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ON $\Lambda(p)$ SETS WITH MINIMAL CONSTANT IN DISCRETE NONCOMMUTATIVE GROUPS

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ABSTRACT. We compute the minimal constants for infinite $\Lambda(2n)$ sets in discrete noncommutative groups and as a consequence we obtain an alternate proof of Leinert's theorem on $\Lambda(\infty)$ sets.

1. **Introduction.** Let G be a discrete group. Let $l^2(G)$ denote the space of square summable complex functions on G with the norm $\|f\|_{l^2} = (\Sigma_x |f(x)|^2)^{1/2}$. *A convolver* of l^2 is a function g on G such that for each $f \in l^2$ the convolution

$$(g * f)(x) = \sum_{y \in G} g(xy^{-1})f(y)$$

is defined and belongs to $l^2(G)$.

In accordance with the terminology of Eymard [3], we shall denote the space of "*convolvers*" by $VN(G)$. The norm of an element of $VN(G)$ will be the norm of the corresponding convolution operator (which is necessarily continuous) on $l^2(G)$. It is clear that $VN(G) \subseteq l^2(G)$. In this paper we study subsets $E \subseteq G$ with the property that every function $g \in l^2(G)$ supported on E is a convolver. The existence of infinite sets E satisfying this property was first established by M. Leinert [7]. He proved that if a set E satisfies a certain condition, which we shall call Leinert's condition, then every square summable function f supported on E is a convolver, and

$$\|f\|_{VN(G)} \leq \sqrt{5}\, \|f\|_{l^2(G)}.$$

The purpose of this paper is to give an alternate proof of Leinert's theorem which improves the constant $\sqrt{5}$. We prove that if E satisfies Leinert's condition, and f is supported on E, then

$$\|f\|_{VN(G)} \leq 2\|f\|_{l^2(G)}.$$

We also show that the constant 2 is the best possible if E is an infinite set. To prove our result we use estimates involving L^p-convolution norm in

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the sense which was considered in [10] and [1].

We remark that sets satisfying Leinert's condition are always subsets of a free group with at least two generators. On the other hand every set with no relation among its members satisfies this condition. (See [7], [8].) These sets have been used in [5] to construct multipliers of $A(G)$ which are not elements of $B(G)$.

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2. For a finitely supported function f defined on G we set

$$\|f\|_{2s}^{2s} = (f * f^*)^s(1) = \operatorname{tr}(f * f^*)^s$$

for $s = 1, 2, \ldots$, where $(f * f^*)^s$ denotes the convolution power. It is not difficult to see that $\|f\|_{2s}$ is a norm. From a theorem of I. Kaplansky ([6, Theorem 1.8.1],[2]) we also have $\lim_{s\to\infty}\|f\|_{2s} = \|f\|_{VN(G)}$.

Definition 1. Let E be a subset of G and n a positive integer. We say that E is of type L_{2n} if for every finite sequence $\{x_i: x_i \in E,\ i = 1, \ldots, 2k,\ k \leq n\}$ the following relation holds:

$$x_{i_1}x_{i_2}^{-1} \cdots x_{i_{2k-1}}x_{i_{2k}}^{-1} \neq 1$$

if $x_{i_j} \neq x_{i_{j+1}}$ for $j = 1, 2, \ldots, 2k-1$.

Definition 2. A set E is said to satisfy Leinert's condition if E is of type L_{2n} for every natural n.

We can now state our main results:

Theorem. (i) *If E is of type L_{2n} in a discrete group G, then E is $\Lambda(2n)$, i.e., $\|f\|_{2n} \leq C_{2n}\|f\|_2$ for every function f with support in E, where $C_{2n}^{2n} = (n+1)^{-1}\binom{2n}{n}$.*

(ii) *If E is an infinite set of type L_{2n}, then*

$$\sup\{\|f\|_{2n} : \operatorname{supp} f \subseteq E,\ \|f\|_2 = 1\} = C_{2n}$$

and C_{2n} is the minimal constant for all infinite $\Lambda(2n)$ sets.

