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ON (λ, μ)-STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES ON INTUITIONISTIC FUZZY NORMED SPACES

Vijay Kumar and M. Mursaleen

Abstract

In this paper, we define (λ, μ)- statistical convergence and (λ, μ)-statistical Cauchy double sequences on intuitionistic fuzzy normed spaces (IFNS in short), where $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$; $\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$. We display example that shows our method of convergence is more general for double sequences in intuitionistic fuzzy normed spaces.

1 Introduction

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [17], independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number theory. Later on, the idea was further investigated from sequence space point of view and linked with summability theory by Fridy [6], Šalát [15], Connor [3] and many others. The idea of statistical convergence is based on the notion of natural density of subsets of $\mathbf{N}$, the set of positive integers.

The natural density δ of a subset $K \subset \mathbf{N}$ is denoted by $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k < n : k \in K\}|;$$

where the vertical bars denote the cardinality of the enclosed set.
Note that if $K \subset \mathbf{N}$ is finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}$, $\delta(K^C) =$ 1- $\delta(K)$.

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A number sequence $x = (x_k)$ is said to be statistical convergent to ξ if for each $\epsilon > 0$, the set $\{k \in \mathbf{N} : |x_k - \xi| \geq \epsilon\}$ has natural density zero, i.e.,
$\delta(\{k \in \mathbf{N} : |x_k - \xi| \geq \epsilon\}) = 0$.
In this case we write $St - \lim x_k = \xi$.

Mursaleen and Osama [10] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences. Besides this, Mursaleen [9] presented a generalization of statistical convergence with the help of λ-summability method and called it λ-statistical convergence. However, Savas [16] extended the idea of λ-statistical convergence for sequences of fuzzy numbers.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. Let $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ of numbers is said to be λ-statistically convergent to a number ξ provided that for each $\epsilon > 0$,

$$\lim_{n \longrightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - \xi| \geq \epsilon\}| = 0.$$

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [19] and intutionistic fuzzy normed spaces [7], [8], [11] and [12]. Infect these spaces play important role in the study of fuzzy and intuitionistic fuzzy topological spaces [1, 2 and 14] which have very important applications in quantum particle physics particularly in connections with both string and ϵ^{∞} theory which were given and studied by El-Naschie [4].

In this paper, we define and study (λ, μ)-statistical convergence and (λ, μ)-statistical Cauchy double sequences on intuitionistic fuzzy normed spaces, where $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two nondecreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$; $\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$.

2 Definitions and Preliminaries

We begin by recalling some notations and definitions which will be used in this paper.

Definition 2.1 [18] A triangular norm or briefly a t-norm is a binary operation on the closed interval $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 1 as a neutral element, i.e., it is the continuous mapping $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ such that for all a, b, c and $d \in [0, 1]$, we have

(i) $a * 1 = a$;
(ii) $a * b = b * a$;
(iii) $c * d \geq a * b$ if $c \geq a$ and $d \geq b$;
(iv) $(a * b) * c = a * (b * c)$.

Definition 2.2 [18] A triangular co-norm or briefly a t-co-norm is a binary operation on the closed interval $[0, 1]$ which is continuous, commutative, associative, non-decreasing and has 0 as a neutral element, i.e., it is the continuous mapping $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ such that for all a, b, c and $d \in [0, 1]$, we have

(i) $a \diamond 0 = a$;
(ii) $a \diamond b = b \diamond a$;
(iii) $c \diamond d \geq a \diamond b$ if $c \geq a$ and $d \geq b$;
(iv) $(a \diamond b) \diamond c = a \diamond (b \diamond c)$.

With the help of the Definition 2.1 and Definition 2.2, Saadati and Park [14] have recently introduced the concept of an intuitionistic fuzzy normed space as follows.

Definition 2.3 [14] The five tuple $(X, \kappa, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (IFNS in short) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm, and κ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions.

For every $x, y \in X$ and $s, t > 0$, we have
(i) $\kappa(x, t) + \nu(x, t) \leq 1$;
(ii) $\kappa(x, t) > 0$;
(iii) $\kappa(x, t) = 1$ if and only if $x = 0$;
(iv) $\kappa(\alpha x, t) = \kappa(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(v) $\kappa(x, t) * \kappa(y, s) \leq \kappa(x + y, t + s)$;
(vi) $\kappa(x, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous ;
(vii) $\lim_{t \longrightarrow \infty} \kappa(x, t) =1$ and $\lim_{t \longrightarrow 0} \kappa(x, t) = 0$;
(viii) $\nu(x, t) < 1$;
(ix) $\nu(x, t) = 0$, if and only if $x = 0$;
(x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(xi) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$;
(xii) $\nu(x, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous and
(xiii) $\lim_{t \longrightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \longrightarrow 0} \nu(x, t) = 1$.

