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On KUS-Algebras

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Abstract

The aim of this paper is to introduce and study new algebraic structure, called KUS-algebra and investigate some of its properties.

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1. Introduction

The notion of BCK-algebras was proposed by Iami and Iseki in 1966. In the same year, K. Is´eki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. There is a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras see[3]. For the general development of BCK/BCI-algebras the ideal theory plays an important role. Y. Komori ([6]) introduced a notion of BCC-algebras, and W. A. Dudek [1] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In([1], [2]), C. Prabpayak and U. Leerawat introduced the concept of KU-algebra . They gave the concept of homomorphism of KU-algebras and investigated some related properties. In this paper the concepts of KUS-algebras, KUS-sub-algebras , KUS-ideals , homomorphism of KUS-algebras are introduced. The relation between some abelian groups and KUS-algebras , the G-part of KUS-algebras are studied and investigated some of its properties.

2. The Structure of KUS-algebras

In this section, we will introduce a new notion called KUS-algebras and study several properties of it.

Definition 2.1([1],[2]). A KU-algebra is a nonempty set X with a constant (0) and a binary operation $(*)$ satisfying the following axioms: for any $x, y, z \in X$,

(i) $(x*y)*[(y*z)*(x*z)]=0$,
(ii) $0*x=x$,
(iii) $x*0=0$,
(iv) $x*y=0$ and $y*x=0$ imply $x=y$.

Lemma 2.2([1],[2]). Every KU-algebra X satisfies the following conditions:

(v) $x*(y*z)=y*(x*z)$, for arbitrary $x, y, z \in X$.
(vi) $x*x=0$, for arbitrary $x \in X$.

Definition 2.3. Let $(X; *,0)$ be an algebra of type (2,0) with a single binary operation $(*)$. Then $(X; *,0)$ is called KUS-algebra if it satisfies the following axioms : for any $x, y, z \in X$,

(kus_1) : $(z*y)*(z*x)=(y*x)$,
(kus_2) : $0*x=x$,
(kus_3) : $x*x=0$,
(kus_4) : $x*(y*z)=y*(x*z)$.

In X we can define a binary relation $(\le)$ by : $x \le y$ if and only if $y*x=0$.

A KU-algebra $(X; *,0)$ is called KUS-algebra if it satisfies:

(vii) $(z*y) * (z*x) = (y*x)$.

For brevity we shall call X a KUS-algebra unless otherwise specified

Example 2.4. Let X = {0, a, b, c, d} in which ($*$) is defined by the following table:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	d	0	a	b	c
b	c	d	0	a	b
c	b	c	d	0	a
d	a	b	c	d	0

It is easy to show that $(X; *,0)$ is KUS-algebra .

Now, we give some properties and theorems of KUS-algebras.

Proposition 2.5. Let X be a KUS-algebra. Then the following holds: for any $x, y, z \in X$,

a) $x * y = 0$ and $y * x = 0$ imply $x = y$,
b) $x * (y * x) = y * 0$,
c) $(x * y) = 0$ implies that $x * 0 = y * 0$,
d) $(x*y) * 0 = y*x$,
e) $x*0 = 0$ implies that $x = 0$,
f) $x = 0 * (0 * x)$,
g) $0 * (x * y) = (0 * x) * (0 * y)$,
h) $x*z = y*z$ implies that $x*0 = y*0$.

Proof:

a) Since $x * y = 0$ and $y * x = 0$, then $x \leq y$ and $y \leq x$ imply $x = y$.
b) $x * (y*x) = y * (x*x) = y*0$.
c) $(0*y) * (x * y) = x * 0$, by (b), and $(0*y) * (x * y) = (0 * y) * 0$, since $(x * y) = 0$. Then $x * 0 = y * 0$.
d) $(x*y) * 0 = (x*y) * (x*x) = y*x$, by (kus_1).
(e), (f) and (g) are clears by (kus_2).
h) $x*0 = x* (z*z) = z* (x*z) = z*(y*z) = y*(z*z) = y*0$. ⌂

Proposition 2.6. Let X be a KUS-algebra . A relation ($\leq$) on X defined by

$x \le y$ if $y * x = 0$. Then $(X, \le)$ is a partially ordered set.

