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# ON LAPLACE CONTINUED FRACTION FOR THE NORMAL INTEGRAL

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**Abstract.** The Laplace continued fraction is derived through a power series. It provides both upper bounds and lower bounds of the normal tail probability $\bar{\Phi}(x)$, it is simple, it converges for $x > 0$, and it is by far the best approximation for $x \geq 3$. The Laplace continued fraction is rederived as an extreme case of admissible bounds of the Mills' ratio, $\bar{\Phi}(x)/\phi(x)$, in the family of ratios of two polynomials subject to a monotone decreasing absolute error. However, it is not optimal at any finite $x$. Convergence at the origin and local optimality of a subclass of admissible bounds are investigated. A modified continued fraction is proposed. It is the sharpest tail bound of the Mills' ratio, it has a satisfactory convergence rate for $x \geq 1$ and it is recommended for the entire range of $x$ if a maximum absolute error of $10^{-4}$ is required.

*Key words and phrases*: Admissibility, approximation, convergence, Mills' ratio, optimality, rational bound.

## 1. Introduction

Let $\phi(x)$, $\Phi(x)$ and $\bar{\Phi}(x)$ be the normal density function, the normal distribution function and the normal tail probability function respectively. The purpose of this article is to find bounds of the Mills' ratio $R(x) = \bar{\Phi}(x)/\phi(x)$, or equivalently of the normal distribution function $\Phi(x)$. Earlier published results for bounds and approximations can be found in Mitrinović (1970), Abramowitz and Stegun (1972), Kendall and Stuart (1977) and Patel and Read (1982). The Laplace (1805) continued fraction,

$$L_n = \frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots\frac{n}{x}, \tag{1.1}$$

is the best approximation to $R(x)$ for $x \geq 3$. However, it has seldom been recognized in the literature as providing an alternating sequence of upper and lower

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bounds of the Mills' ratio. It is known that $L_n$ converges to $R(x)$ for each $x > 0$. The derivation of $L_n$ as a power series can be found in Kendall and Stuart ((1977), p. 145) and they said that "the most useful forms for the calculation of $\Phi(x)$, or equivalently of $R(x)$, however, are continued fractions." Kerridge and Cook (1976) stated that "If accuracy is the only consideration, and speed unimportant, Laplace's continued fraction can hardly be improved on, although many thousand terms may be needed if $x$ is small." The Laplace continued fraction was used even for small $x$ by Sheppard (1939) to produce his highly accurate tables of the normal integral.

The Laplace continued fraction can be expressed as a ratio of two polynomials with degrees $n$ and $n+1$ respectively. In the family of ratios of two polynomials, there is no uniformly best bound of the Mills' ratio. A subclass having the best tail bound is the continued fraction

$$f_n(x; a_n, b_n) = \frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots\frac{n-1}{x+}\frac{b_n}{x+}a_n \tag{1.2}$$

with admissible coefficients $a_n$ and $b_n$, and it shall be shown that

$$\bar{\Phi}(x) = \phi(x)\left\{\frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots\frac{n-1}{x+}\frac{b_n}{x+}a_n\right\} + \epsilon_n \tag{1.3}$$

even at $x = 0$ where $\epsilon_n$ represents a quantity converging to zero as $n \to \infty$. The expression (1.3) is very simple to compute. It requires $n+1$ additions, $n+1$ reciprocals and $n+1$ multiplications. For example, $\{(a_1 + x)^{-1}b_1 + x\}^{-1}\phi(x)$ at $x = 2$ has an error 0.000004 in approximating $\bar{\Phi}(x)$ if the optimal coefficients $a_1 = 1.252$ and $b_1 = 1.215$ (see Section 5) at $x = 2$ and $n = 1$ are used.

The admissibility of the parameters $a_n$ and $b_n$ in (1.2) is investigated in Section 2 where the Laplace continued fraction $L_n$ with $a_n = 0$ and $b_n = n$ is found to be an extreme case. Convergence at the origin and properties of $f_n$ are presented in Section 3. Bounds of the Mills' ratio are compared in Section 4. It is found that $f_3$ is superior to bounds of the Mills' ratio in the literature if $x \geq 0.7$. Asymptotic optimality of $f_n$ is discussed in Section 5. It is remarkable to observe that the simple expression (1.3) with $n = 12$ has a maximum absolute error no more than $10^{-4}$ if $a_n = 2\{(b_n+1)(b_n-n)/b_n\}^{1/2}$ and $b_n = 2n - xn^{1/2} + (x^2-1)/2$. In the last section, the continued fraction $f_n$ is compared with other well-known approximations to $\Phi(x)$. The continued fraction $f_n$ is recommended for desk computation in the range $x \geq 1$.

## 2. Admissible bounds

Let $P(x)/Q(x)$ be a ratio of two polynomials with degrees $n$ and $m$ respectively. The Laplace continued fraction $L_n$ is such a ratio. The ratio is a lower (upper) bound of $\bar{\Phi}(x)/\phi(x)$ if the integral $\int_t^\infty \Delta_n(x)\phi(x)dx$ is nonnegative (nonpositive) for each $t > 0$ where $\Delta_n(x) = 1 + d\{P(x)/Q(x)\}/dx - xP(x)/Q(x)$. The integral constraint is very complicated and it can at best achieve local optimality. We consider in this article a simpler constraint that requires $\Delta_n(x)$ itself to be nonnegative or nonpositive for each $x > 0$ as in Feller ((1968), p. 179). Under this requirement, the bound $P(x)/Q(x)$ has a monotone decreasing absolute error in $x$ and hence it is least favorable at the origin. The lower bound $P(x)/Q(x)$ is said to be admissible if there does not exist a ratio $p(x)/q(x)$ of two polynomials with the degree of the numerator no more than $n$ such that

$$P(x)/Q(x) \leq p(x)/q(x) \leq R(x).$$

The admissible upper bound is defined similarly. The purpose of this section is to establish the following theorem.