In this case (κ, ν) is called intuitionistic fuzzy norm.

Example 2.1 Suppose $(X, \|.\|)$ is a normed space and let $a * b = ab$ and $a \diamond b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$, consider

$$\kappa(x, t) = \frac{t}{t + \|x\|} \ \ and \ \ \nu(x, t) = \frac{\|x\|}{t + \|x\|}.$$

Then $(X, \kappa, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. We define open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \kappa(x - y, t) > 1 - r \ and \ \nu(x - y, t) < r\}, t > 0.$$

With the help of above topological structure, the concepts of usual convergence and Cauchy double sequences in an IFNS $(X, \kappa, \nu, *, \diamond)$ are defined as follows.

Definition 2.4 A double sequence $x = (x_{ij})$ of elements of X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (κ, ν) if for each $\epsilon > 0$ and $t > 0$ there exists a positive integer m such that

$$\kappa(x_{ij} - \xi, t) > 1 - \epsilon \ and \ \nu(x_{ij} - \xi, t) < \epsilon$$

whenever $i, j \geq m$. The element ξ is called the ordinary limit of the sequence (x_{ij}) with respect to the intuitionistic fuzzy norm (κ, ν) and we shall write $(\kappa, \nu)-$

$\lim x_{ij} = \xi$.
Definition 2.5 A double sequence $x = (x_{ij})$ of elements of X is said to be Cauchy with respect to the intuitionistic fuzzy norm (κ, ν) if for each $\epsilon > 0$ and $t > 0$ there exist positive integers $M = M(\epsilon)$ and $N = N(\epsilon)$ such that

$$\kappa(x_{ij} - x_{pq}, t) > 1 - \epsilon \; and \; \nu(x_{ij} - x_{pq}, t) < \epsilon$$

for all $i, p \geq M$ and $j, q \geq N$.
Remark 2.1 [11] Let $(X, \|.\|)$ be a real normed space and let $a * b = ab$ and $a \diamond b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$, consider

$$\kappa(x, t) = \frac{t}{t + \|x\|} \;\; and \;\; \nu(x, t) = \frac{\|x\|}{t + \|x\|}.$$

Then $x_{ij} \longrightarrow \xi$ w.r.t.$\|.\|$ if and only if $x_{ij} \longrightarrow \xi$ w.r.t. (κ, ν).

Mursaleen and Mohiuddine [11] generalized the above notions and defined statistical convergence and statistical Cauchy sequences on IFNS. They used the notion of double natural density of subsets of $\mathbf{N} \times \mathbf{N}$.

Let $K \subset \mathbf{N} \times \mathbf{N}$ and $K(m, n)$ denotes the number of (i, j) in K such that $i \leq m$ and $j \leq n$. Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty} \frac{K(m,n)}{mn}$. In case, the sequence $(\frac{K(m,n)}{mn})$ has a limit then we say that K has a double natural density and is defined by $\lim_{m,n\to\infty} \frac{K(m,n)}{mn} = \delta_2(K)$.
Definition 2.6 Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. A double sequence $x = (x_{ij})$ of elements in X is said to be statistically convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for each $\epsilon > 0$ and $t > 0$ we have

$$\delta_2(\{(i, j) \in \mathbf{N} \times \mathbf{N} : \kappa(x_{ij} - \xi, t) \leq 1 - \epsilon \; or \; \nu(x_{ij} - \xi, t) \geq \epsilon\}) = 0.$$

The element ξ is called the double statistical limit of the sequence (x_{ij}) with respect to the intuitionistic fuzzy norm (κ, ν) and we write $St_{(\kappa,\nu)} - \lim x_{ij} = \xi$.
Definition 2.7 Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. A double sequence $x = (x_{ij})$ of elements in X is said to be statistically Cauchy with respect to the intuitionistic fuzzy norm (κ, ν) if for each $\epsilon > 0$ and $t > 0$ there exist positive integers $M = M(\epsilon)$ and $N = N(\epsilon)$ such that for all $i, p \geq M$ and $j, q \geq N$ we have

$$\delta_2(\{(i, j) \in \mathbf{N} \times \mathbf{N} : \kappa(x_{ij} - x_{pq}, t) \leq 1 - \epsilon \; or \; \nu(x_{ij} - x_{pq}, t) \geq \epsilon\}) = 0.$$