Proof: Let X be a KUS-algebra and let x, y, z $\in$ X, since $x * x = 0$, $x \le x$. Suppose that $x \le y$ and $y \le x$, then $x * y = 0 = y * x$. By proposition (2.5(a)), x = y. Suppose that $x \le y$ and $y \le z$, then $y * x = 0$ and $z*y = 0$. By (kus_1)
$0 = (y * x) * (y * x) = (y * x) * [(z * y) * (z * x)] = 0 * [0 * (z * x)] = z * x$, hence $x \le z$. Thus $(X, \le)$ is a partially ordered set. ⌂

Proposition 2.7. Let X be a KUS-algebra . Then the following holds: for any x, y, z $\in$X,

1. $x * y \le z$ imply $z * y \le x$,
2. $x \le y$ implies that $z * x \le z * y$,
3. $y * [(y * z) * z] = 0$,
4. $(x * z) * (y * z) \le (y * x)$,
5. $x \le y$ and $y \le z$ imply $x \le z$,
6. $x \le y$ implies that $y * z \le x * z$.

Proof:

1. It follows from (kus_4) .
2. By (kus_1), we obtain $[(z * y) * (z * x)] = (y * x)$, but $x \le y$ implies $y * x = 0$, then we get $(z * y) * (z * x) = 0$. i.e., $z * x \le z * y$.
3. It is clear by (kus_4) and (kus_3) .
4. By (kus_3) , (kus_4) and (kus_1) , $(y * x) * [(x * z) * (y * z)] = (x * z) * [(y * x) * (y * z)] = (x * z) * (x * z) = 0$. Thus $(x * z) * (y * z) \le (y * x)$.
5. If $x \le y$, then by (2), $z * x \le z * y$. By applying (kus_2) and ($x \le y$), $z * x = 0 * (z * x) = (y * x) * (z * x) \le z * y = 0$ [by (4) and ($y \le z$)] imply $z * x \le 0$, i.e., $0 * (z * x) = 0$. By (kus_2), $z * x = 0$ and so $x \le z$.
6. If $x \le y$, then $(x * z) * (y * z) = (y * x) = 0$. Hence $y * z \le x * z$. ⌂

Proposition 2.8. Every KUS-algebra X satisfying $x * (x * y) = x * y$ for all x, y $\in$ X is a trivial algebra .

Proof: putting x = y in the equation $x * (x * y) = x * y$, we have $x*0 = 0$ .By (kus_2) , x = 0. Hence X is a trivial algebra .⌂

3. KUS-ideals and Homomorphism of KUS-algebras

In this section we will present some results on images and preimages of homomorphism on KUS-algebras.

Definition 3.1. Let X be a KUS-algebra and let S be a nonempty set of X. S is called a KUS-sub-algebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 3.2([5]). Let I be a nonempty subset of X , I is called an ideal of X if, for all $x, y \in X$

(I_1) $0 \in I$,

(I_2) $x \in I$ and $y * x \in I$ imply $y \in I$.

Definition 3.3([1],[2]). A nonempty subset I of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions : for all $x, y, z \in X$

(KU_1) $0 \in I$,

(KU_2) $x * (y * z) \in I$, $y \in I$ implies $(x * z) \in I$.

Definition 3.4. A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for $x, y, z \in X$,

($Ikus_1$) $(0 \in I)$,

($Ikus_2$) $(z * y) \in I$ and $(y * x) \in I$ imply $(z * x) \in I$.

Example 3.5 . Let X ={0 ,a, b, c} in which ($*$) is defined by the following table:

$*$	**0**	**a**	**b**	**c**
0	**0**	**a**	**b**	**c**
a	**a**	**0**	**c**	**b**
b	**b**	**c**	**0**	**a**
c	**c**	**b**	**a**	**0**

Then (X; $*$,0) is KUS-algebra . It is easy to show that I_1 ={0,a}, I_2 ={0,b}, I_3 ={0,c}, and I_4={0, a, b, c} are KUS-ideals of X .

Proposition 3.6. Let X be a KUS-algebra and I be a nonempty subset of X containing **0**. Then I is a KUS-ideal of X if and only if :

$$(z * y) \in I,\ (z * x) \notin I \text{ imply } (y * x) \notin I, \text{ for all } x, y, z \in X.$$

Proof: Let I be an KUS-ideal of X and $(z * y) \in I$, $(z * x) \notin I$. Suppose that $(y * x) \in I$, since I is a KUS-ideal, then $(z * x) \in I$, a contradiction .