THEOREM 2.1. *Let $n$ be an odd (even) integer. Then a continued fraction $f_n(x; a_n, b_n)$ is an admissible lower (upper) bound of $\bar{\Phi}(x)/\phi(x)$ subject to a monotone decreasing absolute error in the family of ratios of two polynomials with the degree of the numerator no more than $n$ if*

$$(2.1)\quad a_n = 2\{(b_n + 1)(b_n - n)/b_n\}^{1/2} \quad \text{and} \quad n \leq b_n \leq (n^2 + n + 1)^{1/2} + (n - 1).$$

*It is an admissible upper (lower) bound of $\bar{\Phi}(x)/\phi(x)$ under the same conditions if $a_n = (n + 1)^{1/2}$ and $b_n = n$.*

We shall first introduce the following notation. For nonnegative integers $n$ and $m$ with $m \leq n/2$, let

$$c_{n,n-2m} = \{(2m)!/(2^m m!)\}\binom{n}{2m},$$

$$d_{n,n-2m} = \sum_{t=0}^{m}(-1)^t\{(2t)!/(2^t t!)\}c_{n,n-2m+2t},$$

$$C_n(x) = \sum_{m=0}^{[n/2]} c_{n,n-2m}x^{n-2m} \quad \text{and}$$

$$D_n(x) = \sum_{m=0}^{[(n-1)/2]} d_{n,n-2m}x^{n-2m-1}$$

where the symbol $[\alpha]$ represents the largest integer less than or equal to $\alpha$. It can be shown that $c_{n,n} = 1$, $d_{n,n} = 1$,

$$c_{n,n-2m} = c_{n-1,n-1-2m} + (n-1)c_{n-2,n-2m} \quad \text{and}$$
$$d_{n,n-2m} = d_{n-1,n-1-2m} + (n-1)d_{n-2,n-2m}.$$

Furthermore, $c_{2m,0} = (2m-1)c_{2m-2,0}$, $d_{2m,0} = 0$ and $d_{2m+1,1} = 2m d_{2m-1,1}$ if $m \geq 1$. It follows that

$$(2.2)\quad C_n(x) = xC_{n-1}(x) + (n-1)C_{n-2}(x)$$

and

$$D_n(x) = xD_{n-1}(x) + (n-1)D_{n-2}(x).$$

The polynomial $C_n(x)$ can also be expressed by $H_n(ix)/i^n$ where $i$ is the imaginary number and $H_n(x)$ is the $n$-th Hermite polynomial (see Kendall and Stuart ((1977), p. 167)).

LEMMA 2.1. *Let $a_n$ and $b_n$ be nonnegative real numbers and let*

$$P_n(x) = (x + a_n)D_n(x) + b_n D_{n-1}(x) \quad \text{and}$$
$$Q_n(x) = (x + a_n)C_n(x) + b_n C_{n-1}(x).$$

*Then the continued fraction $f_n(x; a_n, b_n)$ in (1.2) can be expressed by $P_n(x)/Q_n(x)$.*

PROOF. It is trivial that $f_1 = P_1(x)/Q_1(x)$. By (2.2),

$$Q_{n+1}(x; a_{n+1}, b_{n+1}) = (x + a_{n+1})Q_n\{x; b_{n+1}/(x + a_{n+1}), n\} \tag{2.3}$$

and the identity also holds for $P_{n+1}(x)$. The proof is completed by induction. □

LEMMA 2.2. *Let $a_n$ and $b_n$ be nonnegative real numbers and let*

$$\Delta_n(x) = 1 + df_n/dx - xf_n.$$

*Then*

$$\Delta_n(x) = (-1)^{n-1}(n-1)!\{(b_n - n)x^2 + a_n(b_n - 2n)x + b_n(b_n + 1) - na_n^2\}/Q_n(x)^2.$$

PROOF. Suppose instead that $a_n$ is a function of $x$. It shall be established that $\Delta_n(x; a_n, b_n)$ is given by the above identity except we add another term $(-1)^{n-1}(n-1)!b_n(da_n/dx)/Q_n(x)^2$ to the right-hand side. The new identity holds when $n = 1$. By (2.3) and

$$\Delta_{n+1}(x; a_{n+1}, b_{n+1}) = \Delta_n(x; b_{n+1}/(x + a_{n+1}), n),$$

the proof is completed by induction. □

PROOF OF THEOREM 2.1. We shall consider only the case that $n$ is an odd integer. The case of even $n$ follows similarly. The continued fraction $f_n(x; a_n, b_n)$ with nonnegative coefficients $a_n$ and $b_n$ is a lower bound of the Mills' ratio subject to a monotone decreasing absolute error if $\Delta_n(x) \geq 0$ for each $x > 0$. By Lemma 2.2, it is required that $b_n \geq n$, $b_n(b_n + 1) \geq na_n^2$, and $a_n^2 b_n \leq 4(b_n + 1)(b_n - n)$ if $b_n \leq 2n$. Let $a$, $b$, $A$ and $B$ be any nonnegative real numbers. Then