3 (λ, μ)-Statistical Convergence of Double Sequences in IFNS

Recently, the concept of (λ, μ)-statistical convergence for double sequences has been introduced and studied in [13]. In this section we define and study (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces.
Definition 3.1[13] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1;$
$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1.$
Let $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$.
For any set $K \subseteq \mathbf{N} \times \mathbf{N}$, the number

$$\delta_{(\lambda,\mu)}(K) = \lim_{m,n \longrightarrow \infty} \frac{1}{\lambda_n \mu_m} |\{(i,j) \in I_n \times I_m : (i,j) \in K\}|;$$

is called the (λ,μ)-density of the set K, provided the limit exists.

A double sequence $x = (x_{ij})$ of numbers is said to be (λ,μ)-statistically convergent to a number ξ provided that for each $\epsilon > 0$,

$$\lim_{m,n \longrightarrow \infty} \frac{1}{\lambda_n \mu_m} |\{(i,j) \in I_n \times I_m : |x_{ij} - \xi| \geq \epsilon\}| = 0;$$

i.e., the set $K(\epsilon) = \{(i,j) \in I_n \times I_m : |x_{ij} - \xi| \geq \epsilon\}$ has (λ,μ)-density zero. In this case the number ξ is called the (λ,μ)-statistical limit of the sequence $x = (x_{ij})$ and we write $St_{(\lambda,\mu)} - \lim_{i,j \longrightarrow \infty} x_{ij} = \xi$.

Now we define the (λ,μ)-statistical convergence of double sequences with respect to IFNS.

Definition 3.2 Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. A double sequence $x = (x_{ij})$ of elements in X is said to be (λ,μ)-statistically convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (κ,ν) if for each $\epsilon > 0$ and $t > 0$,

$$\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) \leq 1 - \epsilon \ or \ \nu(x_{ij} - \xi, t) \geq \epsilon\}) = 0$$

or equivalently

$$\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) > 1 - \epsilon \ and \ \nu(x_{ij} - \xi, t) < \epsilon\}) = 1.$$

In this case the element ξ is called the (λ,μ)-statistical limit of the sequence (x_{ij}) with respect to the intuitionistic fuzzy norm (κ,ν) and we write $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j \longrightarrow \infty} x_{ij} = \xi$.

Let $St^{(\lambda,\mu)}_{(\kappa,\nu)}(X)$ denotes the set of all (λ,μ)-statistical convergent double sequences with respect to the intuitionistic fuzzy norm (κ,ν).

Definition 3.2, immediately implies the following Lemma.

Lemma 3.1. Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS and $x = (x_{ij})$ be a double sequence in X. Then for each $\epsilon > 0$ and $t > 0$, the following statements are equivalent.

(i) $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j \longrightarrow \infty} x_{ij} = \xi$.

(ii) $\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) \leq 1 - \epsilon\}) = \delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \nu(x_{ij} - \xi, t) \geq \epsilon\}) = 0$.

(iii) $\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) > 1 - \epsilon \ and \ \nu(x_{ij} - \xi, t) < \epsilon\}) = 1$.

(iv) $\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) > 1 - \epsilon\}) = \delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \nu(x_{ij} - \xi, t) < \epsilon\}) = 1$.

(v) $St_{(\lambda,\mu)} - \lim_{i,j \longrightarrow \infty} \kappa(x_{ij} - \xi, t) = 1$ and $St_{(\lambda,\mu)} - \lim_{i,j \longrightarrow \infty} \nu(x_{ij} - \xi, t) = 0$.

Theorem 3.1. Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. If a double sequence $x = (x_{ij})$ is

(λ,μ)-statistically convergent with respect to the intuitionistic fuzzy norm (κ,ν), then its $St^{(\lambda,\mu)}_{(\kappa,\nu)}$-limit is unique.