Corollary. *If $E \subseteq G$ is a set which satisfies Leinert's condition, then $\|f\|_{VN(G)} \leq 2\|f\|_2$ and 2 is the minimal constant for all infinite $\Lambda(\infty)$ sets.*

Proof of (i). Let f be a function of the form

$$f = \sum_{i=1}^{N} \alpha_i \delta_{x_i}, \qquad x_i \in E, \quad \|f\|_2 = 1.$$

For a subset $A \subseteq \Phi^{2n}$, $\Phi = \{1, 2, \ldots, N\}$ we define

$$p_n(A) = \operatorname{tr} \sum_{i \in A; i=(i_1, i_2, \ldots, i_{2n})} \alpha_{i_1} \overline{\alpha}_{i_2} \cdot \ldots \cdot \alpha_{i_{2n-1}} \overline{\alpha}_{i_{2n}} \delta_{x_{i_1}} x_{i_2}^{-1} \cdot \ldots \cdot \delta_{x_{i_{2n-1}}} \delta_{x_{i_{2n}}}^{-1} .$$

Because the set E is of type L_{2n} we note that p_n is a positive measure on subsets of $\Phi^{2n} = \Phi x \cdot \ldots \cdot \Phi$. It follows by induction from the following facts:

(1) $$p_n(\{\mathrm{i}\}) = 0$$

if for $\mathrm{i} = (i_1, i_2, \ldots, i_{2n}) \in \Phi^{2n}$, $i_k \neq i_{k+1}$ for $k = 1, 2, \ldots, 2n-1$, and

(2) $$p_n(\{\mathrm{i}\}) = |\alpha_{i_{k_0}}|^2 p_{n-1}(\{\mathrm{i}'\})$$

if for some $1 \leq k_0 < 2n$, $i_{k_0} = i_{k_0+1}$ and $\mathrm{i}' = (i_1, \ldots, i_{k_0-1}, i_{k_0+2}, \ldots, i_{2n-1}, i_{2n})$. Let

$$S^n = \|f\|_{2n}^{2n} = p_n(\Phi^{2n}),$$

$$A_k = \{\mathrm{i} : \mathrm{i} = (i_1, i_2, \ldots, i_{2n}), i_k = i_{k+1}\}$$

and

$$S_k^n = p_n(A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c),$$

where $A_m^c = \Phi^{2n} \setminus A_m$. Since $p_n(A_k) = S^{n-1}$ for every natural $k < 2n$, so we obtain

(3) $$S^n = S^{n-1} + S_2^n.$$

Since $p_n(A_1^c) = p_n(A_1^c \cap A_2) + p_n(A_1^c \cap A_2^c)$, but $p_n(A_1^c \cap A_2) \leq p_n(A_2) = S^{n-1}$, therefore

(4) $$S_2^n \leq S^{n-1} + S_3^n.$$

Now because $p_n(A_1^c \cap A_2^c) = p_n(A_1^c \cap A_2^c \cap A_3) + S_4^n$ and $p_n(A_1^c \cap A_3) = p_{n-1}(A_1^c)$, we obtain

(5) $$S_3^n \leq S_4^n + S_2^{n-1}.$$

By that same argument we have

(6) $$S_{k+1}^n \leq S_{k+2}^n + S_k^{n-1}.$$

But the set E is of type L_{2n} so from (6) we obtain

(7) $$S_{2n}^n = S_k^n = 0 \quad \text{for } n < k \leq 2n.$$

Applying (6) and (7) we have

$$S_n^n \le S_2^2 \le 1. \tag{8}$$

From (6) we obtain

$$S_{n-1}^n \le S_n^n + S_{n-2}^{n-1}. \tag{9}$$

Since $S^1 = 1$, $S^2 = 2$ and $S_2^3 \le 3$, therefore from (9) we have

$$S_{n-1}^n \le \binom{n}{1} \quad \text{for } n > 2. \tag{10}$$

By this same way, from (6) we obtain

$$S_{n-2}^n \le S_{n-1}^n + S_{n-3}^{n-1} \tag{11}$$

and from (4) and (10) and also $S_2^4 \le 9$ we have

$$S_{n-2}^n \le \binom{n+1}{2} - \binom{n+1}{0} \quad \text{for } n > 3. \tag{12}$$

