On large deviations of sums of independent random variables

Zhishui Hu[12], Valentin V. Petrov[23] and John Robinson[2*]

[1]Department of Statistics and Finance,
University of Science and Technology of China,
Hefei, Anhui 230026, China,
[2]School of Mathematics and Statistics, University of Sydney,
NSW 2006, Australia,
[3]Faculty of Mathematics and Mechanics, St. Petersburg University,
Stary Peterhof, St. Petersburg 198504, Russia

Extensions of some limit theorems are proved for tail probabilities of sums of independent identically distributed random variables satisfying the one-sided or two-sided Cramér's condition. The large deviation x-region under consideration is broader than in the classical Cramér's theorem, and the estimate of the remainder is uniform with respect to x. The corresponding asymptotic expansion with arbitrarily many summands is also obtained.

Running head On large deviations.

Keywords Limit theorems; sums of independent random variables; large deviation; Cramér's condition; asymptotic expansions.

Mathematics Subject Classification Primary 60F10; Secondary 60G50, 62E20.

1 Introduction and results

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with a common distribution $V(x)$, mean 0 and variance 1. Assume that the following

*Address correspondence to John Robinson, School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia; E-mail: johnr@maths.usyd.edu.au

one-sided Cramér's condition is satisfied:

$$Ee^{hX_1} < \infty \qquad \text{for } 0 \le h < H \text{ and some } H > 0. \tag{1.1}$$

Introduce a sequence of conjugate i.i.d. random variables $X_{1h}, X_{2h}, \cdots$ with the d.f.

$$V_h(x) = \frac{1}{R(h)} \int_{-\infty}^{x} e^{hy} dV(y),$$

where $0 \le h < H$ and

$$R(h) = \int_{-\infty}^{\infty} e^{hy} dV(y).$$

The distribution $V_h(x)$ has the same set of the growth points as $V(x)$. Put

$$EX_{1h} = m(h), \qquad \mathrm{Var} X_{1h} = \sigma^2(h), \tag{1.2}$$

where $0 \le h < H$. We have

$$m(h) = \int_{-\infty}^{\infty} x dV_h(x) = \frac{1}{R(h)} \int_{-\infty}^{\infty} x e^{hx} dV(x) = \frac{R'(h)}{R(h)} = \frac{d \log R(h)}{dh},$$
$$\sigma^2(h) = E(X_{1h}^2) - (EX_{1h})^2 = \frac{R''(h)}{R(h)} - \Big(\frac{R'(h)}{R(h)}\Big)^2 = \frac{dm(h)}{dh}.$$

Denote by $F_n(x)$ the d.f. of $\sum_{k=1}^n X_k/\sqrt{n}$. Then we have the following theorem.

Theorem 1.1 *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean 0, variance 1 and*

$$EX_1^4 < \infty. \tag{1.3}$$

Suppose that conditions (1.1) and

$$\limsup_{|t| \to \infty} |Ee^{itX_1}| < 1 \tag{1.4}$$

are satisfied and let $0 < H' < H$. Then

$$1 - F_n(x) = \frac{1}{\sqrt{2\pi}} \exp\{n \log R(h) - nhm(h)\} M(h\sigma(h)\sqrt{n})[1 + O(1/\sqrt{n})] \tag{1.5}$$

holds uniformly in x in the area $0 < x \le m(H')\sqrt{n}$. Here

$$M(y) = e^{y^2/2} \int_y^\infty e^{-t^2/2} dt$$

is the Mills ratio and h is the unique positive root of the equation $m(h) = x/\sqrt{n}$.

Let us introduce a corollary to Theorem 1.1 that corresponds to the two-sided Cramér's condition:

$$Ee^{hX_1} < \infty \qquad \text{for } -H < h < H \text{ and some } H > 0. \tag{1.6}$$

Corollary 1.1 *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables satisfying conditions (1.4) and (1.6), and let $EX_1 = 0$, $EX_1^2 = 1$. If $0 < H' < H$, then relation (1.5) holds uniformly in x in the area $0 < x \le m(H')\sqrt{n}$, where $m(h)$ is defined in (1.2), and h is the unique positive root of the equation $m(h) = x/\sqrt{n}$.*

This result differs from the classical Cramér's theorem on large deviations (see, for example, Theorem 5.23 in Petrov(1995)), in particular, by a stronger uniform estimate of the remainder; the remainder term in Cramér's theorem is $O(x/\sqrt{n})$, and the area is $1 < x = o(\sqrt{n})(n \to \infty)$ instead of the broader one $0 < x \le m(H')\sqrt{n}$.