Proof. Suppose that $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$ and $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \eta$ where $\xi \neq \eta$. Let $\epsilon > 0$ be given. Choose $r > 0$ such that

$$(1-r) * (1-r) > 1-\epsilon \ \textit{and} \ r \diamond r < \epsilon. \tag{1}$$

For any $t > 0$, define

$$K_1(r,t) = \{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t/2) \leq 1 - r\};$$
$$K_2(r,t) = \{(i,j) \in I_n \times I_m : \nu(x_{ij} - \xi, t/2) \geq r\};$$
$$K_3(r,t) = \{(i,j) \in I_n \times I_m : \kappa(x_{ij} - \eta, t/2) \leq 1 - r\}; \ \textit{and}$$
$$K_4(r,t) = \{(i,j) \in I_n \times I_m : \nu(x_{ij} - \eta, t/2) \geq r\}.$$

Since $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$, we have by Lemma 3.1
$\delta_{(\lambda,\mu)}(K_1(r,t)) = \delta_{(\lambda,\mu)}(K_2(r,t)) = 0$ for all $t > 0$.
Furthermore, using $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \eta$, we get
$\delta_{(\lambda,\mu)}(K_3(r,t)) = \delta_{(\lambda,\mu)}(K_4(r,t)) = 0$ for all $t > 0$.
Now, if we take
$K(r,t) = (K_1(r,t) \cup K_3(r,t)) \cap (K_2(r,t) \cup K_4(r,t))$,
then it is easy to observe that $\delta_{(\lambda,\mu)}(K(r,t)) = 0$. But then we have $\delta_{(\lambda,\mu)}(K^C(r,t)) = 1$. Let $(m,n) \in K^C(r,t)$. Now we have the two possibilities.

Case-1 If $(m,n) \in K_1^C(r,t) \cap K_3^C(r,t)$, then we have
$\kappa(x_{mn} - \xi, t/2) > 1 - r$ and $\kappa(x_{mn} - \eta, t/2) > 1 - r$;
and consequently
$\kappa(\xi - \eta, t) \geq \kappa(x_{mn} - \xi, \frac{t}{2}) * \kappa(x_{mn} - \eta, \frac{t}{2}) > (1-r) * (1-r)$.
It follows by (1) that
$\kappa(\xi - \eta, t) > 1 - \epsilon$.
Since $\epsilon > 0$, was selected arbitrary therefore we have $\kappa(\xi - \eta, t) = 1$ for every $t > 0$. But then (iii) of Definition 2.3 implies that $\xi - \eta = 0$ and therefore $\xi = \eta$.

Case-2 If $(m,n) \in K_2^C(r,t) \cap K_4^C(r,t)$, then we have by definition
$\nu(x_{ij} - \xi, t/2) < r$ and $\nu(x_{ij} - \eta, t/2) < r$;
and consequently
$\nu(\xi - \eta, t) < \nu(x_{mn} - \xi, \frac{t}{2}) \diamond \nu(x_{mn} - \eta, \frac{t}{2}) < r \diamond r$.
It follows by (1) that
$\nu(\xi - \eta, t) < \epsilon$.
As $\epsilon > 0$, was selected arbitrary therefore we have $\nu(\xi - \eta, t) = 0$ for every $t > 0$. But then (ix) of Definition 2.3 implies that $\xi = \eta$.

Hence in both cases, we have $\xi = \eta$. This shows that $St^{(\lambda,\mu)}_{(\kappa,\nu)}$-limit of the sequence (x_{ij}) is unique.

Next we show that (κ,ν)-convergence of a double sequence in an IFNS implies its (λ,μ)-statistical convergence, however, the converse is not true in general.

Theorem 3.2. Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. If $x = (x_{ij})$ be a double sequence in X such that $(\kappa,\nu)-\lim_{i,j\longrightarrow\infty} x_{ij}=\xi$, then $St^{(\lambda,\mu)}_{(\kappa,\nu)}-\lim_{i,j\longrightarrow\infty} x_{ij}=\xi$.

Proof. (i) Let $(\kappa,\nu)-\lim_{i,j\longrightarrow\infty} x_{ij}=\xi$. For each $\epsilon>0$ and $t>0$ there exists a positive integer m (say) such that

$\kappa(x_{ij}-\xi,t)>1-\epsilon$ and $\nu(x_{ij}-\xi,t)<\epsilon$

for every $i,j\geq m$. It follows that the set

$$\{(i,j)\in I_n\times I_m : \kappa(x_{ij}-\xi,t)\leq 1-\epsilon \textit{ or } \nu(x_{ij}-\xi,t)\geq\epsilon\}$$

has at most finitely many terms. Since every finite subset of $\mathbf{N}\times\mathbf{N}$ has $\delta_{(\lambda,\mu)}$-density zero, consequently we have

$$\delta_{(\lambda,\mu)}(\{(i,j)\in I_n\times I_m : \kappa(x_{ij}-\xi,t)\leq 1-\epsilon \textit{ or } \nu(x_{ij}-\xi,t)\geq\epsilon\})=0.$$

This shows that $St^{(\lambda,\mu)}_{(\kappa,\nu)}-lim_{i,j\longrightarrow\infty}x_{ij}=\xi$.

