Suppose furthermore that $Rot F_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H^{1,2}_\circ(\Omega, \Gamma) : \ \|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)} .$$

Proof. The proof proceeds now in a standard fashion by contradiction, see e.g [6, 13] for the case of the classical Korn's first inequality. Assume to the contrary that there is a sequence of functions $\phi_k \in H^{1,2}_\circ(\Omega,\Gamma)$ such that

$$\|\phi_k\|^2_{H^{1,2}(\Omega)} = 1 \quad \text{but} \quad \|\nabla\phi_k \cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi_k^T\|^2_{L^2(\Omega)} \to 0\,.$$

Via the Rellich compact embedding of $H^{1,2}(\Omega)$ in $L^2(\Omega)$ there is a subsequence again denoted by ϕ_k and an element $\phi \in H^{1,2}_\circ(\Omega,\Gamma)$ with

$$\begin{aligned} \phi_k &\to \phi \quad \text{strongly in } L^2(\Omega) \\ \phi_k &\rightharpoonup \phi \quad \text{in } H^{1,2}(\Omega)\,. \end{aligned}$$

Due to the convexity of the mapping $H \mapsto \|H\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot H^T\|^2$ we have

$$\|\nabla\phi\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)} \le \liminf_{k\to\infty}\|\nabla\phi_k\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi_k^T\|^2_{L^2(\Omega)} = 0$$

If we apply Theorem 1 this yields $\phi = 0$.
We show now that this subsequence is in fact a Cauchy sequence in the norm

$$\|\nabla u\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla u^T\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$$

on $H^{1,2}(\Omega)$. To see this we note

$$\begin{aligned}
&\|\nabla(\phi_k-\phi_j)\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla(\phi_k-\phi_j)^T\|^2_{L^2(\Omega)} + \|\phi_k-\phi_j\|^2_{L^2(\Omega)} \le \\
&\qquad \underbrace{\|\nabla\phi_k\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi_k^T\|^2_{L^2(\Omega)}}_{\to 0 \quad \text{by assumption}} + \\
&\qquad \underbrace{\|\nabla\phi_j\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi_j^T\|^2_{L^2(\Omega)}}_{\to 0} + \underbrace{\|\phi_k-\phi_j\|^2_{L^2(\Omega)}}_{\to 0 \quad \text{via Rellich}}
\end{aligned}$$

Therefore, ϕ_k is also a Cauchy sequence in $H^{1,2}(\Omega)$. Which means

$$\begin{aligned} &\phi_k \to \phi \quad \text{strongly in } H^{1,2}(\Omega) \text{ and} \\ &\|\phi\|^2_{H^{1,2}(\Omega)} = 1 \end{aligned}$$

contrary to $\phi = 0$. ■

Remark 2 (The general gradient case)
The Theorem shows that if $F_p = \nabla\Psi_p$ it is sufficient to have $F_p, F_p^{-1} \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$. Compare [7] p. 44.

Interestingly enough, the above theorem can be proved using a direct argument in the gradient case $F_p = \nabla\Psi_p,\ \Psi \in C^2(\overline{\Omega}, \mathbb{R}^3)$, which mirrors the simple formula for the first Korn's inequality for functions $\phi \in H_\circ^{1,2}(\Omega)$.

Theorem 4 (Special $H_\circ^{1,2}(\Omega)$ gradient case)
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_p = \nabla\Psi_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ be given with $\det F_p^{-1}(x) = \mu^+ = const. \neq 0$. Then

$$\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,2}(\Omega):\ \|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Proof. For $A \in \mathbb{M}^{3\times 3}$ the Caley-Hamilton theorem tells us that

$$A^3 - tr(A)\cdot A^2 + tr(\mathrm{Adj}A)\cdot A - \det A \cdot 1\!\!1 = 0$$

If $A \in GL(3,\mathbb{R})$ we can multiply this equation with A^{-1}. Taking the *trace* on both sides we then have