$$f_n(x; a, b) \leq f_n(x; A, B)$$

if and only if

$$(b - B)x + (bA - aB) \geq 0. \tag{2.4}$$

Consider a specific lower bound $f_n(x; A, B)$ with $A = (4n+2)^{1/2}$ and $B = 2n$. If $b \geq 2n$ and $b(b+1) \geq na^2$ then (2.4) holds and the lower bound $f_n(x; a, b)$ is not admissible. It follows from (2.4) that admissibility for fixed $b$ requires the largest $a$ and it suffices to consider $n \leq b \leq 2n$ and $a^2 b = 4(b+1)(b-n)$. Since $a/b$ as a function of $b$ has a unique mode $B = (n^2+n+1)^{1/2} + (n-1)$, it follows from (2.4) again that admissibility requires $n \leq b \leq B$. On the other hand, a continued fraction $f_n(x; a_n, b_n)$ with nonnegative coefficients $a_n$ and $b_n$ is an upper bound of the Mills' ratio subject to the same constraint if $\Delta_n(x) \leq 0$ for each $x > 0$ or equivalently $b_n \leq n$ and $na_n^2 \geq b_n(b_n+1)$. It can be shown by (2.4) that admissibility requires $a_n = (n+1)^{1/2}$ and $b_n = n$.

Let $a$ and $b$ be any pair of real numbers. Suppose that $f_n(x; a, b)$ is a lower bound of the Mills' ratio and it is uniformly no smaller than one of the lower bound $f_n(x; a_n, b_n)$ with $a_n$ and $b_n$ satisfying (2.1). Since it is bounded from above by $f_n\{x; (n+1)^{1/2}, n\}$, by (2.4) we have that $(b-n)x + (n+1)^{1/2}b - na \geq 0$ for each $x > 0$. Therefore, $b \geq n$. By (2.4) again, for fixed $b$ it suffices to consider $a \geq 0$. On the other hand, suppose that $f_n(x; a, b)$ is an upper bound of the Mills' ratio and it is uniformly no larger than $f_n\{x; (n+1)^{1/2}, n\}$. Since it is bounded from below by $f_n(x; 0, n)$, by (2.4) we have that $(n-b)x + na \geq 0$ for each $x > 0$. Therefore $a \geq 0$. By (2.4) again, for fixed $a$ it suffices to consider $b \geq 0$.

Let $P(x)$ and $Q(x)$ be any pair of polynomials with the degree of $P(x)$ no more than $n$. Suppose that $P(x)/Q(x)$ is a lower bound of the Mills' ratio and $f_n(x; a, b) \leq P(x)/Q(x)$ for a pair of nonnegative real numbers $a$ and $b$. It suffices to consider $P(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$ and $Q(x) = q_m x^m + \cdots + q_0$ for some integer $m$. Then $P(x)/Q(x) \leq f_n(x; A, B)$ with $A = (n+1)^{1/2}$ and $B = n$. By the fact that a continued fraction

$$\frac{b_0}{x+}\frac{b_1}{x+}\frac{b_2}{x+}\cdots\frac{b_n}{x+}a_n \tag{2.5}$$

can be expressed by $b_0(x^n + a_n x^{n-1} + \cdots)/(x^{n+1} + a_n x^n + \cdots)$,

$$q_m x^{m+n} + (aq_m + q_{m-1})x^{m+n-1} + \cdots \leq x^{2n+1} + (a + p_{n-1})x^{2n} + \cdots$$

and

$$x^{2n+1} + (A + p_{n-1})x^{2n} + \cdots \leq q_m x^{m+n} + (Aq_m + q_{m-1})x^{m+n-1} + \cdots$$

for each $x \geq 0$. It follows that $m = n+1$, $q_m = 1$ and $q_{m-1} = p_{n-1}$. Taking the reciprocal and then subtracting $x$, we have

$$\frac{1}{x+}\frac{2}{x+}\cdots\frac{n-1}{x+}\frac{b}{x+}a \geq \{Q(x) - xP(x)\}/P(x) \geq \frac{1}{x+}\frac{2}{x+}\cdots\frac{n-1}{x+}\frac{B}{x+}A$$

where $Q(x) - xP(x)$ is a polynomial with degree no more than $n-1$. Successive reduction will eatablish that $P(x)/Q(x)$ is the continued fraction $f_n$ in (1.2). The results for the case that $P(x)/Q(x)$ is an upper bound of the Mills' ratio follow similarly. □

## 3. Convergence and properties of $f_n$

The sequence $\{L_n\}$ of Laplace continued fractions in (1.1) is by Theorem 2.1 a sequence of admissible bounds $\{f_n\}$ in (1.2) when $a_n = 0$ and $b_n = n$. It is known that

$$L_1 < L_3 < \cdots < \bar{\Phi}(x)/\phi(x) < \cdots < L_4 < L_2 \tag{3.1}$$

if $x > 0$. By the fact that $L_n - L_{n-1} = (-1)^n/C_n(x)C_{n+1}(x)$ (see Abramowitz and Stegun ((1972), p. 19)), the difference converges to zero for each fixed $x > 0$. It follows that (1.3) holds for each $x > 0$. However, $L_1 = L_3 = \cdots = 0$ and $L_2 = L_4 = \cdots = \infty$ at the origin. Consequently, its convergence rate is very slow for small $x$.