Under the one-sided Cramér's condition (1.1) a series of limit theorems for probabilities of large deviations of sums of independent identically distributed random variables was proved for special x-regions in Petrov(1965), where references on earlier works are given.

Contrary to condition (1.1), the two-sided Cramér's condition (1.6) implies the existence of moments of X_1 of all orders. The one-sided condition (1.1) imposes restrictions on the behavior of the distribution of the random variable X_1 only on the positive half-line.

Our next theorem is an improvement of Theorem 5 in Petrov(1965). In this theorem, we replace the condition (1.4) by the condition that X_1 is nonlattice.

Theorem 1.2 *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. nonlattice random variables with mean 0, variance 1 and $E|X_1|^3 < \infty$. Suppose that condition (1.1) is satisfied and let $0 < H' < H$. Then*

$$\begin{aligned} 1 - F_n(x) \\ &= \frac{1}{\sqrt{2\pi}} \exp\{n \log R(h) - nhm(h)\} M(h\sigma(h)\sqrt{n})[1 + o(1)] \end{aligned} \tag{1.7}$$

holds uniformly in $0 < x \le m(H')\sqrt{n}$*, where* $M(\cdot)$ *is defined in Theorem 1.1 and* h *is the unique positive root of the equation* $m(h) = x/\sqrt{n}$*.*

From Theorem 1.2, we can get the following corollary directly. This corollary is also shown in Theorem A of Höglund (1979).

Corollary 1.2 *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. nonlattice random variables satisfying condition (1.6), and let* $EX_1 = 0,\ EX_1^2 = 1$*. If* $0 < H' < H$*, then relation (1.7) holds uniformly in* x *in the area* $0 < x \le m(H')\sqrt{n}$*, where* h *is the unique positive root of the equation* $m(h) = x/\sqrt{n}$*.*

Remark 1. Since $EX_1 = 0,\ EX_1^2 = 1$, we have $m(0) = 0$ and $\sigma^2(0) = 1$. Under the condition (1.4), X_1 is nonlattice and so is X_{1h}. Thus $\sigma^2(h) = \mathrm{Var}(X_{1h}) > 0$. This together with the fact $\sigma^2(h) = m'(h)$ implies $m(h)$ is strictly increasing. So $m(h) > 0$ for all $0 < h < H$ and the root of the equation $m(h) = x/\sqrt{n}$ is unique for $0 < x \le m(H')\sqrt{n}$. More information about the behavior of $m(h)$ can be found in Petrov(1965).

Let $\gamma_\nu(h)$ be the cumulant of order ν of the random variable $X_{1h}^* = (X_{1h} - m(h))/\sigma(h)$, i.e.

$$\gamma_\nu(h) = \frac{1}{i^\nu}\frac{d^\nu}{dt^\nu} \ln E e^{itX_{1h}^*}\Big|_{t=0}.$$

In particular, $\gamma_1(h) = 0,\ \gamma_2(h) = 1,\ \gamma_3(h) = E(X_{1h}^*)^3$, if $E|X_1|^3 < \infty$ and condition (1.1) is satisfied. Define the so-called Esscher functions: for $\lambda > 0$, put

$$B_k(\lambda) = \frac{\lambda}{\sqrt{2\pi}} \int_0^\infty \exp(-\lambda x - x^2/2) H_k(x)\, dx, \qquad k = 0, 1, \cdots,$$

where $H_k(x)$ is the Chebyshev-Hermite polynomial of degree k. In particular,

$$\begin{aligned} B_0(\lambda) &= (2\pi)^{-1/2}\lambda M(\lambda), \\ B_1(\lambda) &= -\lambda(B_0(\lambda) - (2\pi)^{-1/2}), \\ B_2(\lambda) &= \lambda^2(B_0(\lambda) - (2\pi)^{-1/2}). \end{aligned}$$

For more about the Esscher functions, see, for instance, Section 2.1 of Jensen(1995). We have the following extension of Theorem 1.1.