$$tr(A^2) - tr(A)^2 + 2\, tr(\mathrm{Adj}A) = 0\,. \tag{3}$$

This formula remains valid for general $A \in \mathbb{M}^{3\times 3}$. Now

$$\begin{aligned}
\|\nabla\phi\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi^T\|^2 &= 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2 + 2tr((\nabla\phi\cdot F_p^{-1}(x))^2)\\
&= 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2 - 4tr(\mathrm{Adj}(\nabla\phi\cdot F_p^{-1}(x))) +\\
&\quad\ 2tr((\nabla\phi\cdot F_p^{-1}(x)))^2\\
&\geq 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2 - 4tr(\mathrm{Adj}(\nabla\phi\cdot F_p^{-1}(x)))
\end{aligned}$$

where use has been made of the identity 3. Assume that $\phi \in C_0^\infty(\Omega)$ and look at

$$\begin{aligned}
tr(\mathrm{Adj}(\nabla\phi\cdot F_p^{-1}(x))) &= \langle \mathrm{Adj}(\nabla\phi\cdot F_p^{-1}(x)), 1\!\!1\rangle\\
&= \langle \mathrm{Adj}(\nabla\phi), \mathrm{Adj}F_p^{-T}(x)\rangle\\
&= \langle \mathrm{Adj}(\nabla\phi), \det F_p^{-1}\cdot F_p^T\rangle\\
&= \mu\langle \mathrm{Adj}(\nabla\phi), F_p^T\rangle\\
&= \mu\langle \mathrm{Adj}(\nabla\phi), \nabla\Psi_p^T\rangle\,.
\end{aligned}$$

However, the Piola-Identity (see [6, p.39])

$$div\ Cof(\nabla\Psi_p) = div\ \mathrm{Adj}\nabla\Psi_p^T = 0$$

together with the divergence theorem implies that ($\mu = const.$)

$$\int_\Omega \mu\ \langle \mathrm{Adj}(\nabla\phi), \nabla\Psi_p^T\rangle\ dx = \mu\int_\Omega \langle \mathrm{Adj}(\nabla\phi), \nabla\Psi_p^T\rangle\ dx = 0$$

if $\phi \in C_0^\infty(\Omega)$. Therefore upon integrating we get

$$\begin{aligned}\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)} &\geq 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2_{L^2(\Omega)}\\ &\geq 2\ \lambda_{min,\overline{\Omega}}(F_p^{-1}F_p^{-T})\cdot\|\nabla\phi\|^2_{L^2(\Omega)}\end{aligned}$$

where $\lambda_{min,\overline{\Omega}}(F_p^{-1}F_p^{-T})$ denotes a lower bound for the smallest eigenvalues of $F_p^{-1}(x)\cdot F_p^{-T}(x)$ on $\overline{\Omega}$. An application of Poincare's inequality gives the result for $\phi \in C_0^\infty(\Omega)$. But $C_0^\infty(\Omega)$ is dense in $H_\circ^{1,2}(\Omega)$. ■

More can be said in another special case:

Theorem 5 (Special $H_\circ^{1,2}(\Omega)$ gradient case with Ψ_p a diffeomorphism) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $F_p = \nabla\Psi_p \in C^1(\overline{\Omega}, \mathbb{M}^{3\times 3})$ be given with $\det F_p^{-1}(x) \geq \mu^+$ and let $\Psi_p : \overline{\Omega} \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a C^1-diffeomorphism. Then*

$$\exists\, c^+ > 0 \quad \forall\,\phi \in H_\circ^{1,2}(\Omega):\ \|\nabla\phi\cdot F_p^{-1}(x) + F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)} \geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.$$