Conversely, assume that $(z*y) \in I$, $(z*x) \notin I$ imply $(y*x) \notin I$, for all x, y, z $\in$ X. If $(z*y) \in I$, $(y*x) \in I$. It is clear that $(z*x) \in I$. then I is a KUS-ideal of X. ⌂

Corollary 3.7.

Let X be a KUS-algebra and I be a nonempty subset of X containing **0**. Then :

(C_1) I is a KUS-ideal of X if and only if $(z*y) \in I$, $(y*x) \notin I$ imply $(z*x) \notin I$, for all x, y, z $\in$ X .

(C_2) I is a KUS-ideal of X if and only if $(z*y) \in I$, $(z*x) \in I$ imply $(y*x) \in I$, for all x, y, z $\in$ X.

(C_3) I is a KUS-ideal of X if and only if $(z*x) \in I$, $(y*x) \in I$ imply $(z*y) \in I$, for all x, y, z $\in$ X .

Proposition 3.8. Every KUS-ideal of KUS-algebra X is a KUS-sub-algebra.

Proof: For all x, y, z $\in$ X, let I be a KUS-ideal of a KUS-algebra X such that x, y $\in$I, then $(0*x) = x \in I$,$(0*y) = y \in I$. Hence, by corollary (3.7(C_2)) , $x*y \in I$. Therefore I is a KUS-sub-algebra.⌂

Proposition 3. 9. Every KUS-ideal of X is an ideal of X.

Proof: For all x, y, z$\in$X, let I is KUS-ideal of X. By corollary(3.7(C_3)) $(z*x) \in I$ and $(y*x) \in I$ imply $(z*y) \in I$. If $z = 0$,then $(0*x) = x \in I$ and $(y*x) \in I$ imply that $(0*y) = y \in I$ **.** and hence I is an ideal of X. ⌂

In general, the converse of proposition (3.9) is not true . For example:

Example 3.10. Let X ={0,1, 2, 3} be a KUS-algebra with the following Cayley table:

*	0	1	2	3
0	**0**	**1**	**2**	**3**
1	**3**	**0**	**1**	**2**
2	**2**	**3**	**0**	**1**
3	**1**	**2**	**3**	**0**

Then (X; $*$,0) is KUS-algebra . It is easy to show that I ={0, 3} is an ideal of X which is not a KUS-ideal of X , since $3 * 2 = 3 \in I$, $2 * 1 = 3 \in I$ and $3*1 = 2 \notin I$.

Proposition 3.11. Every KUS-ideal of KUS-algebra X is a KU-ideal of X.

Proof: For all x, y, z $\in$ X, let I is KUS-ideal of a KUS-algebra X. If $x*(y*z) \in I$, $y \in I$, then $y = 0*y = (z*z)*y$, by (kus_3).

$= [y*(z*z)]*0$, by proposition(2.5(d)).

$= [z*(y*z)]*0$, by (kus_4).

$= (y*z)*z \in I$, by proposition(2.5(d)).

Then $x*(y*z) \in I$ and $(y*z)*z \in I$ implies $(x*z) \in I$.

Hence I is a KU-ideal of X. ⌂

In general, the converse of proposition (3.11) is not true .

For example: Let X ={0,1, 2, 3} be a KUS-algebra with the following Cayley table

$*$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Then $(X; *, 0)$ is KUS-algebra . It is easy to show that I ={0, 3} is a KU-ideal of X ,which is not a KUS-ideal of X , since $3*2 = 3 \in I$, $2*1 = 3 \in I$ and $3*1 = 2 \notin I$.

Proposition 3.13. Let $\{I_i \mid i \in \Lambda\}$ be a family of KUS-ideals on KUS-algebra X. Then $\bigcap_{i\in\Lambda} I_i$ is a KUS-ideal of X.