The Laplace continued fraction may be written in another form $\{f_n(x; n+1, n)\}$. However, this sequence also fails to converge at the origin. Furthermore, the bound $f_n(x; n+1, n)$ is not admissible. By Theorem 2.1, the superior admissible bound is

$$f_n(x; (n+1)^{1/2}, n) = L_n^r = \frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\cdots\frac{n}{x+}(n+1)^{1/2}.$$

It can be shown that

$$L_2^r < L_4^r < \cdots < \bar{\Phi}(x)/\phi(x) < \cdots < L_3^r < L_1^r.$$

It can also be shown that $L_n^r$ satisfies (1.3) for each $x \geq 0$ including the origin. It is trivial that $L_{2n-1} < L_{2n}^r$ and $L_{2n+1} < L_{2n}^r$ if $x < (2n+1)^{1/2}$, and $L_{2n+1}^r < L_{2n}$, $L_{2n+1}^r < L_{2n+2}$ if $x < (2n+2)^{1/2}$. One would prefer $L_n^r$ to $L_n$.

In the remainder of this article, the continued fraction $f_n(x; a_n, b_n)$ will be reserved for the case

$$a_n = 2\{(b_n+1)(b_n-n)/b_n\}^{1/2} \quad \text{and} \quad n \leq b_n \leq (n^2+n+1)^{1/2} + (n-1).$$

It can be shown that

$$L_{2r-1} \leq f_{2r+1} \leq \bar{\Phi}(x)/\phi(x) \leq f_{2r} \leq L_{2r-2} \tag{3.2}$$

for each $x \geq 0$. By Stirling's formula, $f_n$ satisfies (1.3) including the origin if and only if

$$b_n = 2n + O(n^r) \tag{3.3}$$

with $r < 1$. The coefficient $b_n$, denoted by $b_n(0)$, which minimizes the absolute error $|\bar{\Phi}(x) - f_n\phi(x)|$ at the origin is of the form

$$b_n(0) = (n^2+n+1)^{1/2} + (n-1) \tag{3.4}$$

and $\{f_n\}$, with $b_n = b_n(0)$, satisfies (3.1) and (1.3) including the origin. The corresponding $a_n$ can be expressed by

$$2[\{2(n^2+n+1)^{1/2}+(n-1)\}/3]^{1/2}.$$

The continued fraction $f_n$ has an absolute error uniformly no more than $L^r_{n-1}$ if and only if $b_n^3 \leq 4n(b_n+1)(b_n-n)$. The coefficient $b_n(0)$ satisfies this constraint. The solution to the cubic polynomial is of the form

$$b_n = 4\{n+(n^2+3n)^{1/2}\sin(\theta/3-\pi/6)\}/3 \tag{3.5}$$

where $\theta = \arctan[\{27(8n^2+13n+16)/n\}^{1/2}/(9-4n)]$. The absolute error of $f_n$ with $b_n$ in (3.5) is less than that of $f_n$ with $b_n$ in (3.4) if $x \geq 1$. One would prefer $f_n$ with $b_n$ in (3.4) or (3.5) to $L_n$ and $L^r_n$.

## 4. Bounds of Mills' ratio

The first three orders of $L_n$, $L^r_n$ or $f_n$ are respectively

$$(x+a_1)/(x^2+a_1x+b_1),$$
$$(x^2+a_2x+b_2)/\{x^3+a_2x^2+(b_2+1)x+a_2\} \quad \text{and}$$
$$\{x^3+a_3x^2+(b_3+2)x+2a_3\}/\{x^4+a_3x^3+(b_3+3)x^2+3a_3x+b_3\}.$$

It will be shown that these bounds are superior to bounds of the Mills' ratio in the literature where the Laplace continued fraction $L_n$ has seldom been recognized as a bound. The upper bound and the lower bound of the Mills' ratio in Gordon (1941) are $L_0 = 1/x$ and $L_1 = x/(x^2+1)$ respectively. The lower bound and the upper bound of the Mills' ratio in Gross and Hosmer (1978) are $L_1$ and $L_2 = (x^2+2)/(x^3+3x)$ respectively. For lower bounds of the Mills' ratio, $L_1$ is superior to the Laplace (1785) asymptotic expansion $1/x - 1/x^3$ (see Feller ((1968), p. 175 and p. 193)), $L^r_2$ is superior to $\{(x^2+4)^{1/2}-x\}/2$ of Birnbaum (1942), and $f_3\{x; a_3, b_3(0)\}$ is better than $\pi/\{(\pi-1)x+(x^2+2\pi)^{1/2}\}$ of Boyd (1959) if $x \geq 0.1$. For upper bounds of the Mills' ratio, $L^r_3$ is superior to $4/\{3x+(x^2+8)^{1/2}\}$ of Sampford (1953), $f_2\{x; a_2, b_2(0)\}$ is better than $2/\{x+(x^2+2)^{1/2}\}$ of Komatu (1955), $2/\{x+(x^2+8/\pi)^{1/2}\}$ of Pollak (1956) if $x \geq 0.1$ and $\pi/[2x+\{2\pi+(\pi-2)^2x^2\}^{1/2}]$ of Boyd (1959) if $x \geq 0.7$. Continued fractions $f_n$ and $L^r_n$ provide the sharpest tail bounds of the Mills' ratio in the literature if $x \geq 0.7$.