Proof. The proof uses the fact, that under the assumption that $\Psi_p : \overline{\Omega} \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a diffeomorphism, the map $x \mapsto \Psi_p(x) =: \xi$ induces a change of variables. Indeed if $\phi \in C_0^\infty(\Omega)$ we can uniquely define a function ϕ_e by setting

$$\phi(x) = \phi_e(\Psi_p(x))\,.$$

We then get $\nabla\phi(x) = \nabla_\xi\phi_e(\Psi_p(x))\cdot\nabla_x\Psi_p(x)$ or $\nabla\phi(x)\cdot\nabla_x\Psi_p^{-1}(x) = \nabla_\xi\phi_e(\Psi_p(x))$. For ϕ_e we obtain by the simple $H_\circ^{1,2}(\Omega)$ case of Korn's first inequality that

$$\int_{\xi\in\Psi_p(\Omega)} \|\nabla\phi_e(\xi) + \nabla\phi_e(\xi)^T\|^2\,d\xi \geq 2\int_{\xi\in\Psi_p(\Omega)} \|\nabla_\xi\phi_e(\xi)\|^2\,d\xi$$

since $\phi_e(\xi) = 0$ if $\xi \in \partial\Psi_p(\Omega)$. Now on applying the change of variables formula we obtain

$$\begin{aligned}&\int_\Omega \|\nabla\phi_e(\Psi_p(x)) + \nabla\phi_e(\Psi_p(x))^T\|^2\ \det\nabla\Psi_p(x)\,dx\\ &\qquad\geq 2\int_\Omega \|\nabla_\xi\phi_e(\Psi_p(x))\|^2\ \det\nabla\Psi_p(x)\,dx\,.\end{aligned}$$

By assumption $\det\nabla\Psi_p(x)$ is strictly positive. Hence we can conclude that

$$\begin{aligned}&\max_\Omega(\det\nabla\Psi_p(x))\int_\Omega \|\nabla\phi_e(\Psi_p(x)) + \nabla\phi_e(\Psi_p(x))^T\|^2\,dx \geq\\ &\qquad 2\min_\Omega((\det\nabla\Psi_p(x))\int_\Omega \|\nabla_\xi\phi_e(\Psi_p(x))\|^2\,dx\,.\end{aligned}$$

Therefore

$$
\begin{aligned}
\|\nabla\phi\cdot F_p^{-1}(x)+F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)} &\geq 2\frac{\min_\Omega(\det\nabla\Psi_p(x))}{\max_\Omega(\det\nabla\Psi_p(x))}\ \|\nabla\phi\cdot F_p^{-1}(x)\|^2_{L^2(\Omega)}\\
&\geq 2\frac{\min_\Omega(\det\nabla\Psi_p(x))}{\max_\Omega(\det\nabla\Psi_p(x))}\ \lambda_{min,\overline{\Omega}}(F_p^{-1}F_p^{-T})\ \|\nabla\phi\|^2_{L^2(\Omega)}\\
&= 2\frac{\min_\Omega(\det\nabla\Psi_p(x)^{-1})}{\max_\Omega(\det\nabla\Psi_p(x)^{-1})}\ \lambda_{min,\overline{\Omega}}(F_p^{-1}F_p^{-T})\ \|\nabla\phi\|^2_{L^2(\Omega)}\\
&\geq 2\frac{\mu^+}{\max_\Omega(\det F_p(x)^{-1})}\ \lambda_{min,\overline{\Omega}}(F_p^{-1}F_p^{-T})\ \|\nabla\phi\|^2_{L^2(\Omega)}\,.
\end{aligned}
$$

An application of Poincare's inequality together with the density of $C_0^\infty(\Omega)$ in $H_\circ^{1,2}(\Omega)$ will give the result. ■

For $n = 2$ space dimensions we can prove exactly the same theorem as above but there is another theorem which might be interesting in its own right because it can handle incompatible plastic configurations with much less regularity:

Theorem 6
Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and let $F_p\in L^\infty(\overline{\Omega},\mathbb{M}^{2\times2})$ be given with $\det F_p^{-1}(x)=\mu=const.\neq 0$. Then

$$
\exists\, c^+>0\quad \forall\,\phi\in H_\circ^{1,2}(\Omega):\ \|\nabla\phi\cdot F_p^{-1}(x)+F_p^{-T}(x)\cdot\nabla\phi^T\|^2_{L^2(\Omega)}\geq c^+\ \|\phi\|^2_{H^{1,2}(\Omega)}\,.
$$

