On Layered Stable Processes

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Abstract

Layered stable (multivariate) distributions and processes are defined and studied. A layered stable process combines stable trends of two different indices, one of them possibly Gaussian. More precisely, in short time, it is close to a stable process while, in long time, it approximates another stable (possibly Gaussian) process. We also investigate the absolute continuity of a layered stable process with respect to its short time limiting stable process. A series representation of layered stable processes is derived, giving insights into both the structure of the sample paths and of the short and long time behaviors. This series is further used for sample paths simulation.

1 Introduction and preliminaries

Stable processes form one of the simplest class of Lévy processes without Gaussian component. They have been thoroughly studied by many authors and have been used in several fields of applications, such as statistical physics, queueing theory, mathematical finance. One of their major attractions is the scaling property induced by the structure of the corresponding Lévy measure. Sato [13] and Samorodnitsky and Taqqu [11] contain many basic facts on stable distributions and processes. Recent generalizations of stable processes can also be found, for example, in Barndorff-Nielsen and Shepard [2] and in Rosiński [10]. These new classes are also of great interest in applications and have moreover motivated our study.

In the present paper, we introduce and study further generalizations which we call *layered stable* distributions and processes. They are defined in terms of the structure of their Lévy measure whose radial component behaves asymptotically as an inverse polynomial of different orders near zero and at infinity. *The inner and outer (stability) indices* correspond respectively to these orders of polynomial decay. This simple layering leads to the following properties: The outer index determines the moment properties (Proposition 2.3), while the variational properties depend on the inner index (Proposition 2.5). On the other hand, the inner and outer indices also correspond to short and long time behavior of the sample paths. In short time, a layered stable process behaves like a stable process with the corresponding inner index (Theorem 3.1). The long time behavior has two modes depending on the outer index. When the outer index is strictly smaller than two, a layered stable process is close to a stable process with this index, while behaving like a Brownian motion if the outer index is strictly greater than two (Theorem 3.2). In relation to the short time behavior, we investigate the mutual absolute continuity of a layered stable process and of its short time limiting stable process (Theorem 4.1). A shot noise series representation reveals the nature of layering and also gives direct insights into the properties of layered

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stable processes. We present typical sample paths of a layered stable process, which are simulated via the series representation for various combinations of stability indices in order to cover all the types of short and long time behavior.

Let us begin with some general notations which will be used throughout the text. $\mathbb{R}^d$ is the d-dimensional Euclidean space with the norm $\|\cdot\|$, $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$, $\mathcal{B}(\mathbb{R}^d_0)$ is the Borel σ-field of $\mathbb{R}^d_0$, and $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$. A' is the transpose of the matrix A, while $\|\cdot\|_o$ is the operator norm of the linear transformation $A \in \mathbb{R}^{d\times d}$, i.e., $\|A\|_o = \sup_{\|x\|\le 1} \|Ax\|$. $f(x) \sim g(x)$ indicates that $f(x)/g(x) \to 1$, as $x \to x_0 \in [-\infty, \infty]$, while $f(x) \asymp g(x)$ is used to mean that there exist two positive constants c_1 and c_2 such that $c_1 g(x) \le f(x) \le c_2 g(x)$, for all x in an approximate set. $\mathcal{L}(X)$ is the law of the random vector X, while $\stackrel{\mathcal{L}}{=}$ and $\stackrel{\mathcal{L}}{\to}$ denote, respectively, equality and convergence in distribution, or of the finite dimensional distributions when random processes are considered. $\stackrel{d}{\to}$ is used for the weak convergence of random processes in the space $\mathbb{D}([0,\infty), \mathbb{R}^d)$ of càdlàg functions from $[0,\infty)$ into $\mathbb{R}^d$ equipped with the Skorohod topology. $\stackrel{v}{\to}$ denotes convergence in the vague topology. For any $r > 0$, T_r is a transformation of measures on $\mathbb{R}^d$ given, for any positive measure ρ, by $(T_r\rho)(B) = \rho(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. $\mathbb{P}|_{\mathcal{F}_t}$ is the restriction of a probability measure $\mathbb{P}$ to the σ-field $\mathcal{F}_t$, while ΔX_t denotes the jump of X at time t, that is, $\Delta X_t := X_t - X_{t-}$. Finally, and throughout, all the multivariate or matricial integrals are defined componentwise.

Recall that an infinitely divisible probability measure μ on $\mathbb{R}^d$, without Gaussian component, is called *stable* if its Lévy measure is given by

$$\nu(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d_0),$$

where $\alpha \in (0,2)$ is the stability index and where σ is a finite positive measure on S^{d-1}. It is well known that the characteristic function of μ is given by

$$\begin{aligned} \widehat{\mu}(y) &= \exp\left[i\langle y, \eta\rangle + \int_{\mathbb{R}^d_0} (e^{i\langle y,z\rangle} - 1 - i\langle y, z\rangle \mathbf{1}_{\{\|z\|\le 1\}}(z))\nu_\alpha(dz)\right] \qquad (1.1)\\ &= \begin{cases} \exp\left[i\langle y, \tau_\alpha\rangle - c_\alpha \int_{S^{d-1}} |\langle y, \xi\rangle|^\alpha \left(1 - i \tan\frac{\pi\alpha}{2} \mathrm{sgn}\langle y, \xi\rangle\right) \sigma(d\xi)\right], & \text{if } \alpha \ne 1,\\ \exp\left[i\langle y, \tau_1\rangle - c_1 \int_{S^{d-1}} \left(|\langle y, \xi\rangle| + i\frac{2}{\pi}\langle y, \xi\rangle \ln|\langle y, \xi\rangle|\right) \sigma(d\xi)\right], & \text{if } \alpha = 1, \end{cases} \end{aligned}$$

for some $\eta \in \mathbb{R}^d$, and where $c_\alpha = |\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}|$ when $\alpha \ne 1$ while $c_1 = \pi/2$, with moreover $\tau_\alpha = \eta - \frac{1}{1-\alpha}\int_{S^{d-1}} \xi\sigma(d\xi)$ when $\alpha \ne 1$ and $\tau_1 = \eta - (1-\gamma)\int_{S^{d-1}} \xi\sigma(d\xi)$, $\gamma(= 0.5772...)$ being the Euler constant. A Lévy process $\{X_t : t \ge 0\}$ such that $\mathcal{L}(X_1) \sim \mu$ is called a *stable process*. Stable processes enjoy the *selfsimilarity* property, i.e., for any $a > 0$,

$$\{X_{at} : t \ge 0\} \stackrel{\mathcal{L}}{=} \{a^{1/\alpha} X_t + bt : t \ge 0\},$$

for some $b \in \mathbb{R}^d$. Next, we recall a shot noise series representation of stable processes on a fixed finite horizon $[0, T]$, $T > 0$. Related results can be found, for example, in Theorem 1.4.5 of Samorodnitsky and Taqqu [11]. The centering constants given below are obtained in Proposition 5.5 of Rosiński [10].