Other complicated bounds of the Mills' ratio are also investigated. Laplace asymptotic expansions, the six rational bounds in Shenton ((1954), p. 188) and the three rational lower bounds in Ruben ((1963), p. 362) can be expressed by the continued fraction (2.5). While the leading coefficients $b_0, b_1, b_2, b_3, \ldots$ have the pattern $1, 1, 2, 3, \ldots$, there is always a first negative coefficient, say $b_{n+1}$, with two exceptions. These bounds can be expressed as ratios of two polynomials with the degree of the numerator no less than $n+1$. Laplace asymptotic expansions are inferior to $L_0, L_1, L_2, L_3, \ldots$, except that the first one $1/x$ is $L_0$. The six rational bounds of Shenton (1954) are inferior to $L_3$, $L_5$, $L_4$, $L_6$, $L_5$ and $L_6$ respectively. The three rational lower bounds of Ruben (1963) are inferior to $L_1$, $L_3$ and $L_5$

respectively except that the first one is $L_1$. The sequence of lower bounds of the Mills' ratio of the form $(x^2+2r+2)^{-1/2}$ in (2.10) of Ruben (1964) are inferior to $L_2^r, L_4^r, L_6^4, \ldots$. Even though the numbers of terms in $L_n^r$ are larger, expressions and computations are, however, much simpler.

When $x$ is small, say $x \le 1$, the Laplace (1785) series

$$\Phi(x) = \Phi(0) + \phi(0)\sum_{t=0}^{n}(-1)^t x^{2t+1}/2^t(2t+1)t! + \varepsilon_n \tag{4.1}$$

and the Laplace (1812) series, rediscovered by Pólya (1949),

$$\Phi(x) = \Phi(0) + \phi(x)\sum_{t=0}^{n} x^{2t+1}/1\cdot 3\cdot 5\cdots(2t+1) + \varepsilon_n \tag{4.2}$$

are very good bounds of $\Phi(x)$. If the same number of terms is used, the absolute error of (4.1) is less than that of (4.2) in the range $x \le 2$. While the Laplace series is alternatively an upper bound and a lower bound, the Laplace-Pólya series is always a lower bound. These bounds will be compared to other approximations to $\Phi(x)$ in the last section.

## 5. Asymptotic optimality

Consider admissilble bounds $f_n(x;a,b)$ with the coefficients $a$ and $b$ satisfying (2.1). The coefficient $b$ is said to be optimal at a fixed $x_0$ if it has the smallest absolute error $|f_n(x_0;a,b) - R(x_0)|$. Since $a/b$ is unimodal with the unique mode $b_n(0)$, by (2.4) we have that every coefficient $b$, $n < b \le b_n(0)$, is optimal at a unique finite $x$, $x \ge 0$, with the exception of $b = n$ when $f_n = L_n$. By (2.4), $f_n\{x; a_n, b_n(0)\}$ has smaller absolute error than $L_n$ if $x < 2[\{2(n^2+n+1)^{1/2} + (n+2)\}/3]^{1/2}$. Suppose that one has a specific $x_0$ in mind. The optimal $b_n$ is the solution to the equation

$$x_0 = \{3n + 2(n-1)b_n - b_n^2\}/\{b_n(b_n+1)(b_n-n)\}^{1/2}$$

in the range of $n < b_n \le b_n(0)$. The expression of this solution is rather complicated. However, it can be expressed asymptotically by

$$b_n(x_0) = 2n - x_0 n^{1/2} + (x_0^2 - 1)/2 + \varepsilon_n \tag{5.1}$$

with $\varepsilon_n = O(n^{-1/2})$. The solution $b_n(x_0)$ is of the form (3.3). Some selected values of $b_n(x_0)$ are provided in Table 1. When $x_0 = 0$, $b_n(0)$ is given in (3.4) and the corresponding $\varepsilon_n$ lies between 0 and $3/(8n+4)$. It is interesting to note that $b_n$ in (3.5) can be expressed by $b_n(2^{1/2}) + \varepsilon_n$. The value $b_n(x_0)$ is monotone decreasing in $x_0$. However, the three-term expression in (5.1) is not. To remedy this deficiency, one may use

$$b_n(x_0) = (3n-1)/2 + \varepsilon_n \quad \text{if} \quad x_0 > n^{1/2} \tag{5.2}$$

or expand $\varepsilon_n$ in (5.1) to several extra terms. In view of (3.2) and the fact that $L_n$ is a very good approximation for large $x$, the use of the three terms in (5.1) is adequate.

Table 1. Optimal values of $b_n(x_0)/n$.

| $n$ | $x_0$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1.732 | 1.376 | 1.215 | 1.135 | 1.090 | 1.064 | 1.047 |
| 2 | 1.823 | 1.465 | 1.283 | 1.184 | 1.126 | 1.091 | 1.068 |
| 3 | 1.869 | 1.526 | 1.335 | 1.224 | 1.157 | 1.115 | 1.087 |
| 4 | 1.896 | 1.571 | 1.377 | 1.259 | 1.185 | 1.137 | 1.105 |
| 5 | 1.914 | 1.605 | 1.412 | 1.289 | 1.210 | 1.157 | 1.121 |
| 10 | 1.954 | 1.703 | 1.524 | 1.396 | 1.305 | 1.239 | 1.190 |
| 20 | 1.976 | 1.783 | 1.631 | 1.512 | 1.418 | 1.344 | 1.286 |
| 50 | 1.990 | 1.860 | 1.748 | 1.652 | 1.569 | 1.498 | 1.437 |
| 100 | 1.995 | 1.901 | 1.815 | 1.739 | 1.670 | 1.608 | 1.552 |