Example 3.1. Let $(\mathbf{R},|.|)$ denotes the space of real numbers with the usual norm, and let $a*b=ab$ and $a\diamond b=min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in\mathbf{R}$ and every $t>0$, consider

$$\kappa_0(x,t)=\frac{t}{t+|x|} \quad \textit{and} \quad \nu_0(x,t)=\frac{|x|}{t+|x|},$$

then $(\mathbf{R},\kappa_0,\nu_0,*,\diamond)$ is an IFNS. Now define a sequence $x=(x_{ij})$ as follows.

$$x_{ij}=\begin{cases} ij, & \text{if } n-[\sqrt{\lambda_n}]+1\leq i\leq n \text{ and } m-[\sqrt{\mu_m}]+1\leq j\leq m\\ 0 & \textit{otherwise.}\end{cases}$$

For $\epsilon>0$ and $t>0$, let

$$\begin{aligned} K(\epsilon,t) &= \{(i,j)\in I_n\times I_m : \kappa_0(x_{ij},t)\leq 1-\epsilon \textit{ or } \nu_0(x_{ij},t)\geq\epsilon\}\\ &= \{(i,j)\in I_n\times I_m : \frac{t}{t+|x_{ij}|}\leq 1-\epsilon \textit{ or } \frac{|x_{ij}|}{t+|x_{ij}|}\geq\epsilon\}\\ &= \{(i,j)\in I_n\times I_m : |x_{ij}|\geq\frac{t\epsilon}{1-\epsilon}>0\}\\ &= \{(i,j)\in I_n\times I_m : x_{ij}=ij\}\\ &= \{(i,j)\in I_n\times I_m : n-[\sqrt{\lambda_n}]+1\leq i\leq n \textit{ and } m-[\sqrt{\mu_m}]+1\leq j\leq m\};\end{aligned}$$

and so we get

$$\frac{1}{\lambda_n\mu_m}|K(\epsilon,t)|\leq\frac{1}{\lambda_n\mu_m}|\{(i,j)\in I_n\times I_m : n-[\sqrt{\lambda_n}]+1\leq i\leq n \textit{ and } m-[\sqrt{\mu_m}]+1\leq j\leq m\}|\leq\frac{\sqrt{\lambda_n}\sqrt{\mu_m}}{\lambda_n\mu_m}.$$

Taking limit n and m both approaches to ∞, we get

$$\delta_{(\lambda,\mu)}(K(\epsilon,t))=\lim_{n,m\longrightarrow\infty}\frac{1}{\lambda_n\mu_m}|K(\epsilon,t)|\leq\lim_{n,m\longrightarrow\infty}\frac{\sqrt{\lambda_n}\sqrt{\mu_m}}{\lambda_n\mu_m}=0.$$

This shows that
$St^{(\lambda,\mu)}_{(\kappa_0,\nu_0)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \theta$, where θ denotes the zero element in X.
But the sequence is not (κ_0,ν_0)-convergent to θ as

$$\kappa_0(x_{ij},t) = \frac{t}{t+|x_{ij}|} = \begin{cases} \frac{t}{t+ij}, & \text{if } n-[\sqrt{\lambda_n}]+1 \le i \le n \text{ and } m-[\sqrt{\mu_m}]+1 \le j \le m \\ 1 & \textit{otherwise.} \end{cases} \le 1;$$

and

$$\nu_0(x_{ij},t) = \frac{|x_{ij}|}{t+|x_{ij}|} = \begin{cases} \frac{ij}{t+ij}, & \text{if } n-[\sqrt{\lambda_n}]+1 \le i \le n \text{ and } m-[\sqrt{\mu_m}]+1 \le j \le m \\ 0 & \textit{otherwise.} \end{cases} \ge 0.$$

Now we give an important characterization of (λ,μ)-statistical convergent double sequences in an IFNS.