Proof. For $A\in\mathbb{M}^{2\times2}$ the Caley-Hamilton theorem tells us that

$$
A^2-tr(A)\cdot A-\det A\cdot 1\!\!1=0\,.
$$

Hence, taking the trace on both sides

$$
tr(A^2)-tr(A)^2=2\det A
$$

which gives for $\phi\in C_0^\infty(\Omega)$

$$
\begin{aligned}
\|\nabla\phi\cdot F_p^{-1}(x)+F_p^{-T}(x)\cdot\nabla\phi^T\|^2 &= 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2+2tr((\nabla\phi\cdot F_p^{-1}(x))^2)\\
&= 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2+2tr((\nabla\phi\cdot F_p^{-1}(x)))^2-\\
&\qquad 4\det(\nabla\phi\cdot F_p^{-1}(x))\\
&\geq 2\|\nabla\phi\cdot F_p^{-1}(x)\|^2-4\mu\det(\nabla\phi)\,.
\end{aligned}
$$

Because $\det(\nabla\phi)$ is a divergence, integrating over Ω and application of Poincare's inequality will give the desired result, because $C_0^\infty(\Omega)$ is dense in $H_\circ^{1,2}(\Omega)$. ■

5 Concluding Remarks

In case of analyzing the form $\|F_p(x) \cdot \nabla\phi + \nabla\phi^T \cdot F_p^T(x)\|^2$ instead of $\|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^2$ we can do the same calculations as in Lemma 5. But we see that with Lemma 4 and invertible B we can directly solve for $RotA$ and we only have to check that $L_{1\!\!1} : \mathbb{R}^9 \mapsto \mathbb{R}^9$ is bijective. This can directly be seen by looking again at the computations which were done in the proof of Corollary 2. Altogether the whole analysis done so far carries over to this case.
The same type of coerciveness holds as well for forms of the type

$$\|G_p \cdot \nabla\phi \cdot F_p + F_p^T \cdot \nabla\phi^T \cdot G_p^T\|^2$$

with $F_p, G_p \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$. If we write

$$\|G_p \cdot \nabla\phi \cdot F_p + F_p^T \cdot \nabla\phi^T \cdot G_p^T\|^2 = \|G_p \cdot (\nabla\phi \cdot F_p \cdot G_p^{-T} + G_p^{-1} \cdot F_p^T \cdot \nabla\phi^T) \cdot G_p^T\|^2$$

we see immediately that we can always reduce the above case to the case

$$\|\nabla\phi \cdot C(x) + C^T(x) \cdot \nabla\phi^T\|^2$$

with $C \in C^1(\overline{\Omega}, GL(3,\mathbb{R}))$ since $\|G \cdot X \cdot G^T\|$ and $\|X\|$ are equivalent norms on $\mathbb{M}^{3\times 3}$ if $G \in GL(3,\mathbb{R})$. This remark shows that we have Korn's first inequality in the case with elastic rotations as well.
A generalization of our main theorem to $L^p(\Omega)$ spaces with $1 < p < \infty$, i.e

$$\begin{aligned} &\exists\, c^+ > 0 \quad \forall\, \phi \in H_\circ^{1,p}(\Omega,\Gamma) \\ &\qquad \|\nabla\phi \cdot F_p^{-1}(x) + F_p^{-T}(x) \cdot \nabla\phi^T\|^p_{L^p(\Omega)} \geq c^+\ \|\phi\|^p_{H^{1,p}(\Omega)} \end{aligned}$$

seems to be straightforward, because we get the generalization of Korn's second inequality in our situation and the $L^p(\Omega)$ setting by Theorem 6, in [4, p.530]. But to proceed from Korn's second inequality to Korn's first inequality we did not make use of any specific $L^2(\Omega)$ property.
The question remains to be settled whether the awkward smoothness assumptions made for F_p and the part of the boundary Γ are sharp. Less smoothness is of course of utmost importance in real applications.

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