Table 2. Absolute errors of $L_n$ and $f_n$ as approximations to $\Phi(x)$†.

| | degree $n$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0.5 $L_n$ | $\bar{0}.17$ | $\bar{0}.18$ | $\bar{0}.11$ | $\bar{0}.10$ | $\bar{1}.72$ | $\bar{1}.69$ | $\bar{1}.51$ | $\bar{1}.48$ |
| $f_n$ | $\bar{1}.30$ | $\bar{2}.14$ | $\bar{3}.33$ | $\bar{3}.13$ | $\bar{4}.62$ | $\bar{4}.34$ | $\bar{4}.20$ | $\bar{4}.13$ |
| 1.0 $L_n$ | $\bar{1}.38$ | $\bar{1}.23$ | $\bar{1}.13$ | $\bar{2}.89$ | $\bar{2}.58$ | $\bar{2}.40$ | $\bar{2}.28$ | $\bar{2}.20$ |
| $f_n$ | $\bar{1}.38$ | $\bar{3}.52$ | $\bar{4}.82$ | $\bar{4}.22$ | $\bar{5}.78$ | $\bar{5}.33$ | $\bar{5}.16$ | $\bar{6}.81$ |
| 1.5 $L_n$ | $\bar{2}.70$ | $\bar{2}.31$ | $\bar{2}.15$ | $\bar{3}.77$ | $\bar{3}.42$ | $\bar{3}.24$ | $\bar{3}.14$ | $\bar{4}.88$ |
| $f_n$ | $\bar{3}.56$ | $\bar{4}.53$ | $\bar{5}.85$ | $\bar{5}.20$ | $\bar{6}.62$ | $\bar{6}.22$ | $\bar{7}.89$ | $\bar{7}.39$ |
| 2.0 $L_n$ | $\bar{2}.12$ | $\bar{3}.39$ | $\bar{3}.15$ | $\bar{4}.63$ | $\bar{4}.28$ | $\bar{4}.14$ | $\bar{5}.68$ | $\bar{5}.35$ |
| $f_n$ | $\bar{3}.18$ | $\bar{5}.25$ | $\bar{6}.19$ | $\bar{7}.51$ | $\bar{7}.17$ | $\bar{8}.64$ | $\bar{8}.25$ | $\bar{8}.10$ |
| 2.5 $L_n$ | $\bar{3}.17$ | $\bar{4}.44$ | $\bar{4}.13$ | $\bar{5}.47$ | $\bar{5}.18$ | $\bar{6}.70$ | $\bar{6}.30$ | $\bar{6}.13$ |
| $f_n$ | $\bar{3}.20$ | $\bar{5}.72$ | $\bar{6}.50$ | $\bar{7}.54$ | $\bar{8}.79$ | $\bar{8}.15$ | $\bar{9}.35$ | $\overline{10}.95$ |
| 3.0 $L_n$ | $\bar{4}.20$ | $\bar{5}.43$ | $\bar{5}.10$ | $\bar{6}.31$ | $\bar{7}.96$ | $\bar{7}.33$ | $\bar{7}.12$ | $\bar{8}.45$ |
| $f_n$ | $\bar{4}.76$ | $\bar{5}.31$ | $\bar{6}.23$ | $\bar{7}.25$ | $\bar{8}.34$ | $\bar{9}.58$ | $\bar{9}.12$ | $\overline{10}.26$ |
| 3.5 $L_n$ | $\bar{5}.21$ | $\bar{6}.36$ | $\bar{7}.74$ | $\bar{7}.18$ | $\bar{8}.47$ | $\bar{8}.14$ | $\bar{9}.42$ | $\bar{9}.14$ |
| $f_n$ | $\bar{4}.17$ | $\bar{6}.68$ | $\bar{7}.49$ | $\bar{8}.50$ | $\bar{9}.64$ | $\overline{10}.99$ | $\overline{10}.18$ | $\overline{11}.36$ |
| 4.0 $L_n$ | $\bar{6}.18$ | $\bar{7}.25$ | $\bar{8}.44$ | $\bar{9}.88$ | $\bar{9}.20$ | $\overline{10}.49$ | $\overline{10}.13$ | $\overline{11}.38$ |
| $f_n$ | $\bar{5}.28$ | $\bar{6}.10$ | $\bar{8}.66$ | $\bar{9}.61$ | $\overline{10}.72$ | $\overline{10}.10$ | $\overline{11}.16$ | $\overline{12}.30$ |

† $\bar{0}.17$ stands for 0.17 and $\bar{1}.30$ for 0.030.