Theorem 3.3. Let $(X,\kappa,\nu,*,\diamond)$ be an IFNS. Then, $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$, if and only if, there exists a subset $K = \{(i,j) : i,j = 1,2,3,\ldots\} \subseteq \mathbf{N}\times\mathbf{N}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $(\kappa,\nu) - \lim_{(i,j)\in K, i,j\longrightarrow\infty} x_{ij} = \xi$.

Proof. **Necessity-** Suppose that $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$. For any $t > 0$ and $r = 1,2,3,\ldots$, let

$$M_{\kappa,\nu}(r,t) = \{(i,j) \in I_n \times I_m : \kappa(x_{ij}-\xi,t) > 1-\frac{1}{r} \textit{ and } \nu(x_{ij}-\xi,t) < \frac{1}{r}\} \textit{ and}$$
$$K_{\kappa,\nu}(r,t) = \{(i,j) \in I_n \times I_m : \kappa(x_{ij}-\xi,t) \le 1-\frac{1}{r} \textit{ or } \nu(x_{ij}-\xi,t) \ge \frac{1}{r}\}.$$

Since $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$, it follows that
$\delta_{(\lambda,\mu)}(K_{\kappa,\nu}(r,t)) = 0.$
Furthermore, for $t > 0$ and $r = 1,2,3,\ldots$, we observe
$M_{\kappa,\nu}(r,t) \supset M_{\kappa,\nu}(r+1,t)$ and

$$\delta_{(\lambda,\mu)}(M_{\kappa,\nu}(r,t)) = 1. \tag{2}$$

Now, we have to show that, for $(i,j) \in M_{\kappa,\nu}(r,t)$, $(\kappa,\nu) - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$. Suppose, for $(i,j) \in M_{\kappa,\nu}(r,t)$, (x_{ij}) is not convergent to ξ with respect (κ,ν)-norm. Then there exists some $\beta > 0$ such that
$\{(i,j) \in \mathbf{N}\times\mathbf{N} : \kappa(x_{ij}-\xi,t) \le 1-\beta \, or \, \nu(x_{ij}-\xi,t) \ge \beta\}$
for infinitely many terms x_{ij}.
Let,
$M_{\kappa,\nu}(\beta,t) = \{(i,j) \in I_n \times I_m : \kappa(x_{ij}-\xi,t) > 1-\beta \, and \, \nu(x_{ij}-\xi,t) < \beta\}$
and
$\beta > \frac{1}{r}, r = 1,2,3,\ldots.$
Then we have
$\delta_{(\lambda,\mu)}(M_{\kappa,\nu}(\beta,t)) = 0.$
Furthermore, $M_{\kappa,\nu}(r,t) \subset M_{\kappa,\nu}(\beta,t)$ implies that $\delta_{(\lambda,\mu)}(M_{\kappa,\nu}(r,t)) = 0$. In this way we obtained a contradiction to (2) as $\delta_{(\lambda,\mu)}(M_{\kappa,\nu}(r,t)) = 1$. Hence, $(\kappa,\nu) - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$.

Sufficiency- Suppose that there exists a subset $K = \{(i, j) : i, j = 1, 2, 3, \ldots\} \subseteq \mathbf{N} \times \mathbf{N}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $(\kappa, \nu) - \lim_{(i,j)\in K, i,j\longrightarrow\infty} x_{ij} = \xi$. But then for every $\epsilon > 0$ and $t > 0$ we can find out a positive integer m such that

$\kappa(x_{ij} - \xi, t) > 1 - \epsilon$ and $\nu(x_{ij} - \xi, t) < \epsilon$

for all $i, j \geq m$. If we take,

$K_{\kappa,\nu}(\epsilon, t) = \{(i, j) \in I_n \times I_m : \kappa(x_{ij} - \xi, t) \leq 1 - \epsilon \, or \, \nu(x_{ij} - \xi, t) \geq \epsilon\}$,

then it is easy to see that

$K_{\kappa,\nu}(\epsilon, t) \subseteq \mathbf{N} \times \mathbf{N} - \{(i_{m+1}, j_{m+1}), (i_{m+2}, j_{m+2}), \ldots\}$

and consequently

$\delta_{(\lambda,\mu)}(K_{\kappa,\nu}(\epsilon, t)) \leq 1 - 1 = 0.$

Hence, $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$.

Finally, we define (λ, μ)- statistically Cauchy double sequences in an IFNS and establish the Cauchy convergence criteria in these spaces.