Lemma 1.1. *Let $T > 0$. Let $\{T_i\}_{i\ge 1}$ be a sequence of iid uniform random variables on $[0,T]$, let $\{\Gamma_i\}_{i\ge 1}$ be an arrival times of a standard Poisson process, and let $\{V_i\}_{i\ge 1}$ a sequence of iid random vectors in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. Also let*

$$z_0 = \begin{cases} 0, & if\ \alpha \in (0,1),\\ \int_{S^{d-1}} \xi\sigma(d\xi)/\sigma(S^{d-1}), & if\ \alpha \in [1,2), \end{cases}$$

and

$$b_T = \begin{cases} 0, & if\ \alpha \in (0,1),\\ \sigma(S^{d-1})T(\gamma + \ln(\sigma(S^{d-1})T)), & if\ \alpha = 1,\\ \left(\frac{\alpha}{\sigma(S^{d-1})T}\right)^{-1/\alpha} \zeta(1/\alpha), & if\ \alpha \in (1,2), \end{cases}$$

where ζ denotes the Riemann zeta function. Then, the stochastic process

$$\left\{\sum_{i=1}^{\infty}\left[\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} V_i\,\mathbf{1}(T_i\le t)-\left(\frac{\alpha i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} z_0\frac{t}{T}\right]+b_T z_0\frac{t}{T}:t\in[0,T]\right\},$$

converges almost surely uniformly in t to an α-stable process $\{X_t : t \in [0,T]\}$ satisfying $\mathbb{E}[e^{i\langle y,X_T\rangle}] = \widehat{\mu}(y)^T$, where $\widehat{\mu}$ given by (1.1) with

$$\eta=\begin{cases}\frac{1}{1-\alpha}\int_{S^{d-1}}\xi\sigma(d\xi), & \text{if } \alpha\neq 1,\\ 0, & \text{if } \alpha=1.\end{cases}$$

2 Definition and basic properties

We first define a layered stable multivariate distribution by precising the structure of its Lévy measure in polar coordinates.

Definition 2.1. *Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$ and without Gaussian component. Then, μ is called layered stable if its Lévy measure on $\mathbb{R}_0^d$ is given by*

$$\nu(B)=\int_{S^{d-1}}\sigma(d\xi)\int_0^{\infty}\mathbf{1}_B(r\xi)q(r,\xi)dr,\quad B\in\mathcal{B}(\mathbb{R}_0^d), \tag{2.1}$$

where σ is a finite positive measure on S^{d-1}, and q is a measurable function from $(0,\infty)\times S^{d-1}$ to $(0,\infty)$ such that for each $\xi\in S^{d-1}$,

$$q(r,\xi)\sim c_1(\xi)r^{-\alpha-1},\quad \text{as } r\to 0, \tag{2.2}$$

and

$$q(r,\xi)\sim c_2(\xi)r^{-\beta-1},\quad \text{as } r\to\infty, \tag{2.3}$$

where c_1 and c_2 are integrable (with respect to σ) functions on S^{d-1}, and where $(\alpha,\beta)\in(0,2)\times(0,\infty)$.

$q(\cdot,\cdot)$ is called *the q-function* of μ, or of its Lévy measure ν. Clearly, ν is well defined as a Lévy measure since it behaves like an α-stable Lévy measure near the origin while decaying like a β-Pareto density when sufficiently far away from the origin. α and β are respectively called *the inner and outer (stability) indices* of μ, or of ν.

For convenience, we henceforth use the notations σ_1 and σ_2 for the finite positive measures on S^{d-1} defined respectively by

$$\sigma_1(B):=\int_B c_1(\xi)\sigma(d\xi),\quad B\in\mathcal{B}(S^{d-1}), \tag{2.4}$$

and

$$\sigma_2(B):=\int_B c_2(\xi)\sigma(d\xi),\quad B\in\mathcal{B}(S^{d-1}), \tag{2.5}$$

while ν_σ^α is used for the positive measure on $\mathbb{R}_0^d$ given by

$$\nu_\sigma^\alpha(B):=\int_{S^{d-1}}\sigma(d\xi)\int_0^{\infty}\mathbf{1}_B(r\xi)\frac{dr}{r^{\alpha+1}},\quad B\in\mathcal{B}(\mathbb{R}_0^d), \tag{2.6}$$

where $\alpha\in(0,\infty)$ and where σ is a finite positive measure on S^{d-1}. Note that if $\alpha\in(0,2)$, ν_σ^α is simply an α-stable Lévy measure, while not well defined as a Lévy measure when $\alpha\geq 2$.

Example 2.2. The following layered stable Lévy measure is simple, yet interesting:

$$\begin{aligned}\nu(B) &= \int_B \mathbf{1}_{\{\|z\|\leq 1\}}(z)\nu_\sigma^\alpha(dz) + \int_B \mathbf{1}_{\{\|z\|>1\}}(z)\nu_\sigma^\beta(dz) \\ &= \int_{S^{d-1}} \sigma(d\xi)\int_0^\infty \mathbf{1}_B(r\xi)\frac{dr}{r^{\alpha+1}\mathbf{1}_{(0,1]}(r) + r^{\beta+1}\mathbf{1}_{(1,\infty)}(r)}, \quad B\in\mathcal{B}(\mathbb{R}_0^d). \end{aligned} \tag{2.7}$$

The corresponding q-function is given by

$$q(r,\xi) = \sigma(S^{d-1})^{-1}(r^{-\alpha-1}\mathbf{1}_{(0,1]}(r) + r^{-\beta-1}\mathbf{1}_{(1,\infty)}(r)), \quad \xi\in S^{d-1},$$

which is independent of ξ. The measure ν consists of two disjoint domains of stability, and this construction results in two layers for the radial component associated with each respective stability index. The name "*layered stable*" originates from this special structure.

Recall that an infinitely divisible probability measure μ on $\mathbb{R}^d$ is said to be of class L_0, or selfdecomposable if for any $b>1$, there exists a probability measure ϱ_b such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. Equivalently, the Lévy measure of μ has the form

$$\int_{S^{d-1}} \sigma(d\xi)\int_0^\infty \mathbf{1}_B(r\xi)k_\xi(r)\frac{dr}{r}, \quad B\in\mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and where $k_\xi(r)$ is a nonnegative function measurable in $\xi\in S^{d-1}$ and decreasing in $r>0$. Clearly, the Lévy measure (2.7) induces a selfdecomposable measure. Moreover, the classes L_m, $m=1,2,\ldots$, are defined recursively as follows; $\mu\in L_m$ if for every $b>1$, there exists $\varrho_b\in L_{m-1}$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. Clearly, $L_0\supset L_1\supset L_2\supset\cdots$. Let $h_\xi(u) := k_\xi(e^{-u})$, be the so-called *h-function* of μ, or of its Lévy measure. Then, alternatively, $\mu\in L_0$ is shown to be in L_m if and only if $h_\xi(u)\in C^{m-1}$ and $h^{(j)}\geq 0$, for $j=0,1,\ldots,m-1$. (See Sato [12] for more details.) The h-function of the Lévy measure (2.7) is given by

$$h_\xi(u) = e^{\alpha u}\mathbf{1}_{(0,\infty)}(u) + e^{\beta u}\mathbf{1}_{(-\infty,0]}(u),$$

which is in C^0 but not in C^1. Therefore, the infinitely divisible probability measure induced by (2.7) is in L_1, but not in L_2.

The following result asserts that a layered stable distribution has the same probability tail behavior as β-Pareto distributions, or β-stable distributions if $\beta\in(0,2)$.

Proposition 2.3. (Moments) *Let μ be a layered stable distribution with Lévy measure ν given by (2.1) and let σ_2 be the measure (2.5). If $\sigma_2(S^{d-1})\neq 0$, then*

$$\int_{\mathbb{R}^d}\|x\|^p\mu(dx)\begin{cases} <\infty, & p\in(0,\beta), \\ =\infty, & p\in[\beta,\infty).\end{cases}$$

Moreover, $\int_{\mathbb{R}^d}\|x\|^p\mu(dx)<\infty$, $p\geq\beta$ and $\int_{\mathbb{R}^d}e^{\theta\|x\|}\mu(dx)<\infty$, $\theta>0$ if and only if $\sigma_2(S^{d-1})=0$.