Let $a_n = 2\{(b_n+1)(b_n-n)/b_n\}^{1/2}$ and $b_n = 2n - xn^{1/2} + (x^2-1)/2$. The normal integral $\Phi(x)$ is approximated by

$$\Phi(x) = 1 - \phi(x)\left\{\frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots\frac{(n-1)}{x+}\frac{b_n}{x+}a_n\right\} + \varepsilon_n. \tag{5.3}$$

Absolute errors of (5.3) and those corresponding to $L_n$ for $x = 0.5, 1.0, 1.5, \ldots, 4.0$ and $n = 1, 2, \ldots, 8$ are provided in Table 2. The notation $\bar{a}.bc$, used by Ruben (1964), is to be interpreted as $0.bc \times 10^{-a}$, e.g. $\bar{0}.17 = 0.17 \times 10^{-0} = 0.17$ and $\bar{1}.30 = 0.30 \times 10^{-1} = 0.030$. The values of these two approximations are upper bounds of $\Phi(x)$ if $n$ is odd and lower bound if $n$ is even. The approximation (5.3) is superior to its predecessor $L_n$. If (5.2) is also used, then absolute errors of $f_n$ will remain the same if $x \leq n^{1/2}$ and they are identical to those of $L_n$ if $n = 1$ and $x \geq 1$. However, it is superior for almost all other combination of $x$ and $n$ in Table 2. For instance, its asbolute errors at $n = 2$ and $x \geq 1.5$ are $\bar{4}.54$, $\bar{5}.16$, $\bar{6}.14$, $\bar{7}.86$, $\bar{7}.21$, $\bar{8}.31$; at $n = 8$ and $x \geq 3.0$ are $\overline{10}.25$, $\overline{11}.25$, $\overline{12}.15$. While $L_n^r$ is a good bound of the Mills' ratio, it is not as efficient as $f_n$ in approximation $\Phi(x)$. Its absolute errors when $n = 8$ are $\bar{2}.27$, $\bar{3}.23$, $\bar{4}.15$, $\bar{6}.86$, $\bar{7}.42$, $\bar{8}.18$, $\overline{10}.68$ and $\overline{11}.22$ respectively at $x = 0.5, 1.0, \ldots, 4.0$.

## 6. Approximations to $\Phi(x)$

There is a vast literature in approximating the normal integral. For example, the result in Hastings ((1955), p. 185)

$$\bar{\Phi}(x) = \frac{1}{2}\left(\sum_{t=0}^{4} c_t x^t\right)^{-4} + \varepsilon$$

not only has a fixed maximum absolute error 0.00025 but also requires a table of six significant digit coefficients. The rational approximation in Gray and Schucany (1968) is inferior to $f_7$ of the same degree in (5.3) except in a very small neighbourhood where the error changes sign. Convergent approximations having simple expressions are $L_n$, $f_n$ in (5.3), Shenton's (1954) continued fraction, the Laplace-Pólya series in (4.2), the Laplace series in (4.1) and Kerridge and Cook's (1976) expansion. Shenton (1954) proposed a continued fraction

$$\Phi(x) = \Phi(0) + \phi(x)\left\{\frac{x}{1-}\frac{x^2}{3+}\frac{2x^2}{5-}\frac{3x^2}{7+}\frac{4x^2}{9-}\frac{5x^2}{11+}\cdots\frac{nx^2}{2n+1}\right\} + \varepsilon_n$$

and Kerridge and Cook (1976) derived an expansion

$$\Phi(x) = \Phi(0) + x\phi(x/2)\sum_{t=0}^{n}\frac{\theta_{2t}(x/2)}{2t+1} + \varepsilon_n$$

where $\theta_0(x) = 1$, $\theta_1(x) = x^2$ and $\theta_n(x) = x^2\{\theta_{n-1}(x) - \theta_{n-2}(x)\}/n$ for $n \geq 2$. While $L_n$, $f_n$, the Laplace-Pólya series and the Laplace series are bounds of the normal integral, the Kerridge and Cook expansion is not.

Table 3. Degree $n$ or number of terms required for accuracy $10^{-d}$.

| | | degree $n$ | | terms | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $d$ | $L_n$ | $f_n$ | Shenton | Laplace -Pólya | Laplace | Kerridge and Cook |
| 0.5 | 4 | 84 | 5 | 3 | 3 | 3 | 2 |
| | 5 | 132 | 9 | 4 | 4 | 3 | 3 |
| | 6 | 190 | 17 | 4 | 4 | 4 | 3 |
| | 7 | 259 | 28 | 5 | 5 | 5 | 7 |
| 1.0 | 4 | 20 | 3 | 5 | 5 | 5 | 3 |
| | 5 | 32 | 5 | 5 | 6 | 5 | 3 |
| | 6 | 47 | 8 | 6 | 7 | 6 | 4 |
| | 7 | 64 | 12 | 7 | 8 | 7 | 4 |
| 1.5 | 4 | 8 | 2 | 6 | 7 | 7 | 2 |
| | 5 | 14 | 3 | 7 | 8 | 8 | 4 |
| | 6 | 20 | 5 | 8 | 9 | 9 | 5 |
| | 7 | 28 | 7 | 9 | 10 | 10 | 6 |
| 2.0 | 4 | 4 | 2 | 8 | 9 | 9 | 5 |
| | 5 | 7 | 2 | 9 | 10 | 10 | 6 |
| | 6 | 11 | 3 | 10 | 12 | 12 | 6 |
| | 7 | 15 | 4 | 11* | 13 | 13 | 7 |
| 2.5 | 4 | 2 | 2 | 10 | 12 | 12 | 5 |
| | 5 | 4 | 2 | 11 | 13 | 14 | 6 |
| | 6 | 6 | 3 | 12 | 15 | 15 | 7 |
| | 7 | 9 | 4 | 13* | 16 | 17 | 7 |
| 3.0 | 4 | 1 | 1 | 13 | 14 | 16 | 5 |
| | 5 | 2 | 2 | 14 | 16 | 17 | 8 |
| | 6 | 4 | 3 | 15* | 18 | 19 | 9 |
| | 7 | 5 | 4 | 16* | 19 | 21 | 10 |
| 3.5 | 4 | 1 | 1 | 15 | 17 | 20 | 8 |
| | 5 | 1 | 2 | 17 | 19 | 22 | 9 |
| | 6 | 2 | 2 | 18 | 21 | 24 | 10 |
| | 7 | 3 | 3 | 19 | 23 | 25 | 10 |
| 4.0 | 4 | 1 | 1 | 18 | 20 | 25 | 8 |
| | 5 | 1 | 1 | 20* | 22 | 27 | 10 |
| | 6 | 1 | 2 | 21* | 25 | 29* | 12 |
| | 7 | 2 | 2 | 22* | 26 | 30* | 13 |

*Specified accuracy not attainable in single precision programming.