And now by the induction argument we obtain

$$S_{n-k}^n \le \binom{n+k-1}{k} - \binom{n+k-1}{k-2} \quad \text{for } k \ge 2. \tag{13}$$

Since the following equality is true:

$$\frac{1}{n}\binom{2n-2}{n-1} + \binom{2n-3}{n-2} - \binom{2n-3}{n-4} = \frac{1}{n+1}\binom{2n}{n} \tag{14}$$

we obtain from (13) and (3), by induction,

$$S^n \le \frac{1}{n+1}\binom{2n}{n} \tag{15}$$

Proof of (ii). Let E be an infinite set of type L_{2n}; $E = \{x_1, x_2, \ldots,\}$ and $f_N = N^{-1/2}\sum_{i=1}^N \delta_{x_i}$. We prove by induction that

$$S^n(f_N) = \|f_N\|_{2n}^{2n} = C_{2n}^{2n} + R_n(N), \tag{16}$$

where $\lim_{N\to\infty} R_n(N) = 0$. That fact follows from the formula

$$S^n = p_n\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n p_n(A_k) - \sum_{i_1 < i_2} p_n(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^{n-1} p_n(A_1 \cap A_2 \cap \cdots \cap A_n). \tag{17}$$

Note also that if $i_m + 1 \ne i_{m+1}$ for $m = 1, 2, \ldots, k-1$, then

$$p_n\left(\bigcap_{m=1}^k A_{i_m}\right) = S^{n-k} \tag{18}$$

and

$$p_n\left(\bigcap_{m=1}^{n} A_{i_m}\right) \to 0 \qquad (N \to \infty), \tag{19}$$

if for some $m < n$, $i_m + 1 = i_{m+1}$. In order to prove (19), it suffices to note that

$$p_n(A_1 \cap A_2) \to 0 \qquad (N \to \infty), \tag{20}$$

but

$$p_n(A_1 \cap A_2) = N^{-1}\|f_N\|_{2n-2}^{2n-2} \to 0 \qquad (N \to \infty). \tag{21}$$

We shall prove the induction step in (16) if we show that

$$K = \sum_{k \in Z} (-1)^k D^k B_n^{n-k} \tag{22}$$

equals zero, where $D^k = C_{2k}^{2k}$ and B_n^m denote the number of subsequences of the sequence $(1, 2, \ldots, n)$ of the form $(k_1, k_2, \ldots, k_m)$ where for every $1 \le s < m - 1$, $k_s + 1 \ne k_{s+1}$. It is easy to see that

$$B_n^m = \binom{n+1-m}{m}. \tag{23}$$

Applying the following formulas (see [4])

$$\sum_{v=0}^{n} (-1)^v \binom{a}{v} = (-1)^n \binom{a-1}{n}, \tag{24}$$

$$\sum_{k \in Z} (-1)^k \binom{n}{k}\binom{2k+1}{v} = 0, \tag{25}$$

we obtain

$$K = \sum_{k \in Z} (-1)^k \frac{1}{k+1}\binom{2k}{k}\binom{k+1}{n-k} = \frac{1}{n} \sum_k (-1)^k \binom{n}{k}\binom{2k}{n-1}$$

$$= \frac{1}{n} \sum_k \sum_v (-1)^{n+k+v-1} \binom{n}{k}\binom{2k+1}{v} = 0.$$

The Corollary follows at once from the following inequality (see [11]):

$$\frac{2^{2n-1}}{n} \le \binom{2n}{n} \le 2^{2n-1}.$$

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