Proof: Since $\{I_i \mid i \in \Lambda\}$ be a family of KUS-ideals of X , then $0 \in I_i$, for all $i \in \Lambda$, then $0 \in \bigcap_{i\in\Lambda} I_i$. For any x, y, z $\in$ X , suppose $z*y \in I_i$ and $y*x \in I_i$, for all $i \in \Lambda$, but I_i is a KUS-ideal of X for all $i \in \Lambda$. Then $z*x \in I_i$, for all $i \in \Lambda$, therefore, $z*x \in \bigcap_{i\in\Lambda} I_i$. Hence $\bigcap_{i\in\Lambda} I_i$ is KUS-ideal of KUS-algebra X .⌂

Definition 3.14 ([2]). Let $(X ; *, 0)$ and $(Y; *`, 0`)$ be KUS-algebras , the mapping $f : (X; *, 0) \to (Y; *`, 0`)$ is called a homomorphism if it satisfies:

$f(x*y) = f(x) *` f(y)$ for all x , y $\in$ X.

Theorem 3.15. Let $f : (X; *, 0) \to (Y; *`, 0`)$ be into homomorphism of a KUS-algebras, then :

A) $f(0) = 0'$.

B) f is injective if and only if Ker $f = \{0\}$.
C) $x \leq y$ implies $f(x) \leq f(y)$.

Proof: Clear.

Theorem 3.16. Let $f : (X; *,0) \rightarrow (Y; *`,0`)$ be an into homomorphism of a KUS-algebras, then :
(F_1) If S is a KUS-sub-algebra of X, then f (S) is a KUS-sub-algebra of Y .
(F_2) If I is a KUS-ideal of X, then f (I) is a KUS-ideal in Y .
(F_3) If B is a KUS-sub-algebra of Y, then f^{-1}(B) is a KUS-sub-algebra of X .
(F_4) If J is a KUS- ideal in Y, then f^{-1} (J) is a KUS-ideal in X .
(F_5) Ker f is KUS-ideal of X.
(F_6) Im(f) is a KUS-sub-algebra of Y.

Proof: Clear .

4. The G-part of KUS-algebras

In this section we give some basic definitions , preliminaries and lemmas of G-part in KUS-algebras.

Definition 4.1. Let $(X; *,0)$ be a KUS-algebra. For any nonempty subset S of X, we define $G(S) := \{x \in S \mid x * 0 = x\}$.
In particular, if $S = X$ then we say that G(X) is the G-part of KUS-algebra X.
For any KUS-algebra X, the set $B(X) := \{x \in X \mid x * 0 = 0\}$ is called a p-radical of X.
A KUS-algebra X is said to be p-semi-simple if $B(X) = \{0\}$.
The following property is obvious: $G(X) \cap B(X) = \{0\}$.

Proposition 4.2. If $(X; *,0)$ is a KUS-algebra and x, y∈X, then
$y \in G(X) \Leftrightarrow x * (y * x) = y$.

Proof: By (kus_4) and (kus_3) $x * (y * x) = y * (x * x) = y * 0 = y \Leftrightarrow y \in G(X)$.⌂

Corollary 4.3. If $(X; *,0)$ is a KUS-algebra and x, y∈X, then $y \in B(X)$ $\Leftrightarrow x * (y * x) = 0$.

Proof: By (kus_4) and (kus_3) $x * (y * x) = y * (x * x) = y * 0 = 0 \Leftrightarrow y \in B(X)$.⌂

Proposition 4.4. If $(X; *,0)$ is a KUS-algebra and $b* a = c* a$, then $b*0 = c*0$, where $a, b, c \in X$.

Proof: By (kus_4) and (kus_3), $a* (b*a) = b* (a*a) = b*0$ and $a* (c*a) = c* (a*a) = c*0$. Since $b*a = c*a$, then $b*0 = c*0$.⌂

Corollary 4.5. Let X be a KUS-algebra. Then the right cancellation law holds in G(X).

Proof: Let $a ,b ,c \in G(X)$ with $b*a = c*a$. By proposition (4.4), $b*0 = c*0$. Since $b, c \in G(X)$, we obtain $b = c$.⌂

Proposition 4.6. Let $(X; *,0)$ be a KUS-algebra. Then
$x \in G(X)$ if and only if $x*0 \in G(X)$.

Proof: If $x \in G(X)$, then $x*0 = x$ and $(x*0) *0 = x*0$. Hence $x*0 \in G(X)$. Conversely, if $x*0 \in G(x)$, then $(x*0) *0 = x*0$. By applying corollary (4.5), we obtain $x*0 = x$. Therefore $x \in G(X)$.⌂

Corollary 4.7. Let X be a KUS-algebra. Then the left cancellation law holds in G(X).