Definition 3.3. Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. A double sequence $x = (x_{ij})$ of elements in X is said to be (λ, μ)-statistically Cauchy with respect to the intuitionistic fuzzy norm (κ, ν) if for each $\epsilon > 0$ and $t > 0$, there exist positive integers $n = n(\epsilon)$ and $m = m(\epsilon)$ such that for all $i, p \geq n$ and $j, q \geq m$

$$\delta_{(\lambda,\mu)}(\{(i, j) \in I_n \times I_m : \kappa(x_{ij} - x_{pq}, t) \leq 1 - \epsilon \; or \; \nu(x_{ij} - x_{pq}, t) \geq \epsilon\}) = 0$$

or equivalently

$$\delta_{(\lambda,\mu)}(\{(i, j) \in I_n \times I_m : \kappa(x_{ij} - x_{pq}, t) > 1 - \epsilon \; and \; \nu(x_{ij} - x_{pq}, t) < \epsilon\}) = 1.$$

Theorem 3.4. Let $(X, \kappa, \nu, *, \diamond)$ be an IFNS. A double sequence $x = (x_{ij})$ in X is (λ, μ)-statistically convergent, if and only if, it is (λ, μ)-statistically Cauchy with respect to the intuitionistic fuzzy norm (κ, ν).

Proof. First suppose that there exists $\xi \in X$ such that $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$. Let $\epsilon > 0$ be given. Choose $r > 0$ such that (1) is satisfied. For any $t > 0$, define

$$A(r, t) = \{(i, j) \in I_n \times I_m : \kappa(x_{ij} - \xi, \frac{t}{2}) \leq 1 - r \; or \; \nu(x_{ij} - \xi, \frac{t}{2}) \geq r\} \; and$$
$$A^C(r, t) = \{(i, j) \in I_n \times I_m : \kappa(x_{ij} - \xi, \frac{t}{2}) > 1 - r \; and \; \nu(x_{ij} - \xi, \frac{t}{2}) < r\}.$$

Since $St^{(\lambda,\mu)}_{(\kappa,\nu)} - \lim_{i,j\longrightarrow\infty} x_{ij} = \xi$, it follows that

$\delta_{(\lambda,\mu)}(A(r, t)) = 0$, and consequently, $\delta_{(\lambda,\mu)}(A^C(r, t)) = 1$.

Let $(p, q) \in A^C(r, t)$. Then

$$\kappa(x_{pq} - \xi, \frac{t}{2}) > 1 - r \; and \; \nu(x_{pq} - \xi, \frac{t}{2}) < r. \tag{3}$$

If we take,

$$B(\epsilon, t) = \{(i, j) \in I_n \times I_m : \kappa(x_{ij} - x_{pq}, t) \leq 1 - \epsilon \; or \; \nu(x_{ij} - x_{pq}, t) \geq \epsilon\};$$

then to prove the result it is sufficient to prove that $B(\epsilon,t) \subseteq A(r,t)$. Let $(m,n) \in B(\epsilon,t)$, then

$$\kappa(x_{mn}-x_{pq},t) \leq 1-\epsilon \ or \ \nu(x_{mn}-x_{pq},t) \geq \epsilon. \tag{4}$$

Case 1 If $\kappa(x_{mn}-x_{pq},t) \leq 1-\epsilon$, then we have $\kappa(x_{mn}-\xi,\frac{t}{2}) \leq 1-r$ and therefore $(m,n) \in A(r,t)$. As otherwise i.e., if $\kappa(x_{mn}-\xi,\frac{t}{2}) > 1-r$, then by (1), (3) and (4) we get

$$\begin{aligned} 1-\epsilon \geq \kappa(x_{mn}-x_{pq},t) &\geq \kappa(x_{mn}-\xi,\frac{t}{2}) * \kappa(x_{pq}-\xi,\frac{t}{2}) \\ &> (1-r)*(1-r) > 1-\epsilon; \end{aligned}$$

which is not possible. Hence $B(\epsilon,t) \subset A(r,t)$.

Case 2 If $\nu(x_{mn}-x_{pq},t) \geq \epsilon$, then we have $\nu(x_{mn}-\xi,\frac{t}{2}) \geq r$ and therefore $(m,n) \in A(r,t)$. As otherwise i.e., if $\nu(x_{mn}-\xi,\frac{t}{2}) < r$, then by (1), (3) and (4) we get

$$\begin{aligned} \epsilon \leq \nu(x_{mn}-x_{pq},t) &\leq \nu(x_{mn}-\xi,\frac{t}{2}) \diamond \nu(x_{pq}-\xi,\frac{t}{2}) \\ &< r \diamond r < \epsilon; \end{aligned}$$

which is not possible. Hence $B(\epsilon,t) \subset A(r,t)$.