For desk computation and single precision programming, we are interested in absolute error no more than $10^{-4}$, $10^{-5}$, $10^{-6}$ and $10^{-7}$. Degree $n$ or number of terms required to achieve the desired accuracy for the six convergent approximations in the order of simplicity in computation are provided in Table 3 for $x = 0.5, 1.0, \ldots, 4.0$. It is found that $f_n(x)\phi(x)$ is the best approximation for the normal tail probability $\bar{\Phi}(x)$ in the range $x \geq 1.5$. For $x \leq 1$, the last four approximations in Table 3 are very efficient. It is remarkable to observe that the simple expression (5.3) has a maximum absolute error no more than $10^{-4}$ for the entire range of $x \geq 0$ in $n = 12$.

The last four approximations in Table 3 are also computed at $x = 0.5, 1.0, \ldots,$ 4.0 and $n = 1, 2, \ldots, 8$. Shenton's continued fraction is better than $f_n$ in Table 2 if $x = 0.5$ or $x = 1.0$ and $n \geq 5$. The same result holds true for both the Laplace-Pólya series and the Laplace series except $n \geq 6$ at $x = 1.0$. These three approximations are not recommended for $x \geq 2$ because of large errors. For example, at $x = 4$ and $n = 8$, the Shenton continued fraction has as absolute error 0.49, the Laplace-Pólya series 0.24 and the Laplace series 21.33. The Kerridge and Cook expansion is better than $f_n$ in Table 2 if $x = 0.5$, $x = 1.0$ or $x = 1.5$ and $n \geq 5$. However, it is not so easy to compute. For the remaining $x$ and $n$ in Table 2, $f_n$ with $b_n$ in (5.1) and (5.2) has the smallest absolute error. The continued fraction $f_n$ in (5.3) is recommended for desk computation in the range of $x \geq 1$.

Table 4. Degree $n$ or number of terms required for accuracy $10^{-d}$.

| | | degree $n$ | | terms | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $d$ | $L_n$ | $f_n$ | Shenton | Laplace -Pólya | Laplace | Kerridge and Cook |
| 1 | 10 | 131 | 35 | 9 | 10 | 10 | 6 |
| | 15 | 297 | 110 | 12 | 14 | 13 | 9 |
| 2 | 10 | 32 | 11 | 14 | 16 | 16 | 9 |
| | 15 | 73 | 34 | 18 | 21 | 21 | 13 |
| 3 | 10 | 13 | 8 | 19 | 24 | 25 | 13 |
| | 15 | 31 | 18 | 25 | 30 | 31 | 17 |
| 4 | 10 | 6 | 5 | 26 | 32 | 35 | 15 |
| | 15 | 16 | 12 | * | 39 | * | 21 |
| 5 | 10 | 2 | 3 | 34 | 41 | 48 | 20 |
| | 15 | 9 | 8 | * | 50 | * | 24 |
| 6 | 10 | 1 | 2 | * | 51 | 63 | 23 |
| | 15 | 4 | 5 | * | 61 | * | 30 |

*Specified accuracy not attainable in double precision programming.

The six approximations are also compared at accuracy of $10^{-10}$ and $10^{-15}$ when double precision programming is used. The results for $x = 1, 2, \ldots, 6$ are provided in Table 4. It is found that $f_n$ is the best if $x \geq 3$. If (5.2) is also used, the eight degrees for $x = 3$, 4, 5 and 6 are 7, 18, 4, 12, 2, 8, 1, and 4 respectively, and they are 6, 16, 4, 10, 2, 7, 1, and 3 respectively if $\varepsilon_n$ in (5.1) is approximated by $x_0(6 - x_0^2)/8n^{1/2} + (x_0^3 - 4x_0^2 + 2x_0 - 4)/8n$. For $x \leq 1$, the last four approximations in Table 4 are very efficient. Double precision computation carries 16 significant digits during the course of execution. If one subtracts two numbers with the matching leading digits then the accuracy of the difference will suffer by the same number of digits. That is why both the Shenton continued fraction and the Laplace series fail to attain the specified accuracies in Table 4. On the other hand, the Laplace continued fraction $L_n$ is tested at $x = 0.001$. It is found that its absolute error will eventually be reduced to less than $10^{-15}$ once $n \geq 2.934 \times 10^8$. The accuracy of $f_n$ and Laplace continued fraction $L_n$ are superior because the effects of rounding errors are not cumulative. While the values of $f_n$ in (5.3) and the Laplace-Pólya series lie between 0 and 1, the values of Shenton's (1954) continued fraction, the Laplace series and Kerridge and Cook's (1976) expansion may be less than 0 or greater than 1.

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