Proof: Let $a, b ,c \in G(X)$ with $a* b = a* c$. Then $a* y = a*(y*0) = y*(a*0) = y*a$, for any $y \in G(X)$. By proposition (4.6), $b = a* (b*a)=a* (a*b) = a* (a*c)$
$= a* (c*a) = c$, we obtain $b = c$.⌂

Proposition 4.8. Let $(X;*,0)$ be a KUS-algebra, for all $x, y , z \in G(X)$, then

L_1) $y*x = z$ imply $x*z = y$,

L_2) $x* (0*y) = y* (0*x)$.

Proof:

L_1) Since $y*x = z$,then

$x*z = (y*x) * (y*z)$, by (kus_1).
$= z* (y*z)$, since $(y*x = z)$.
$= y* (z*z)$, by (kus_4).
$= y*0 = y$, since $y \in G(X)$.

L_2) Since $((y* (0*x)) *x) *0 = x* (y* (0*x))$, by proposition(2.5(d)).
$= y* (x* (0*x))$, by (kus_4).
$= y* (0*0)$, by proposition(2.5(b)).

$= y * 0 = y$, since $y \in G(X)$,

we have $((y * (0 * x)) * x) * 0 = y$. It follows from that $(y * (0 * x)) * x = 0 * y$ and hence $y * (0 * x) = x * (0 * y)$ by (L_1) . ⌂

The following theorem state the relation between KUS-algebra and abelian group

Theorem 4.9. Let $(X; \cdot, {-1}, e)$ be an abelian group. If $x * y = x^{-1} \cdot y$ is defined and $0 = e$, then $(X; *, 0)$ is a KUS-algebra .

Proof: We only show that conditions (kus_1) and (kus_4) of KUS-algebra are satisfied. For the case of (kus_1) , since X is an abelian group, we have

$(z * y) * (z * x) = (z^{-1} \cdot y)^{-1} \cdot (z^{-1} \cdot x) = (y^{-1} \cdot z) \cdot (z^{-1} \cdot x) = y^{-1} \cdot (z \cdot z^{-1}) \cdot x = y^{-1} \cdot x = (y * x)$

For the case of (kus_4) , we also have

$x * (y * z) = x^{-1} \cdot (y^{-1} \cdot z) = (x^{-1} \cdot y^{-1}) \cdot z = (y^{-1} \cdot x^{-1}) \cdot z = y^{-1} \cdot (x^{-1} \cdot z) = y * (x * z)$.⌂

Theorem 4.10. Let $(X; *, 0)$ be a KUS-algebra. If $x \cdot y = x * y$ is defined , $x^{-1} = x$ and $e = 0$, then the structure $(X ; \cdot , -1 , e)$ is an abelian group.

Proof: We only show that the structure $(X ; \cdot , -1 , e)$ satisfies the conditions of associative and commutative with respect to the operation $(\cdot)$.

For associative , we have

$x \cdot (y \cdot z) = x * (y * z)$
$= (0 * x) * (y * (0 * z))$, by (kus_2).
$= (0 * x) * (z * (0 * y))$, by proposition $(4.8(L_2))$.
$= z * ((0 * x) * (0 * y))$, by (kus_4).
$= z * (x * y)$, by (kus_2).
$= ((x * y) * z) * 0$, by proposition(2.5(d)).
$= (x * y) * z$
$= (x \cdot y) \cdot z$.

For commutative, we also have

$x \cdot y = x * y$
$= x * (0 * y)$
$= y * (0 * x)$, by proposition $(4.8(L_2))$.
$= y * x = y \cdot x$. ⌂

Definition 4.11. Let $(X; *, 0)$ be a KUS-algebra satisfying $(x * y) * (z * u) = (x * z) * (y * u)$, for any x, y, z and $u \in X$ is called a medial of KUS-algebra.

Proposition 4.12. Let $(X; *,0)$ be a medial of KUS-algebra. Then $G(X)$ is a KUS-sub-algebra of X .

Proof: Let $x, y \in G(X)$, then $x*0 = x$ and $y*0 = y$. Hence

$$\begin{aligned}(x*y)*0 &= (x*y)*(0*0) \\ &= (x*0)*(y*0) \\ &= x*y \text{ .⌂}\end{aligned}$$

Corollary 4.13. Let $(X; *,0)$ be a medial of KUS-algebra. Then $B(X)$ is a KUS-sub-algebra of X .