Theorem 1.3 *Let $0 < H' < H$ and let $k \geq 3$ be an integer. Suppose that $E|X_1|^{k+1} < \infty$. Then under the conditions of Theorem 1.1,*

$$1 - F_n(x) = \exp\{n \log R(h) - nhm(h)\}\Big\{\frac{B_0(\sqrt{n}h\sigma(h))}{\sqrt{n}h\sigma(h)} + \sum_{\nu=1}^{k-2}\sum n^{-\nu/2}\prod_{m=1}^{\nu}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}(h)}{(m+2)!}\right)^{k_m}\frac{B_{\nu+2s}(\sqrt{n}h\sigma(h))}{\sqrt{n}h\sigma(h)} + O(n^{-(k-1)/2})\Big\} \quad (1.8)$$

holds uniformly in x in the area $0 < x \leq m(H')\sqrt{n}$, where h is the unique positive root of the equation $m(h) = x/\sqrt{n}$, the inner summation is extended over all non-negative integer solutions $(k_1, k_2, \cdots, k_\nu)$ of the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$ and $s = k_1 + k_2 + \cdots + k_\nu$.

Relation (1.8) is an extension of (2.2.6) in Jensen(1995), where the density function of X_1 is required.

This paper is organized as follows. A smoothness condition on the conjugate variables is studied in Section 2. Proofs of theorems and corollaries in Section 1 are offered in Section 3. Throughout the paper we shall use $A, A_1, A_2, ...$ to denote absolute positive constants whose values may differ at each occurrence.

2 Smoothness for conjugate random variables

Results on large deviations frequently use a smoothness condition on the conjugate variables. We will show that it is enough to assume the usual C-condition of Cramér.

Theorem 2.1 *Let X be a random variable with distribution function $V(x)$ such that for $0 \leq h < H$,*

$$R(h) = Ee^{hX} = \int_{-\infty}^{\infty} e^{hx} dV(x) < \infty. \quad (2.1)$$

For any $0 < c < C \leq \infty$, if

$$\sup_{c \leq |t| < C} |Ee^{itX}| < 1, \quad (2.2)$$

then, for any positive $H' < H$,

$$\sup_{0 \leq h \leq H'} \sup_{c \leq |t| < C} |Ee^{(h+it)X}/Ee^{hX}| < 1. \quad (2.3)$$

Theorem 2.1 can be obtained by using a method similar to that in the proof of Lemma 7.3 in Zhou and Jing (2006). Here we give another proof which discloses the inner relationship between X and the corresponding conjugate random variable. The proof will depend on the following lemma which is closely modelled on a result of Weber and Kokic (1997).

Lemma 2.1 *Let X be a random variable with distribution function $V(x)$. For any $0 < c < C \le \infty$, the following are equivalent:*

(i) there exist $\varepsilon > 0$ and $\delta > 0$ such that for all real y and all t satisfying $c \le |t| < C$, the set

$$A(y,t,\varepsilon) = \{x : |xt - y - 2k\pi| > \varepsilon, \text{ for all integers } k\}$$

has $P(X \in A(y,t,\varepsilon)) > \delta$;

(ii) $\sup_{c\le|t|<C} |Ee^{itX}| < 1$.