Thus in all cases, we have $B(\epsilon,t) \subset A(r,t)$. Since $\delta_{(\lambda,\mu)}(A(r,t)) = 0$, it follows that $\delta_{(\lambda,\mu)}(B(\epsilon,t)) = 0$. This shows that $x = (x_{ij})$ is (λ,μ)-statistically Cauchy with respect to the intuitionistic fuzzy norm (κ,ν).

Conversely, suppose that $x = (x_{ij})$ is (λ,μ)-statistically Cauchy but not (λ,μ)-statistically convergent with respect to the intuitionistic fuzzy norm (κ,ν). Then there exist positive integers p and q such that if we take

$$\begin{aligned} A(\epsilon,t) &= \{(i,j) \in I_n \times I_m : \kappa(x_{ij}-x_{pq},t) \leq 1-\epsilon \ \ or \ \ \nu(x_{ij}-x_{pq},t) \geq \epsilon\} \ and \\ B(\epsilon,t) &= \{(i,j) \in I_n \times I_m : \kappa(x_{ij}-\xi,\frac{t}{2}) > 1-\epsilon \ \ and \ \ \nu(x_{ij}-\xi,\frac{t}{2}) < \epsilon\}, \end{aligned}$$

then $\delta_{(\lambda,\mu)}(A(\epsilon,t)) = \delta_{(\lambda,\mu)}(B(\epsilon,t)) = 0$ and consequently

$$\delta_{(\lambda,\mu)}(A^C(\epsilon,t)) = \delta_{(\lambda,\mu)}(B^C(\epsilon,t)) = 1. \tag{5}$$

Since

$$\begin{aligned} \kappa(x_{ij}-x_{pq},t) &\geq 2\kappa(x_{ij}-\xi,\frac{t}{2}) > 1-\epsilon \ and \\ \nu(x_{ij}-x_{pq},t) &\leq 2\nu(x_{ij}-\xi,\frac{t}{2}) < \epsilon, \end{aligned}$$

if

$$\kappa(x_{ij}-\xi,\frac{t}{2}) > \frac{1-\epsilon}{2} \ and \ \nu(x_{ij}-\xi,\frac{t}{2}) < \frac{\epsilon}{2}$$

respectively, we have

$$\delta_{(\lambda,\mu)}(\{(i,j) \in I_n \times I_m : \kappa(x_{ij}-x_{pq},t) > 1-\epsilon \ \ and \ \ \nu(x_{ij}-x_{pq},t) < \epsilon\}) = 0$$

i.e., $\delta_{(\lambda,\mu)}(A^C(\epsilon,t)) = 0$. But then we obtained a contradiction to (5) as $\delta_{(\lambda,\mu)}(A^C(\epsilon,t)) = 1$. Hence, (x_{ij}) is (λ,μ)-statistically convergent with respect to the intuitionistic fuzzy norm (κ,ν).

On combine Theorem 3.3 and Theorem 3.4, we have the following.

Theorem 3.5. Let $(X,\kappa,\nu,*,\diamond)$ be an IFNS and $x = (x_{ij})$ be a double sequence in X. Then, the following statements are equivalent:

(i) x is a (λ,μ)-statistically convergent with respect to the intuitionistic fuzzy norm (κ,ν).

(ii) x is a (λ,μ)-statistically Cauchy with respect to the intuitionistic fuzzy norm (κ,ν).

(iii) there exists a subset $K = \{(i,j) : i,j = 1,2,3,\ldots\} \subseteq \mathbf{N}\times\mathbf{N}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $(\kappa,\nu) - \lim_{(i,j)\in K, i,j\longrightarrow\infty} x_{ij} = \xi$.

4 Conclusions

In this paper, we introduced a more general type of convergence, namely, (λ,μ)-statistical convergence for double sequences on IFNS. We did so with the help of two non-decreasing sequences $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ of positive real numbers such that each tending to ∞ and

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$;

$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$.

We observe that, if $(\lambda_n) = (n)$ and $(\mu_m) = (m)$ for every n and m, then (λ,μ)-density is reduced to double natural density δ_2 and consequently (λ,μ)-statistical convergence of double sequences in IFNS coincides with its statistical convergence.

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