Proof. By Theorem 25.3 of Sato [13], it is enough to show that the restriction of ν to the set $\{z\in\mathbb{R}_0^d : \|z\|>1\}$ has the corresponding moment properties.

First, assume $\sigma_2(S^{d-1})\neq 0$. Observe that $\int_{\|z\|>1}\|z\|^p\nu(dz) = \int_{S^{d-1}}\sigma(d\xi)\int_1^\infty r^p q(r,\xi)dr$, and then by (2.3), the right hand side is bounded from above and below by constant multiples of $\sigma_2(S^{d-1})\int_1^\infty r^p\frac{dr}{r^{\beta+1}}$ if $p\in(0,\beta)$, while it is otherwise clearly infinite.

Next, assume $\sigma_2(S^{d-1})=0$ and let $p\in[\beta,\infty)$. Then, there exists $M>0$ such that $\int_{\|z\|>1}\|z\|^p\nu(dz)\asymp\int_{S^{d-1}}\sigma(d\xi)\int_1^M r^p q(r,\xi)dr$ and $\int_{\|z\|>1}e^{\theta\|z\|}\nu(dz)\asymp\int_{S^{d-1}}\sigma(d\xi)\int_1^M e^{\theta r}q(r,\xi)dr$. Conversely, if $\sigma_2(S^{d-1})\neq 0$ and $p\in[\beta,\infty)$, then $\int_{\|z\|>1}\|z\|^p\nu(dz)=+\infty$ as already shown and, again by (2.3), $\int_{\|z\|>1}e^{\theta\|z\|}\nu(dz) = \int_{S^{d-1}}\sigma(d\xi)\int_1^\infty e^{\theta r}q(r,\xi)dr = +\infty$. □

Let us define the associated Lévy processes.

Definition 2.4. *A Lévy process, without Gaussian component, is called layered stable if its Lévy measure is given by (2.1).*

Henceforth, $\{X_t^{LS} : t \geq 0\}$ denotes a layered stable process in $\mathbb{R}^d$. Its characteristic function at time 1 is given by

$$\mathbb{E}[e^{i\langle y, X_1^{LS}\rangle}] = \exp\left[i\langle y,\eta\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle \mathbf{1}_{\{\|z\|\leq 1\}}(z))\nu(dz)\right], \tag{2.8}$$

where ν is the Lévy measure given by (2.1) and $\eta \in \mathbb{R}^d$. For convenience of notation, we write $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$ when (2.8) holds. Similarly, for $\alpha \in (0,2)$, $\{X_t^{(\alpha)} : t \geq 0\}$ denotes an α-stable Lévy process. Its characteristic function at time 1 is given by

$$\mathbb{E}[e^{i\langle y, X_1^{(\alpha)}\rangle}] = \begin{cases} \exp\left[i\langle y,\eta\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1)\nu_\sigma^\alpha(dz)\right], & if\ \alpha \in (0,1), \\ \exp\left[i\langle y,\eta\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle \mathbf{1}_{\{\|z\|\leq 1\}}(z))\nu_\sigma^1(dz)\right], & if\ \alpha = 1, \\ \exp\left[i\langle y,\eta\rangle + \int_{\mathbb{R}_0^d}(e^{i\langle y,z\rangle} - 1 - i\langle y,z\rangle)\nu_\sigma^\alpha(dz)\right], & if\ \alpha \in (1,2), \end{cases} \tag{2.9}$$

where ν_σ^α is given by (2.6), and we write $\{X_t^{(\alpha)} : t \geq 0\} \sim S_\alpha(\sigma;\eta)$ when (2.9) holds.

A layered stable process shares the variational properties of a stable process with inner index α.

Proposition 2.5. (p-th variation) *Let* $X := \{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$.
(i) If $\sigma_1(S^{d-1}) > 0$, *then* X *is a.s. of finite variation on every interval of positive length if and only if* $\alpha \in (0,1)$.
(ii) If $\sigma_1(S^{d-1}) > 0$, $(\alpha,\beta) \in [1,2) \times (1,\infty)$ *and* $\eta = -\int_{S^{d-1}} \xi\sigma(d\xi) \int_1^\infty rq(r,\xi)dr$, *then* X *is a.s. of finite* p*-th variation on every interval of positive length if and only if* $p > \alpha$.
(iii) If $\sigma_1(S^{d-1}) = 0$, *then it is a.s. of finite variation on every interval of positive length.*

Proof. (i) Recall that the radial component of the layered stable Lévy measure near the origin behaves like the one of an α-stable Lévy measure. The first claim then follows immediately from Theorem 3 of Gikhman and Skorokhod [4].

(ii) Since X is now centered, Théorème III b of Bretagnolle [3] directly applies.

(iii) Letting ν be the Lévy measure of X, there exists $\epsilon \in (0,1)$ such that $\nu(\{z \in \mathbb{R}_0^d : \|z\| \leq \epsilon\}) < \infty$ and so $\int_{\|z\|\leq 1} \|z\|^p \nu(dz) < \infty$, $p \geq 1$. As in (i), the result follows from Theorem 3 of Gikhman and Skorokhod [4]. □

Let us now consider a series representation for a general layered stable process $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$. Fix $T > 0$. Let $\{T_i\}_{i\geq 1}$ be a sequence of iid uniform random variables on $[0,T]$, let $\{\Gamma_i\}_{i\geq 1}$ be Poisson arrivals with rate 1, and let $\{V_i\}_{i\geq 1}$ be a sequence of iid random vectors in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. Assume moreover that the random sequences $\{T_i\}_{i\geq 1}$, $\{\Gamma_i\}_{i\geq 1}$, and $\{V_i\}_{i\geq 1}$ are all mutually independent. Also, let

$$\overleftarrow{q}(u,\xi) := \inf\{r > 0 : q([r,\infty),\xi) < u\},$$

and let $\{b_i\}_{i\geq 1}$ be a sequence of constants given by

$$b_i = \int_{i-1}^{i} \mathbb{E}[\overleftarrow{q}(s/T, V_1)V_1 \mathbf{1}(\overleftarrow{q}(s/T, V_1) \leq 1)]ds,$$

Then, by Theorem 5.1 of Rosiński [9] with the help of the LePage's method [8], the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left[\overleftarrow{q}(\Gamma_i/T, V_i) V_i \mathbf{1}(T_i \le t) - b_i \frac{t}{T} \right] : t \in [0,T] \right\}, \tag{2.10}$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is $LS_{\alpha,\beta}(\sigma, q; 0)$.

Example 2.6. The Lévy measure (2.7) leads to a very illustrative series representation. Indeed,

$$\overleftarrow{q}(r,\xi) = \left(\frac{\beta r}{\sigma(S^{d-1})} \right)^{-1/\beta} \mathbf{1}_{(0,\sigma(S^{d-1})/\beta]}(r) + \left(\frac{\alpha r}{\sigma(S^{d-1})} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} \mathbf{1}_{(\sigma(S^{d-1})/\beta,\infty)}(r),$$

and so the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left[\left(\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\beta} \mathbf{1}_{(0,\sigma(S^{d-1})T/\beta]}(\Gamma_i) \right.\right.\right. \tag{2.11}$$
$$\left.\left.\left. + \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} \mathbf{1}_{(\sigma(S^{d-1})T/\beta,\infty)}(\Gamma_i) \right) V_i \mathbf{1}(T_i \le t) - b_i z_0 \frac{t}{T} \right] : t \in [0,T] \right\},$$

where

$$b_i = \left(\frac{\beta}{\sigma(S^{d-1})T} \right)^{-1/\beta} \frac{(i \wedge \sigma(S^{d-1})T/\beta)^{1-1/\beta} - ((i-1) \wedge \sigma(S^{d-1})T/\beta)^{1-1/\beta}}{1-1/\beta},$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is $LS_{\alpha,\beta}(\sigma, q; 0)$, with $z_0 = \int_{S^{d-1}} \xi \sigma(d\xi)/\sigma(S^{d-1})$. This series representation directly reveals the nature of layering; all jumps with absolute size greater than 1 are due to the β-stable shot noise series $\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})}\right)^{-1/\beta} V_i$, while smaller jumps come from $\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})} + 1 - \frac{\alpha}{\beta}\right)^{-1/\alpha} V_i$, which resembles α-stable jumps.