Proof. We first show that (i) implies (ii). Without loss of generality take $\varepsilon < \pi/2$ and assume that (i) holds. Given any t with $c \le |t| < C$, choose, for any x, integers $k(t,x)$ such that $z(t,x) = xt - 2\pi k(t,x) \in [0, 2\pi)$. Then for all $y \in [0, 2\pi)$ and all t from the area $c \le |t| < C$,

$$P(d(z(t,X) - y) > \varepsilon) > \delta,$$

where $d(z) = \min\{|z|, |z + 2\pi|, |z - 2\pi|\}$. So

$$\begin{aligned} Ee^{itX} &= \int_{-\infty}^{\infty} e^{i(z(t,x)-y)} dV(x) \\ &= \int_{-\infty}^{\infty} \cos(z(t,x) - y) dV(x) + i \int_{-\infty}^{\infty} \sin(z(t,x) - y) dV(x). \end{aligned} \tag{2.4}$$

Choose $y = y(t) \in [0, 2\pi)$ such that

$$\frac{\sin(y(t))}{\cos(y(t))} = \frac{E \sin z(t,X)}{E \cos z(t,X)},$$

then $E\sin(z(t,X) - y(t)) = 0$. Now for all t with $c \le |t| < C$,

$$\begin{aligned} &E\cos(z(t,X) - y(t)) \\ &\le P(d(z(t,X) - y(t)) \le \varepsilon) + P(d(z(t,X) - y(t)) > \varepsilon)\cos\varepsilon \\ &= 1 - (1 - \cos\varepsilon) P(d(z(t,X) - y(t)) > \varepsilon) \\ &\le 1 - \delta(1 - \cos\varepsilon). \end{aligned} \tag{2.5}$$

Next, we choose $\tilde{y}(t) \in [0, 2\pi)$ such that $|\tilde{y}(t) - y(t)| = \pi$. Similarly to (2.5), we have

$$-E\cos(z(t,X) - y(t)) = E\cos(z(t,X) - \tilde{y}(t)) \leq 1 - \delta(1 - \cos\varepsilon). \quad (2.6)$$

Hence by (2.4), (2.5) and (2.6),

$$\sup_{c \leq |t| < C} |Ee^{itX}| \leq 1 - \delta(1 - \cos\varepsilon)$$

and (ii) follows.

Now we will show (ii) implies (i) by showing that complement of (i) implies that for any $\eta > 0$,

$$\sup_{c \leq |t| < C} |Ee^{itX}| > 1 - \eta.$$

The complement of (i) may be stated as follows: for all $\varepsilon > 0$ and $\delta > 0$, there exist $y = y(\varepsilon, \delta)$ and $t = t(\varepsilon, \delta)$ with $c \leq |t| < C$ such that the set

$$A^c(y, t, \varepsilon) = \{x : |xt - y - 2k\pi| \leq \varepsilon \text{ for some integer } k\}$$

has

$$P(X \in A^c(y, t, \varepsilon)) \geq 1 - \delta. \quad (2.7)$$

Write

$$\begin{aligned} Ee^{itX} &= e^{iy}\int_{A^c(y,t,\varepsilon)} dV(x) + \int_{A^c(y,t,\varepsilon)} (e^{itx} - e^{i(y+2\pi k(t,x))})\, dV(x) \\ &\quad + \int_{A(y,t,\varepsilon)} e^{itx} dV(x) \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

Then

$$|Ee^{itX}| \geq |J_1| - |J_2| - |J_3|. \quad (2.8)$$

Using (2.7) and the inequality $|e^{iz} - 1| \leq |z|$, we have

$$|J_1| \geq 1 - \delta, \qquad |J_3| \leq \delta, \quad (2.9)$$

and

$$|J_2| \leq \Big|\int_{A^c(y,t,\varepsilon)} (e^{i(tx - y - 2\pi k(t,x))} - 1)\, dV(x)\Big| \leq \varepsilon. \quad (2.10)$$

So it follows from (2.8)-(2.10) that

$$|Ee^{itX}| \geq 1 - 2\delta - \varepsilon > 1 - \eta,$$

by choosing $\delta > 0$ and $\varepsilon > 0$ such that $2\delta + \varepsilon < \eta$.