Proof: Let $x, y \in B(X)$, then $x*0 = 0$ and $y*0 = 0$. Hence
$(x*y)*0 = (x*y)*(0*0) = (x*0)*(y*0) = 0*0 = 0$.⌂

Theorem 4.14. In a KUS-algebra $(X; *,0)$, X is medial of KUS-algebra if and only if it satisfies : for any $x, y, z \in G(X)$,

M_1) $(x*y) = (y*x)$,

M_2) $(x*y)*z = (x*z)*y$.

Proof: Suppose $(X; *,0)$ is medial of KUS-algebra and $x, y, z \in G(X)$. Then :

M_1) $y*x = 0*(y*x) = (x*x)*(y*x) = (x*y)*(x*x) = (x*y)*0 = (x*y)$.

M_2) $(x*y)*z = (x*y)*(z*0) = (x*z)*(y*0) = (x*z)*y$.

Conversely , assume that the conditions (M_1) and (M_2) hold .Then :

$$\begin{aligned}(x*y)*(z*u) &= (z*u)*(x*y) \text{ by } (M_1) \\ &= (z*(x*y))*u \text{ by } (M_2) \\ &= ((x*y)*z)*u \text{ by } (M_1) \\ &= ((x*z)*y)*u \text{ by } (M_2) \\ &= ((x*z)*u)*y \text{ by } (M_2) \\ &= (u*(x*z))*y \text{ by } (M_1) \\ &= (u*y)*(x*z) \text{ by } (M_2) \\ &= (y*u)*(x*z) \text{ by } (M_1) \\ &= (x*z)*(y*u) \text{ by } (M_1) \text{ .}\end{aligned}$$

Therefore X is a medial of KUS-algebra . ⌂

Proposition 4.15. Let $(X; *,0)$ be a medial of KUS-algebra. Then $G(X)$ is an KUS-ideal of X.

Proof: Since $0*0 = 0$ by (kus_3), hence $0 \in G(X)$. Next , let $x, y, z \in G(X)$ be such that $(z*y) \in G(X)$ and $(y*x) \in G(X)$, then $(z*y)*0 = (z*y)$ and $(y*x)*0 = y*x$.

$(z*x)*0 = (z*x)*[(y*x)*[(z*y)*(z*x)]]$, by ($kus_1$)and ($kus_3$).
$= (y*x)*[(z*x)*[(z*y)*(z*x)]]$, by ($kus_4$) .
$= (y*x)*(z*y)$, by proposition(4.2).
$= z*[(y*x)*y]$, by (kus_4) .
$= z*[(y*y)*x]$, by theorem (4.14(M_2)).
$= z*(0*x) = z*x$.
This means that G(X) is a KUS-ideal of X. This completes the proof .⌂

Corollary 4.16. Let $(X; *,0)$ be a medial of KUS-algebra. Then B(X) is a KUS-ideal of X.

Proof: Since $0*(0*0) = 0$, by corollary(4.3), $0 \in B(X)$. Let $x, y, z \in B(X)$ be such that $(z*y) \in B(X)$ and $(y*x) \in B(X)$, then $(z*y)*0= 0$ and $(y*x)*0= 0$.
$(z*x)*0 = (z*x)*[(y*x)*[(z*y)*(z*x)]]$, by ($kus_1$)and ($kus_3$).
$= (z*x)*[0*[0*(z*x)]]$
$= (z*x)*(z*x)$, by (kus_2).
$= 0$, by (kus_3) .
Therefore B(X) is a KUS-ideal of X. ⌂

Proposition 4.17. If S is a KUS-sub-algebra of a KUS-algebra $(X; *,0)$, then $G(X) \cap S = G(S)$.

Proof: It is obvious that $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $x*0 = x$ and $x \in S \subseteq X$. Then $x \in G(X)$ and so $x \in G(X) \cap S$, which proves the proposition. ⌂

If X is an associative KUS-algebra , then for any $x \in B(X)$, $0 = (x*x)*x = x*(x*x) = 0*x = x$ .Thus B(X) is a zero ideal i.e., $B(X) =\{0\}$. Hence any associative KUS-algebra X is p-semi-simple.

Theorem 4.18 . The G-part $(G(X); *)$ of an associative medial of KUS-algebra X is a group in which every element is an involution .