3 Short and long time behavior

We now present one of the two main results of this section by giving the short time behavior of a layered stable process. The results of this section were motivated by Section 3 of Rosiński [10]. Recall that σ_1 and σ_2 are the finite positive measures respectively given in (2.4) and (2.5), and that for any $r > 0$, T_r transforms the positive measure ρ, via $(T_r\rho)(B) = \rho(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. For convenience, we will use the notation $\nu_{\sigma,q}^{\alpha,\beta}$ for the Lévy measure of a layered stable process $LS_{\alpha,\beta}(\sigma, q; \eta)$ throughout this section.

Theorem 3.1. *Short time behavior: Let $\{X_t^{LS} : t \ge 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$, let*

$$\eta_{\alpha,\beta} = \begin{cases} \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r q(r,\xi) dr, & if\ \alpha \in (0,1), \\ -\int_{S^{d-1}} \xi \sigma(d\xi) \int_1^\infty r q(r,\xi) dr, & if\ (\alpha,\beta) \in (1,2) \times (1,\infty), \\ 0, & otherwise, \end{cases}$$

and let

$$b_{\alpha,\beta} = \begin{cases} \frac{1}{\alpha-1} \int_{S^{d-1}} \xi \sigma_1(d\xi), & if\ (\alpha,\beta) \in (1,2) \times (0,1], \\ 0, & otherwise. \end{cases}$$

Then,

$$\{h^{-1/\alpha}(X_{ht}^{LS} + ht\eta_{\alpha,\beta}) - tb_{\alpha,\beta} : t \ge 0\} \xrightarrow{d} \{X_t^{(\alpha)} : t \ge 0\}, \quad as\ h \to 0,$$

where $\{X_t^{(\alpha)} : t \ge 0\} \sim S_\alpha(\sigma_1; 0)$.

Proof. Since a layered stable process is a Lévy process, by a theorem of Skorohod (see Theorem 15.17 of Kallenberg [7]), it suffices to show the weak convergence of its marginals at time 1. To this end, we will show the proper convergence of the generating triplet of the infinitely divisible law, following Theorem 15.14 of Kallenberg [7].

For the convergence of the Lévy measure, we need to show that as $h \to 0$,

$$h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta}) \xrightarrow{v} \nu_{\sigma_1}^{\alpha},$$

or equivalently,

$$\lim_{h\to 0}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) = \int_{\mathbb{R}_0^d} f(z)\nu_{\sigma_1}^{\alpha}(dz),$$

for all bounded continuous function $f : \mathbb{R}_0^d \to \mathbb{R}$ vanishing in a neighborhood of the origin. Letting f be such a function with $|f| \le C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}_0^d : \|z\| \le \epsilon\}$, for some $\epsilon > 0$, we get by (2.2),

$$\begin{aligned}
\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(h^{-1/\alpha}r\xi)hq(r,\xi)dr\\
&= \int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(r\xi)h^{1+1/\alpha}q(h^{1/\alpha}r,\xi)dr\\
&\to \int_{S^{d-1}}c_1(\xi)\sigma(d\xi)\int_0^\infty f(r\xi)\frac{dr}{r^{\alpha+1}},
\end{aligned}$$

as $h \to 0$, where the last convergence holds true since for $h \in (0,1)$,

$$\left|\int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(h^{-1/\alpha}r\xi)hq(r,\xi)dr\right| \le C\left|\int_{S^{d-1}}\sigma(d\xi)\int_\epsilon^\infty q(r,\xi)dr\right| < \infty.$$

For the convergence of the Gaussian component, we need to show that for each $\kappa > 0$,

$$\int_{\|z\|\le\kappa} zz'h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) \to \int_{\|z\|\le\kappa} zz'\nu_{\sigma_1}^{\alpha}(dz),$$

as $h \to 0$. Again, by (2.2),

$$\begin{aligned}
\int_{\|z\|\le\kappa} zz'h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^{h^{1/\alpha}\kappa} r^2h^{1-2/\alpha}q(r,\xi)dr\\
&= \int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^{\kappa} r^2h^{1+1/\alpha}q(h^{1/\alpha}r,\xi)dr\\
&\to \int_{S^{d-1}}\xi\xi'\sigma_1(d\xi)\int_0^{\kappa} r^2\frac{dr}{r^{\alpha+1}}\\
&= \int_{\|z\|\le\kappa} zz'\nu_{\sigma_1}^{\alpha}(dz),
\end{aligned}$$

where the passage to the limit is justified since, for $h \in (0,1)$,

$$\left\|\int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^{h^{1/\alpha}\kappa} r^2h^{1-2/\alpha}q(r,\xi)dr\right\|_o \le \left\|\int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^{\kappa} r^2q(r,\xi)dr\right\|_o < \infty.$$

For the convergence of the drift part, assume first that $(\alpha,\beta) \notin (1,2)\times(0,1]$. For a σ-finite positive measure ν on $\mathbb{R}_0^d$, let

$$C_\alpha(\nu) := \begin{cases} \int_{\|z\|\le 1} z\nu(dz), & \text{if } \alpha \in (0,1),\\ 0, & \text{if } \alpha = 1,\\ -\int_{\|z\|>1} z\nu(dz), & \text{if } \alpha \in (1,2). \end{cases} \tag{3.1}$$

Clearly, $\eta_{\alpha,\beta} = C_\alpha(\nu_{\sigma,q}^{\alpha,\beta})$ and we then show that as $h \to 0$,

$$C_\alpha(h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})) - \int_{\kappa<\|z\|\le 1} zh(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) \to C_\alpha(\nu_{\sigma_1}^\alpha) - \int_{\kappa<\|z\|\le 1} z\nu_{\sigma_1}^\alpha(dz),$$

for each $\kappa > 0$. Letting

$$B = \begin{cases} \{z \in \mathbb{R}_0^d : \|z\| \le \kappa\}, & \text{if } \alpha \in (0,1), \\ \{z \in \mathbb{R}_0^d : \kappa < \|z\| \le 1\}, & \text{if } \alpha = 1, \\ \{z \in \mathbb{R}_0^d : \|z\| > \kappa\}, & \text{if } (\alpha,\beta) \in (1,2)\times(1,\infty), \end{cases}$$

we have as $h \to 0$,

$$\begin{aligned} \int_{\mathbb{R}_0^d} \mathbf{1}_B(z) zh(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^\infty \mathbf{1}_B(h^{-1/\alpha} r\xi) rh^{1-1/\alpha} q(r,\xi) dr \\ &= \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) rh^{1+1/\alpha} q(h^{1/\alpha} r,\xi) dr \\ &\to \int_{S^{d-1}} \xi\sigma_1(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r\frac{dr}{r^{\alpha+1}}, \end{aligned}$$

where the convergence holds true since for $h \in (0,1)$, and with the help of (2.2),

$$\left\| \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) rh^{1+1/\alpha} q(h^{1/\alpha} r,\xi) dr \right\| \asymp \left\| \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) rq(r,\xi) dr \right\| < \infty.$$

Finally, assume $(\alpha,\beta) \in (1,2)\times(0,1]$. Then, as $h \to 0$,

$$-b_{\alpha,\beta} - \int_{\kappa<\|z\|\le 1} zh(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(dz) \to -\int_{\|z\|>\kappa} z\nu_{\sigma_1}^\alpha(dz),$$

for each $\kappa > 0$, where the convergence holds true as before. This completes the proof. □

Our next result is also important. Unlike in short time, the long time behavior of a layered stable process depends on its outer stability index β. This behavior is akin to a β-stable process if $\beta \in (0,2)$, while akin to a Brownian motion whenever $\beta \in (2,\infty)$.