The proof of Lemma 2.1 is complete. $\square$

Proof of Theorem 2.1. We prove Theorem 2.1 by showing that the complement of (2.3) implies that

$$\sup_{c \leq |t| < C} |Ee^{itX}| = 1. \tag{2.11}$$

The complement of (2.3) is: there exists H' with $0 < H' < H$ such that

$$\sup_{0 \leq h \leq H'} \sup_{c \leq |t| < C} |Ee^{(h+it)X}/Ee^{hX}| = 1.$$

So there exists h_0 with $0 \leq h_0 \leq H'$ such that

$$\sup_{c \leq |t| < C} |Ee^{(h_0+it)X}/Ee^{h_0X}| = 1. \tag{2.12}$$

Let

$$V_{h_0}(x) = \frac{1}{R(h_0)} \int_{-\infty}^{x} e^{h_0 u} dV(u)$$

and let X_{h_0} be the conjugate r.v. with distribution function $V_{h_0}(x)$. Then it follows from (2.12) and Lemma 2.1 that for all $\varepsilon > 0$ and $\delta > 0$, there exist y and t with $c \leq |t| < C$ such that

$$P(X_{h_0} \in A(y, t, \varepsilon)) \leq \delta,$$

where $A(y, t, \varepsilon)$ is defined in Lemma 2.1. For any $\delta' > 0$, choose $B > 0$ such that $P(|X| \geq B) < \delta'/2$. Then

$$\begin{aligned} P(X \in A(y,t,\varepsilon)) &\leq P(X \in A(y,t,\varepsilon) \cap (-B,B)) + \delta'/2 \\ &= R(h_0) \int_{A(y,t,\varepsilon)\cap(-B,B)} e^{-h_0 x} dV_{h_0}(x) + \delta'/2 \\ &\leq R(h_0) e^{|h_0|B} P(X_{h_0} \in A(y,t,\varepsilon)) + \delta'/2 \\ &\leq R(h_0) e^{|h_0|B} \delta + \delta'/2 < \delta', \end{aligned}$$

by choice of $\delta < \delta' e^{-|h_0|B}/(2R(h_0))$. So we have shown that the complement of (i) holds for X: for all $\varepsilon > 0$ and $\delta' > 0$, there exist y and t with $c \leq |t| < C$ such that

$$P(X \in A(y,t,\varepsilon)) < \delta'.$$

This, from Lemma 2.1, implies (2.11).

The proof of Theorem 2.1 is complete. □

3 Proof of results

Proof of Theorem 1.1. From Remark 1, we know that $\sigma^2(h) > 0$ for $0 \leq h < H$. Since $\sigma^2(h)$ is continuous on $[0, H)$ and $\sigma^2(0) = 1$, there exist positive constants c_1 and c_2, not depending on h, such that

$$c_1 \leq \sigma^2(h) \leq c_2 \qquad \text{for } 0 \leq h \leq H'. \tag{3.1}$$

Also from Remark 1, the equation

$$m(h) = x/\sqrt{n} \tag{3.2}$$

has a unique positive root $h \in (0, H']$ for any positive $x \leq m(H')\sqrt{n}$.

Conditions (1.1) and (1.3) imply that

$$\sup_{0 \leq h \leq H'} E|X_{1h}|^p < \infty, \qquad 0 < p \leq 4. \tag{3.3}$$

Indeed, for $0 < p \leq 4$,

$$\begin{aligned}\sup_{0 \leq h \leq H'} E|X_{1h}|^p &= \sup_{0 \leq h \leq H'} \int_{-\infty}^{\infty} |x|^p e^{hx} dV(x) \\ &\leq \int_{-\infty}^{0} |x|^p dV(x) + \sup_{0 \leq h \leq H'} \left\{ \int_0^{c_3} + \int_{c_3}^{\infty} \right\} x^p e^{hx} dV(x) \\ &\leq E|X_1|^p + c_3^p e^{c_3 H'} + \int_{c_3}^{\infty} e^{\delta x} dV(x) < \infty\end{aligned}$$

if $H' < \delta < H$ and c_3 is a positive constant, not depending on h, such that $x^p e^{H'x} \leq e^{\delta x}$ for $x \geq c_3$.