Proof:
Let X be an associative KUS-algebra and $x, y \in G(X)$. Then $(x*y)*0 = x*y$. Hence $x*y \in G(X)$ by proposition(4.12), i.e., G(X) is closed under $(*)$. For any $x \in G(X)$, we have $x*0 = x$. By (kus_2), $0*x = x$ holds in a KUS-algebra X. Therefore $0*x = x*0 = x$ in the G-part G(X) of an associative KUS-algebra X. This means that $(G(X); *)$ is a monoid. Moreover, $x*x = 0$ shows that x has an inverse and x is an involution . Hence $(G(X); *)$ is a group which every element is an involution. ⌂

Proposition 4.19. An associative KUS-algebra X satisfying $x*0= x$ for any $x \in X$ is commutative, i.e., $x*y = y*x$ for any $x, y \in X$.

Proof: For any $x, y \in X$, $y*x = y*(x*0) = x*(y*0) = x*y$, proving the proposition .⌂

Corollary 4.20. The G-part (G(X); $*$) of an associative medial of KUS-algebra X is an abelian group in which every element is an involution .

Proof: It follows immediately from proposition (4.19) and Theorem (4.18) . ⌂

Theorem 4.21. Let (X; $*$,0) be a KUS-algebra. If G(X) = X, then X is p-semi-simple.

Proof: Assume that G(X) = X. Since , $\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)$. Hence X is p-semi-simple. ⌂

Theorem 4.22. If (X; $*$,0) is a KUS-algebra of order 3, then $|G(X)| \neq 3$, that is,

$G(X) \neq X$.

Proof: For the sake of convenience, let $X = \{0,a,b\}$ be a KUS-algebra. Assume that $|G(X)| = 3$, that is, $G(X) = X$. Then $0*0 = 0$, $a*0 = a$, and $b*0 = b$. From (kus_3)and (kus_2), then $x*x = 0$ and $0*x = x$, it follows that $a*a = 0$, $b*b = 0$, $0*a = a$, and $0*b = b$. Now let $a*b = 0$. Then 0, a, and b are candidates of the computation.
If $b*a = 0$, then $b*a = 0 = a*b$ and so $a*(b*a) = b*(a*a)$. Hence $a*0 = b*0$. By the cancellation law in G(X), a = b, a contradiction.
If $b*a = a$, then $a = b*a = b*(a*0) = a*(b*0) = a*b = 0$, a contradiction.

For the case $b*a = b$, we have $b = b*a = b*(a*0) = a*(b*0) = a*b = 0$, which is also a contradiction.

Next, if $a*b = a$, then $b*[(b*a)*a] = b*(a*a) = b*0 = b \neq 0$. This leads to the conclusion that Proposition (2.7(3)) does not hold, a contradiction.

Finally, let $a*b = b$.

If $b*a = 0$, then $b = a*b = a*(b*0) = b*(a*0) = b*a = 0$, a contradiction.

If $b*a = a$, $b = b*0 = b*(a*a) = a*(b*a) = a*a = 0$, a contradiction.

For the case $b*a = b$, we have $a = a*0 = a*(b*b) = b*(a*b) = b*b = 0$, which is again a contradiction. This completes the proof. ⌂

Proposition 4.23. If (X; $*$,0) is a KUS-algebra of order 2, then in every case the G-part G(X) of X is an KUS-ideal of X.

Proof: Let |X| = 2. Then either G(X) = {0} or G(X) = X. In either case, G(X) is an ideal of X. ⌂

Theorem 4.24. Let (X; $*$,0) be a KUS-algebra of order 3. Then G(X) is an KUS-ideal of X if and only if |G(X)| = 1.

Proof: Let X := {0,a,b} be a KUS-algebra. If |G(X)| = 1, then G(X) = {0} is the trivial ideal of X.

Conversely, assume that G(X) is an ideal of X. By Theorem (4.22), we know that either |G(X)| = 1 or |G(X)| = 2. Suppose that |G(X)| = 2. Then either G(X) = {0,a}or G(X) = {0,b}. If G(X) = {0,a}, then $b * a \notin$ G(X) because G(X) is an ideal of X.

Hence $a * b = b$. Then $a = a * 0 = a * (b * b) = b * (a * b) = b * b = 0$, which is a contradiction. Similarly, G(X) = {0,b} leads to a contradiction. Therefore |G(X)| ≠ 2 and so |G(X)| = 1. ⌂

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