Theorem 3.2. *Long time behavior: Let* $\{X_t^{LS} : t \ge 0\} \sim LS_{\alpha,\beta}(\sigma,q;0)$.
(i) Let $\beta \in (0,2)$*, let*

$$\eta_{\alpha,\beta} = \begin{cases} \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^1 rq(r,\xi)dr, & \text{if } (\alpha,\beta) \in (0,1)\times(0,1), \\ -\int_{S^{d-1}} \xi\sigma(d\xi) \int_1^\infty rq(r,\xi)dr, & \text{if } \beta \in (1,2), \\ 0, & \text{otherwise}, \end{cases}$$

and let

$$b_{\alpha,\beta} = \begin{cases} \frac{1}{1-\beta} \int_{S^{d-1}} \xi\sigma_2(d\xi), & \text{if } (\alpha,\beta) \in [1,2)\times(0,1), \\ 0, & \text{otherwise}. \end{cases}$$

Then,

$$\{h^{-1/\beta}(X_{ht}^{LS} + ht\eta_{\alpha,\beta}) + tb_{\alpha,\beta} : t \ge 0\} \xrightarrow{d} \{X_t^{(\beta)} : t \ge 0\}, \quad \text{as } h \to \infty,$$

where $\{X_t^{(\beta)} : t \ge 0\} \sim S_\beta(\sigma_2;0)$.
(ii) Let $\beta \in (2,\infty)$ *and let*

$$\eta = -\int_{S^{d-1}} \xi\sigma(d\xi) \int_1^\infty rq(r,\xi)dr. \tag{3.2}$$

Then,

$$\{h^{-1/2}(X_{ht}^{LS}+ht\eta):t\geq 0\}\xrightarrow{d}\{W_t:t\geq 0\},\quad \text{as } h\to\infty, \tag{3.3}$$

where $\{W_t:t\geq 0\}$ *is a centered Brownian motion with covariance matrix* $\int_{\mathbb{R}_0^d} zz'\nu_{\sigma,q}^{\alpha,\beta}(dz)$.

Proof. The claim (i) can be proved as (i) in Theorem 3.1. For the convergence of the Lévy measure, we will show that

$$\lim_{h\to\infty}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(dz)=\int_{\mathbb{R}_0^d} f(z)\nu_{\sigma_2}^{\beta}(dz),$$

for all bounded continuous function $f:\mathbb{R}_0^d\to\mathbb{R}$ vanishing in a neighborhood of the origin. Letting f be such a function with $|f|\leq C<\infty$ and $f(z)\equiv 0$ on $\{z\in\mathbb{R}_0^d:\|z\|\leq\epsilon\}$, for some $\epsilon>0$, we get by (2.3),

$$\begin{aligned}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(h^{-1/\beta}r\xi)hq(r,\xi)dr\\ &= \int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(r\xi)h^{1+1/\beta}q(h^{1/\beta}r,\xi)dr\\ &\to \int_{S^{d-1}}c_2(\xi)\sigma(d\xi)\int_0^\infty f(r\xi)\frac{dr}{r^{\beta+1}},\end{aligned}$$

as $h\to\infty$, where the last convergence holds true because of (2.3) and since for sufficiently large $h>0$,

$$\begin{aligned}\left|\int_{S^{d-1}}\sigma(d\xi)\int_0^\infty f(h^{-1/\beta}r\xi)hq(r,\xi)dr\right| &= \left|\int_{S^{d-1}}\sigma(d\xi)\int_{h^{1/\beta}\epsilon}^\infty f(h^{-1/\beta}r\xi)hq(r,\xi)dr\right|\\ &\leq hC\int_{S^{d-1}}\sigma(d\xi)\int_{h^{1/\beta}\epsilon}^\infty q(r,\xi)dr\\ &\asymp hC\sigma_2(S^{d-1})\int_{h^{1/\beta}\epsilon}^\infty\frac{dr}{r^{\beta+1}}\\ &= C\sigma_2(S^{d-1})\frac{\epsilon^{-\beta}}{\beta}<\infty.\end{aligned}$$

For the convergence of the Gaussian component, we have as $h\to\infty$ and for each $\kappa>0$,

$$\begin{aligned}\int_{\|z\|\leq\kappa} zz'h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^\kappa r^2h^{1+1/\beta}q(h^{1/\beta}r,\xi)dr\\ &\to \int_{S^{d-1}}\xi\xi'\sigma_2(d\xi)\int_0^\kappa r^2\frac{dr}{r^{\beta+1}}\\ &= \int_{\|z\|\leq\kappa} zz'\nu_{\sigma_2}^{\beta}(dz),\end{aligned}$$

where the passage to the limit is justified next. Let $h\in(\kappa^{-\beta},+\infty)$ and write

$$\begin{aligned}\left\|\int_{\|z\|\leq\kappa} zz'h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(dz)\right\|_{\text{o}} &\leq \left\|\int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^1 h^{1-2/\beta}r^2q(r,\xi)dr\right\|_{\text{o}}\\ &\quad+\left\|\int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_1^{h^{1/\beta}\kappa} h^{1-2/\beta}r^2q(r,\xi)dr\right\|_{\text{o}}.\end{aligned}$$

The first term of the right hand side above is clearly bounded by $\kappa^{2-\beta}\|\int_{S^{d-1}}\xi\xi'\sigma(d\xi)\int_0^1 r^2q(r,\xi)dr\|_{\text{o}}$, while the second term is also bounded since for $h \in (\kappa^{-\beta}, +\infty)$,

$$\begin{aligned}\left\| \int_{S^{d-1}} \xi\xi' \sigma(d\xi) \int_1^{h^{1/\beta}\kappa} h^{1-2/\beta} r^2 q(r,\xi) dr \right\|_{\mathrm{o}} &\asymp h^{1-2/\beta} \int_1^{h^{1/\beta}\kappa} r^2 \frac{dr}{r^{\beta+1}} \left\| \int_{S^{d-1}} \xi\xi' \sigma_2(d\xi) \right\|_{\mathrm{o}} \\ &= \frac{\kappa^{2-\beta} - h^{1-2/\beta}}{2-\beta} \left\| \int_{S^{d-1}} \xi\xi' \sigma_2(d\xi) \right\|_{\mathrm{o}} < \infty.\end{aligned}$$