Define by $F_{nh}(x)$ the d.f. of $\sum_{k=1}^n X_{kh}^*/\sqrt{n}$, where $X_{kn}^* = (X_{kh} - m(h))/\sigma(h)$. By the same argument as that in the proof of Theorem 5.23 in Petrov(1995), we have

$$1 - F_n(x) = e^{\Lambda_n(h)} \int_0^{\infty} e^{-\sqrt{n}h\sigma(h)y} dF_{nh}(y), \tag{3.4}$$

where $\Lambda_n(h) = n \log R(h) - nhm(h)$ and h is the unique positive root of the equation (3.2). By a theorem of Osipov (see, for example, Theorem 1 of Chapter 6 in Petrov (1975) or Theorem 5.18 in Petrov (1995)), we have

$$F_{nh}(y) = \Phi(y) + \frac{P_1(y)}{\sqrt{n}} + Q_n(y) \tag{3.5}$$

where

$$\begin{aligned} P_1(y) &= (1/(6\sqrt{2\pi}))(1-y^2)e^{-y^2/2}E(X_{1h}^*)^3, \\ |Q_n(y)| &\le A\Big\{n^{-1/2}(1+|y|)^{-3}E|X_{1h}^*|^3 I(|X_{1h}^*| \ge \sqrt{n}(1+|y|)) \\ &\quad + n^{-1}(1+|y|)^{-4}E|X_{1h}^*|^4 I(|X_{1h}^*| < \sqrt{n}(1+|y|)) \\ &\quad + n^6(1+|y|)^{-4}\Big(\sup_{|t|\ge T(h)} |Ee^{itX_{1h}^*}| + 1/(2n)\Big)^n\Big\} \end{aligned} \tag{3.6}$$

for all y and n, where $T(h) = 1/(12E|X_{1h}^*|^3)$.

By (3.1) and (3.3), we have $1/(12\,\sigma(h)E|X_{1h}^*|^3) \ge c$ for $0 < h \le H'$, where c is a positive constant not depending on h. Thus in the last summand on the right side of (3.6) we can replace X_{1h}^* by X_{1h} and the area $|t| \ge T(h)$ by the area $|t| \ge c$.

It follows from (3.4) and (3.6) that

$$1 - F_n(x) = e^{\Lambda_n(h)}\Big(I_1 + I_2 + I_3\Big), \tag{3.7}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\sqrt{n}h\sigma(h)y - y^2/2}\,dy, \\ I_2 &= \frac{1}{\sqrt{n}}\int_0^\infty e^{-\sqrt{n}h\sigma(h)y}\,dP_1(y), \\ I_3 &= \int_0^\infty e^{-\sqrt{n}h\sigma(h)y}\,dQ_n(y). \end{aligned}$$

Since for $\sqrt{n}h\sigma(h) \ge 1$,

$$h\sigma(h)\sqrt{n}I_1 = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y-y^2/(2nh^2\sigma^2(h))}dy \ge \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y-y^2/2}dy > 0,$$

and for $0 < \sqrt{n}h\sigma(h) < 1$,

$$I_1 \ge \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y-y^2/2}dy > 0,$$

we have

$$I_1 = \frac{1}{\sqrt{2\pi}} M(h\sigma(h)\sqrt{n}) \geq \frac{A_1}{1+\sqrt{n}h\sigma(h)}. \tag{3.8}$$

Note that

$$\begin{aligned} 6\sqrt{2\pi n} I_2 &= E(X_{1h}^*)^3 \int_0^\infty (y^3 - 3y) e^{-\sqrt{n}h\sigma(h)y - y^2/2} dy \\ &= E(X_{1h}^*)^3 B_3(\sqrt{n}h\sigma(h))/(\sqrt{n}h\sigma(h)), \end{aligned}$$

where

$$B_3(\lambda) = (\lambda/\sqrt{2\pi}) \int_0^\infty (y^3 - 3y) e^{-\lambda y - y^2/2} dy.$$

Therefore

$$\big|B_3(\lambda)/\lambda\big| \leq \int_0^\infty (y^3 + 3y) e^{-y^2/2} dy < \infty, \qquad \text{for } \lambda > 0.$$

Note that $B_3(\lambda) = O(\lambda^{-1})$ as $\lambda \to +\infty$ (see, for instance, Lemma 2.1.2 of Jensen(1995)). Thus $|B_3(\lambda)/\lambda| \leq A_2/(1+\lambda)$ holds uniformly for $\lambda > 0$. Then it follows from (3.3) that

$$|I_2| = O\left(\frac{1}{\sqrt{n}(1+\sqrt{n}h\sigma(h))}\right) \tag{3.9}$$

uniformly in x in the area $0 < h \leq H'$.