Finally, we study the convergence of the drift part. Assume first that $(\alpha, \beta) \notin [1,2) \times (0,1)$. Let $C_\beta(\nu)$ be the constant defined as in (3.1) but depending on β and ν. Clearly, $\eta_{\alpha,\beta} = C_\beta(\nu_{\sigma,q}^{\alpha,\beta})$. We will then show that as $h \to \infty$,

$$C_\beta(h(T_{h^{-1/\beta}} \nu_{\sigma,q}^{\alpha,\beta})) - \int_{\kappa < \|z\| \le 1} z h(T_{h^{-1/\beta}} \nu_{\sigma,q}^{\alpha,\beta})(dz) \to C_\beta(\nu_{\sigma_2}^\beta) - \int_{\kappa < \|z\| \le 1} z \nu_{\sigma_2}^\beta(dz),$$

for each $\kappa > 0$. Letting

$$B = \begin{cases} \{z \in \mathbb{R}_0^d : \|z\| \le \kappa\}, & \text{if } \beta \in (0,1), \\ \{z \in \mathbb{R}_0^d : \kappa < \|z\| \le 1\}, & \text{if } \beta = 1, \\ \{z \in \mathbb{R}_0^d : \|z\| > \kappa\}, & \text{if } \beta \in (1,2), \end{cases}$$

we have by (2.3) that

$$\begin{aligned}\int_{\mathbb{R}_0^d} \mathbf{1}_B(z) z h(T_{h^{-1/\beta}} \nu_{\sigma,q}^{\alpha,\beta})(dz) &= \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^\infty \mathbf{1}_B(h^{-1/\beta} r\xi) r h^{1-1/\beta} q(r,\xi) dr \\ &= \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r h^{1+1/\beta} q(h^{1/\beta} r, \xi) dr \\ &\to \int_{S^{d-1}} \xi \sigma_1(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r \frac{dr}{r^{\beta+1}},\end{aligned}$$

as $h \to \infty$, where the convergence holds true since for $h \in (1, \infty)$, and with the help of (2.3),

$$\left\| \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r h^{1+1/\beta} q(h^{1/\beta} r, \xi) dr \right\| \asymp \left\| \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r q(r,\xi) dr \right\| < \infty.$$

Next, let $(\alpha, \beta) \in [1,2) \times (0,1)$. Then, observe that for each $\kappa > 0$, and as $h \to \infty$,

$$-b_{\alpha,\beta} - \int_{\kappa < \|z\| \le 1} z h(T_{h^{-1/\alpha}} \nu_{\sigma,q}^{\alpha,\beta})(dz) \to - \int_{\|z\| > \kappa} z \nu_{\sigma_1}^\alpha(dz),$$

where the convergence holds true as before. This completes the proof of (i).

(ii) The random vector $h^{-1/2} X_h^{LS}$ is infinitely divisible with generating triplet

$$\left(- \int_{\|z\| \ge 1} z h(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})(dz), 0, h(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta}) \right).$$

Letting f be a bounded continuous function from $\mathbb{R}_0^d$ to $\mathbb{R}$ such that $|f| \le C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}^d : \|z\| \le \epsilon\}$, for some $\epsilon > 0$, the Lévy measure $h(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})$ converges vaguely to zero as $h \to \infty$

since for sufficiently large $h > 0$,

$$\begin{aligned}\left|\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})(dz)\right| &= \left|\int_{S^{d-1}} \sigma(d\xi)\int_{h^{1/2}\epsilon}^{\infty} f(h^{-1/2}r\xi)hq(r,\xi)dr\right| \\ &\leq hC\int_{S^{d-1}}\sigma(d\xi)\int_{h^{1/2}\epsilon}^{\infty} hq(r,\xi)dr \\ &\asymp hC\int_{S^{d-1}} c_2(\xi)\sigma(d\xi)\int_{h^{1/2}\epsilon}^{\infty}\frac{dr}{r^{\beta+1}} \\ &= h^{1-\beta/2}C\sigma_2(S^{d-1})\frac{\epsilon^{-\beta}}{\beta}\to 0, \end{aligned} \tag{3.4}$$

as $h \to \infty$. For the convergence of the Gaussian component, we have as $h \to +\infty$ and for each $\kappa > 0$,

$$\int_{\|z\|\leq\kappa} zz'h(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})(dz) = \int_{\|z\|\leq h^{1/2}\kappa} zz'\nu_{\sigma,q}^{\alpha,\beta}(dz) \to \int_{\mathbb{R}_0^d} zz'\nu_{\sigma,q}^{\alpha,\beta}(dz), \tag{3.5}$$

which is clearly well defined since $\int_{\mathbb{R}_0^d}\|z\|^2\nu_{\sigma,q}^{\alpha,\beta}(dz) < \infty$. Finally, for sufficiently large $h > 0$,

$$\begin{aligned}\left\|\int_{\|z\|>\kappa} zh(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})(dz)\right\| &= h^{1/2}\left\|\int_{S^{d-1}}\xi\sigma(d\xi)\int_{h^{1/2}\kappa}^{\infty} rq(r,\xi)dr\right\| \\ &\asymp h^{1/2}\int_{h^{1/2}\kappa}^{\infty} r\frac{dr}{r^{\beta+1}}\left\|\int_{S^{d-1}}\xi\sigma_2(d\xi)\right\| \\ &= h^{1-\beta/2}\frac{\kappa^{1-\beta}}{\beta-1}\left\|\int_{S^{d-1}}\xi\sigma_2(d\xi)\right\|, \end{aligned} \tag{3.6}$$

As $h \to \infty$, (3.6) converges to zero and this concludes the proof of (ii). □

For $\beta = 2$, layered stable processes do not seem to possess any nice long time behavior, and this can be seen from the improper convergence of the Lévy measure, i.e., as $h \to \infty$, $h(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,2})$ converges vaguely to

$$\int_{S^{d-1}}\sigma_2(d\xi)\int_0^{\infty}\mathbf{1}_B(r\xi)\frac{dr}{r^{2+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

which is not well defined as a Lévy measure. However, additional assumptions on σ_2 lead to the weak convergence towards a Brownian motion as β approaches to 2.

Proposition 3.3. *Let $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$ in $\mathbb{R}^d$.*
(i) Let $\beta \in (1,2)$ and let $\eta = -\int_{S^{d-1}}\xi\sigma(d\xi)\int_1^{\infty} rq(r,\xi)dr$. If σ_2 is uniform on S^{d-1} such that $\sigma_2(S^{d-1}) = d(2-\beta)$, then

$$\{h^{-1/\beta}(X_{ht}^{LS} + ht\eta) : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}, \quad \text{as } h \to \infty,\ \beta \uparrow 2,$$

where $\{W_t : t \geq 0\}$ is a d-dimensional (centered) standard Brownian motion. (The limit is taken over $h \to \infty$ first.)
(ii) Let $\beta \in (2,\infty)$ and let η be the constant (3.2). If σ_2 is symmetric such that $\sigma_2(S^{d-1}) = \beta - 2$, then

$$\{h^{-1/2}(X_{ht}^{LS} + ht\eta) : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}, \quad \text{as } h \to \infty,\ \beta \downarrow 2,$$

where $\{W_t : t \geq 0\}$ is a centered Brownian motion with covariance matrix $\int_{\mathbb{R}_0^d} zz'\nu_{\sigma,q}^{\alpha,2}(dz)$. (The limit can be taken either over $h \to \infty$ or over $\beta \downarrow 2$ first.)

Proof. (i) By Theorem 3.1 (i), $h^{-1/\beta}(X_h^{LS} + h\eta) \xrightarrow{\mathcal{L}} X_1^{(\beta)}$, as $h \to \infty$, where $\{X_t^{(\beta)} : t \geq 0\} \sim S_\beta(\sigma_2; 0)$. Then, by E.18.7-18.8 of Sato [13], we get $\mathbb{E}[e^{i\langle y, X_1^{(\beta)}\rangle}] = \exp[-c_{\beta,d}\|y\|^\beta]$, where

$$c_{\beta,d} = \frac{\Gamma(d/2)\Gamma((2-\beta)/2)}{2^\beta \beta \Gamma((\beta+d)/2)} \sigma_2(S^{d-1}).$$

Taking $\beta \uparrow 2$ and since $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we get the result.