It remains to estimate I_3. By (3.6) and the argument below (3.6), we have

$$I_3 = -Q_n(0) - \int_0^\infty Q_n(y) de^{-\sqrt{n}h\sigma(h)y}, \tag{3.10}$$

with

$$|Q_n(y)| \leq A\Big\{E|X_{1h}^*|^4 n^{-1} + n^6 \Big(\sup_{|t|\geq c} |Ee^{itX_{1h}}| + 1/(2n)\Big)^n\Big\}, \tag{3.11}$$

for all y, n and $0 < h \leq H'$.

Condition (1.4) implies that $\sup_{|t|\geq c} |Ee^{itX_1}| < 1$ (see, for instance, p14 in Petrov (1995)). Thus by Theorem 2.1, we have

$$\sup_{0<h\leq H'} \sup_{|t|\geq c} |Ee^{itX_{1h}}| < 1. \tag{3.12}$$

This, together with (3.3), (3.10) and (3.11), implies that

$$I_3 = O(1/n) \tag{3.13}$$

uniformly in x in the area $0 < h \le H'$.

It follows from (3.7) that

$$1 - F_n(x) = e^{\Lambda_n(h)} I_1 \Big(1 + \frac{1}{I_1}(I_2 + I_3)\Big).$$

By (3.1), (3.8)-(3.9) and (3.13), we have

$$I_2/I_1 = O(n^{-1/2}), \qquad I_3/I_1 = O(hn^{-1/2})$$

uniformly in x in the area $0 < x \le m(H')\sqrt{n}$. Since for any positive $x \le m(H')\sqrt{n}$, the equation $m(h) = x/\sqrt{n}$ has the unique positive root $h \le H'$, we have $I_3/I_1 = O(n^{-1/2})$. Therefore

$$\begin{aligned} &1 - F_n(x) \\ &= \frac{1}{\sqrt{2\pi}} \exp\{n \log R(h) - nhm(h)\} M(h\sigma(h)\sqrt{n})[1 + O(1/\sqrt{n})] \end{aligned}$$

uniformly in x in the area $0 < x \le m(H')\sqrt{n}$. □

Proof of Corollary 1.1. This follows immediately from Theorem 1.1. □

Proof of Theorem 1.2. Similarly to the proof of (3.3), we obtain

$$\sup_{0 \le h \le H'} E|X_{1h}|^p < \infty, \qquad 0 < p \le 3. \tag{3.14}$$

Following the mainstream of the proof of Theorem 1.1, we only need to show that

$$F_{nh}(y) - \Phi(y) - \frac{P_1(y)}{\sqrt{n}} = o\Big(\frac{1}{\sqrt{n}}\Big)$$

holds uniformly for all y and $0 < h \le H'$, where $P_1(y)$ is defined below (3.5). This follows from the proof of Theorem 1 and 2 of §42 in Gnedenko and Kolmogorov (1968) with some modification, by using (3.1), (3.14) and Theorem 2.1. □

Proof of Corollary 1.2. This follows immediately from Theorem 1.2. □

Proof of Theorem 1.3. Let $H_m(y)$ be the Chebyshev-Hermite polynomial of degree m defined by the equality

$$H_m(y) = (-1)^m e^{y^2/2} \frac{d^m}{dy^m} e^{-y^2/2},$$

and let

$$P_\nu(y) = -\frac{1}{\sqrt{2\pi}} e^{-y^2/2} \sum H_{\nu+2s-1}(y) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}(h)}{(m+2)!} \right)^{k_m} \tag{3.15}$$

where the summation is extended over all non-negative integer solutions $(k_1, k_2, \cdots, k_\nu)$ of the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$ and $s = k_1 + k_2 + \cdots + k_\nu$.

Similarly to the proof of (3.3),

$$\sup_{0 \le h \le H'} E|X_{1h}|^p < \infty, \qquad 0 < p \le k+1.$$

Using a theorem of Osipov (see, for example, Theorem 1 of Chapter 6 in Petrov (1975) or Theorem 5.18 in Petrov (1995)), we have

$$F_{nh}(y) = \Phi(y) + \sum_{\nu=1}^{k-2} P_\nu(y) n^{-\nu/2} + Q_n(y),$$

where

$$\begin{aligned} |Q_n(y)| \le\ & c(k)\Big\{ n^{-(k-2)/2}(1+|y|)^{-k} E|X_{1h}^*|^k I(|X_{1h}^*| \ge \sqrt{n}(1+|y|)) \\ & + n^{-(k-1)/2}(1+|y|)^{-(k+1)} E|X_{1h}^*|^{k+1} I(|X_{1h}^*| < \sqrt{n}(1+|y|)) \\ & + n^{k(k+1)/2}(1+|y|)^{-(k+1)} \Big(\sup_{|t| \ge T(h)} |Ee^{itX_{1h}^*}| + 1/(2n) \Big)^n \Big\} \end{aligned}$$

for all y and n, where $T(h) = 1/(12E|X_{1h}^*|^3)$ and $c(k)$ is a positive constant depending only on k. By the argument below (3.6) and a simple calculation, we have

$$|Q_n(y)| \le c(k) \Big\{ n^{-\frac{k-1}{2}} E|X_{1h}^*|^{k+1} + n^{\frac{k(k+1)}{2}} \Big(\sup_{|t| \ge c} |Ee^{itX_{1h}^*}| + \frac{1}{2n} \Big)^n \Big\}.$$

Similarly to the proof of (3.7), we have

$$1 - F_n(x) = e^{\Lambda_n(h)} \Big(L_1 + \sum_{\nu=1}^{k-2} L_{2\nu} n^{-\nu/2} + L_3 \Big), \tag{3.16}$$

where $\Lambda_n(h) = n \log R(h) - nhm(h)$ and

$$\begin{aligned}
L_1 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\sqrt{n}h\sigma(h)y - y^2/2} dy, \\
L_{2\nu} &= \int_0^\infty e^{-\sqrt{n}h\sigma(h)y} dP_\nu(y), \qquad \nu = 1, \cdots, k-2, \\
L_3 &= \int_0^\infty e^{-\sqrt{n}h\sigma(h)y} dQ_n(y).
\end{aligned}$$

Using Lemma 2.1.1 in Jensen(1995) and noting that

$$d(H_{\nu-1}(y)e^{-y^2/2})/dy = -H_\nu(y)e^{-y^2/2}, \qquad \nu \geq 1,$$

we have

$$L_1 = B_0(\sqrt{n}h\sigma(h))/(\sqrt{n}h\sigma(h)), \tag{3.17}$$

and

$$\begin{aligned}
L_{2\nu} &= -\frac{1}{\sqrt{2\pi}} \sum \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}(h)}{(m+2)!} \right)^{k_m} \\
&\quad \times \int_0^\infty \exp(-\sqrt{n}h\sigma(h)y) d(H_{\nu+2s-1}(y)e^{-y^2/2}) \\
&= \frac{1}{\sqrt{2\pi}} \sum \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}(h)}{(m+2)!} \right)^{k_m} \int_0^\infty H_{\nu+2s}(y) e^{-\sqrt{n}h\sigma(h)y - y^2/2} dy \\
&= \sum \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}(h)}{(m+2)!} \right)^{k_m} \frac{B_{\nu+2s}(\sqrt{n}h\sigma(h))}{\sqrt{n}h\sigma(h)},
\end{aligned} \tag{3.18}$$

where the summations have the same meaning as that in (3.15). Similarly to the proof of (3.13), we have

$$L_3 = O(n^{-(k-1)/2}) \tag{3.19}$$

uniformly in x in the area $0 < h \leq H'$. Then (1.8) follows from (3.16)-(3.19). The proof of Theorem 1.3 is complete. $\square$

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