(ii) In view of (3.4), we get

$$h^{1-\beta/2} C \sigma_2(S^{d-1}) \frac{\epsilon^{-\beta}}{\beta} = h^{1-\beta/2} C(\beta - 2) \frac{\epsilon^{-\beta}}{\beta} \to 0,$$

as $\beta \downarrow 2$, which shows that the Lévy measure $h(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})$ converges vaguely to zero. Moreover, in view of (3.6), for sufficiently large $h > 0$,

$$h^{1-\beta/2} \frac{\kappa^{1-\beta}}{\beta - 1} \left\| \int_{S^{d-1}} \xi \sigma_2(d\xi) \right\| = 0,$$

by the symmetry of σ_2. In view of (3.5), it remains to show that $\int_{\mathbb{R}_0^d} \|z\|^2 \nu_{\sigma,q}^{\alpha,2}(dz) < \infty$. Observe that

$$\int_{\mathbb{R}_0^d} \|z\|^2 \nu_{\sigma,q}^{\alpha,\beta}(dz) = \int_{S^{d-1}} \sigma(d\xi) \int_0^1 r^2 q(r,\xi) dr + \int_{S^{d-1}} \sigma(d\xi) \int_1^\infty r^2 q(r,\xi) dr,$$

and the first term of the right hand side above is clearly uniformly bounded in $\beta \in [2, \infty)$, and so is the second term, since for every $\beta \in [2, \infty)$,

$$\int_{S^{d-1}} \sigma(d\xi) \int_1^\infty r^2 q(r,\xi) dr \asymp \int_{S^{d-1}} c_2(\xi) \sigma(d\xi) \int_1^\infty r^2 \frac{dr}{r^{\beta+1}} = \frac{\sigma_2(S^{d-1})}{\beta - 2} = 1.$$

This concludes the proof in view of (3.3). □

Remark 3.4. The short time behavior (Theorem 3.1) and the (non-Gaussian) long time behavior (Theorem 3.2 (i)) can also be inferred from the series representation (2.10). For simplicity, consider the symmetric case. Letting $X_t := \sum_{i=1}^\infty \overleftarrow{q}(\Gamma_i/T, V_i) V_i \mathbf{1}(T_i \leq t)$, we have

$$h^{-1/\alpha} X_{ht} = \sum_{i=1}^\infty h^{-1/\alpha} \overleftarrow{q}(\Gamma_i/(hT), V_i) V_i \mathbf{1}(hT_i \leq ht),$$

and so for each $u > 0$ and each $\xi \in S^{d-1}$ such that $c_1(\xi) \in [0, \infty)$, bounded convergence gives

$$\begin{aligned} h^{-1/\alpha} \overleftarrow{q}(h^{-1}u, \xi) &= h^{-1/\alpha} \inf\left\{ r > 0 : \int_r^\infty q(s,\xi) ds < h^{-1} u \right\} \\ &= \inf\left\{ r > 0 : \int_r^\infty h^{1+1/\alpha} q(h^{1/\alpha} s, \xi) ds < u \right\} \\ &\to \inf\left\{ r > 0 : c_1(\xi) \int_r^\infty s^{-\alpha-1} ds < u \right\} = \left(\frac{\alpha u}{c_1(\xi)} \right)^{-1/\alpha}, \end{aligned}$$

as $h \to 0$, which is indeed an α-stable shot noise. The (non-Gaussian) long time behavior can be inferred just similarly.

4 Absolute continuity with respect to short time limiting stable process

Two Lévy processes, which are mutually absolutely continuous, share any almost sure local behavior. The next theorem confirms this fact in relation with the short time behavior result of Theorem 3.1. Indeed, given any layered stable process with respect to some probability measure, one can find a probability measure under which the layered stable process is identical in law to its short time limiting stable process. This result should be compared with Section 4 of Rosiński [10].

Recall that c_1 and c_2 are integrable (with respect to σ) functions on S^{d-1} appearing in (2.2) and (2.3), while σ_1 and σ_2 are the finite positive measures (2.4) and (2.5), respectively. As before, we use the notation $\nu_{\sigma,q}^{\alpha,\beta}$ for the Lévy measure of a layered stable process $X := \{X_t : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$, while ν_σ^α is the measure (2.6).

Theorem 4.1. *Let $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{T}$ be probability measures on $(\Omega, \mathcal{F})$ such that under $\mathbb{P}$ the canonical process $\{X_t : t \geq 0\}$ is a Lévy process in $\mathbb{R}^d$ with $\mathcal{L}(X_1) \sim LS_{\alpha,\beta}(\sigma, q; k_0)$, while under $\mathbb{Q}$ it is a Lévy process with $\mathcal{L}(X_1) \sim S_\alpha(\sigma_1; k_1)$. Moreover, when $\beta \in (0, 2)$ and under $\mathbb{T}$, $\{X_t : t \geq 0\}$ is a Lévy process with $\mathcal{L}(X_1) \sim S_\beta(\sigma_2; \eta)$, for some $\eta \in \mathbb{R}^d$. Then,*
(i) $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$k_0 - k_1 = \begin{cases} \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^1 rq(r,\xi)dr, & \alpha \in (0,1), \\ \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^1 r(q(r,\xi) - c_1(\xi)r^{-\alpha-1})dr, & \alpha = 1, \\ \frac{1}{\alpha-1} \int_{S^{d-1}} \xi\sigma_1(d\xi) + \int_{S^{d-1}} \xi\sigma(d\xi) \int_0^1 r(q(r,\xi) - c_1(\xi)r^{-\alpha-1})dr, & \alpha \in (1,2). \end{cases}$$

(ii) If $\alpha \neq \beta$, then for any choice of $\eta \in \mathbb{R}^d$, $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{T}|_{\mathcal{F}_t}$ are singular for all $t > 0$.
(iii) For each $t > 0$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{U_t},$$

where $\{U_t : t \geq 0\}$ is a Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$U_t := \lim_{\epsilon\downarrow 0} \sum_{\{s\in(0,t]:\|\Delta X_s\|>\epsilon\}} \left[\ln\left(\frac{q(\|\Delta X_s\|, \Delta X_s/\|\Delta X_s\|)}{c_1(\Delta X_s/\|\Delta X_s\|)\|\Delta X_s\|^{-\alpha-1}} \right) - t(\nu_{\sigma,q}^{\alpha,\beta} - \nu_{\sigma_1}^\alpha)(\{z \in \mathbb{R}_0^d : \|z\| > \epsilon\}) \right]. \tag{4.1}$$

In the above right hand side, the convergence holds $\mathbb{P}$-a.s. uniformly in t on every interval of positive length.

Proof. (i) By Theorem 33.1 and Remark 33.3 of Sato [13], it is necessary and sufficient to show that the following three conditions hold;

$$\int_{\{z:|\varphi(z)|\leq 1\}} \varphi(z)^2 \nu_{\sigma_1}^\alpha(dz) < \infty, \tag{4.2}$$

$$\int_{\{z:\varphi(z)>1\}} e^{\varphi(z)} \nu_{\sigma_1}^\alpha(dz) < \infty, \tag{4.3}$$

$$\int_{\{z:\varphi(z)<-1\}} \nu_{\sigma_1}^\alpha(dz) < \infty, \tag{4.4}$$

where the function $\varphi : \mathbb{R}_0^d \to \mathbb{R}$ is defined by $(d\nu_{\sigma,q}^{\alpha,\beta}/d\nu_{\sigma_1}^\alpha)(z) = e^{\varphi(z)}$, that is,

$$\varphi(z) = \ln\left(\frac{q(\|z\|, z/\|z\|)}{c_1(z/\|z\|)\|z\|^{-\alpha-1}} \right), \quad z \in \mathbb{R}_0^d.$$

Now, observe that

$$\lim_{\|z\|\to 0}\varphi(z)=\lim_{\|z\|\to 0}\ln\left(\frac{c_1(z/\|z\|)\|z\|^{-\alpha-1}}{c_1(z/\|z\|)\|z\|^{-\alpha-1}}\right)=0, \tag{4.5}$$

and that as $\|z\|\to\infty$,

$$\varphi(z)\sim\ln\left(\frac{c_2(z/\|z\|)\|z\|^{-\beta-1}}{c_1(z/\|z\|)\|z\|^{-\alpha-1}}\right)=\ln\left(\frac{c_2(z/\|z\|)}{c_1(z/\|z\|)}\right)+(\alpha-\beta)\ln\|z\|\to\begin{cases}-\infty, & \text{if } \alpha<\beta,\\ +\infty, & \text{if } \alpha>\beta.\end{cases} \tag{4.6}$$

The conditions (4.2) and (4.4) are thus immediately satisfied, respectively, by (4.5) and (4.6) with $\alpha<\beta$. In view of (4.6) with $\alpha>\beta$, the condition (4.3) is satisfied since $\int_{\{z:\varphi(z)>1\}}e^{\varphi(z)}\nu^{\alpha}_{\sigma_1}(dz)$ is bounded from above and below by constant multiples of $\int_{\|z\|>1}\frac{q(\|z\|,z/\|z\|)}{c_1(z/\|z\|)\|z\|^{-\alpha-1}}\nu^{\alpha}_{\sigma_1}(dz)=\nu^{\alpha,\beta}_{\sigma,q}(\{z\in\mathbb{R}^d_0:\|z\|>1\})$. When $\alpha=\beta\in(0,2)$, we have, by (4.5) and (4.6),

$$\begin{cases}\lim_{\|z\|\to 0}\varphi(z)=0,\\ \lim_{\|z\|\to\infty}\varphi(z)=\lim_{\|z\|\to\infty}\ln\left(\frac{c_2(z/\|z\|)}{c_1(z/\|z\|)}\right)<\infty.\end{cases}$$

The condition (4.2) is then satisfied since $\int_{\{z:|\varphi(z)|\le 1\}}\varphi(z)^2\nu^{\alpha}_{\sigma_1}(dz)$ is bounded from above and below by constant multiples of $\int_{\|z\|>1}\varphi(z)^2\nu^{\alpha}_{\sigma_1}(dz)$, which is further bounded by $C\nu^{\alpha}_{\sigma_1}(\{z\in\mathbb{R}^d_0:\|z\|>1\})$ for some constant C. The conditions (4.3) and (4.4) are also satisfied since the domains $\{z\in\mathbb{R}^d_0:\varphi(z)>1\}$ and $\{z\in\mathbb{R}^d_0:\varphi(z)<-1\}$ are contained in some compact sets of $\mathbb{R}^d_0$.

(ii) It suffices to show that either one of the following two conditions always fails;

$$\int_{\{z:\psi(z)>1\}}e^{\psi(z)}\nu^{\beta}_{\sigma_2}(dz)<\infty, \tag{4.7}$$

$$\int_{\{z:\psi(z)<-1\}}\nu^{\beta}_{\sigma_2}(dz)<\infty, \tag{4.8}$$

where the function $\psi:S^{d-1}\to\mathbb{R}$ is defined by $(d\nu^{\alpha,\beta}_{\sigma,q}/d\nu^{\beta}_{\sigma_2})(z)=e^{\psi(z)}$, that is,

$$\psi(z)=\ln\left(\frac{q(\|z\|,z/\|z\|)}{c_2(z/\|z\|)\|z\|^{-\beta-1}}\right),\quad z\in\mathbb{R}^d_0.$$

As in the proof of (i), observe that

$$\lim_{\|z\|\to\infty}\psi(z)=\lim_{\|z\|\to\infty}\ln\left(\frac{c_2(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\alpha-1}}\right)=0,$$

and that as $\|z\|\to 0$,

$$\psi(z)\sim\ln\left(\frac{c_1(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\beta-1}}\right)=\ln\left(\frac{c_1(z/\|z\|)}{c_2(z/\|z\|)}\right)+(\beta-\alpha)\ln\|z\|\to\begin{cases}+\infty, & \text{if } \alpha>\beta,\\ -\infty, & \text{if } \alpha<\beta.\end{cases}$$

Therefore, the condition (4.7) fails when $\alpha>\beta$ since

$$\int_{\{z:\psi(z)>1\}}e^{\psi(z)}\nu^{\beta}_{\sigma_2}(dz)=\nu^{\alpha,\beta}_{\alpha,q}(\{z\in\mathbb{R}^d_0:\varphi(z)>1\})=+\infty,$$

while (4.8) fails when $\alpha<\beta$ since $\nu^{\beta}_{\sigma_2}(\{z\in\mathbb{R}^d_0:\psi(z)<-1\})=+\infty$.

(iii) This is a direct consequence of (i) with Theorem 33.2 of Sato [13]. □

Remark 4.2. As in Remark 2.2, let

$$q(r,\xi) = \sigma(S^{d-1})^{-1}(r^{-\alpha-1}\mathbf{1}_{(0,1]}(r) + r^{-\beta-1}\mathbf{1}_{(1,\infty)}(r)),\ \xi \in S^{d-1}.$$

Then, the Lévy process $\{U_t : t \geq 0\}$ given in (4.1) becomes

$$U_t = (\alpha - \beta) \sum_{\{s\in(0,t]:\|\Delta X_s\|>1\}} \ln(\|\Delta X_s\|) - t\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

Intuitively speaking, $(d\mathbb{Q}/d\mathbb{P})|_{\mathcal{F}_t}$ replaces all β-stable jumps of a layered stable process up to time t (i.e., jumps with absolute size greater than 1) by the corresponding α-stable jumps without changing direction. Moreover, when $\alpha < \beta$, the Lévy measure ν of $\mathcal{L}(U_1)$ is concentrated on $(-\infty, 0)$ and is given by

$$\nu(-\infty, y) = \alpha^{-1}\sigma(S^{d-1})e^{\frac{\alpha}{\alpha-\beta}y}, \quad y < 0,$$

while when $\alpha > \beta$, it is concentrated on $(0,\infty)$ and is given by

$$\nu(y,\infty) = \alpha^{-1}\sigma(S^{d-1})e^{\frac{\alpha}{\alpha-\beta}y}, \quad y > 0.$$

Let us next restate the absolute continuity result (Theorem 4.1) based on the fact that a series representation generates sample paths of a Lévy process directly by generating every single jump. For simplicity, we consider the symmetric case. Let $\{Y_t : t \geq 0\}$ be an α-stable process with $\mathcal{L}(Y_1) \sim S_\alpha(\sigma; k_1)$. By Lemma 1.1, there exists a version of $\{Y_t : t \in [0,T]\}$ given by

$$Y'_t = \sum_{i=1}^{\infty} \left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} V_i\mathbf{1}(T_i \leq t) + k_1 t.$$

Also, let $\{X_t : t \geq 0\}$ be a layered stable process with $\mathcal{L}(X_1) \sim LS_{\alpha,\beta}(\sigma, q; k_0)$. In view of the series representation (2.11), there exists a version of $\{X_t : t \in [0,T]\}$ given by

$$X'_t = \sum_{i=1}^{\infty} \Bigg[\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta} \mathbf{1}_{(0,\sigma(S^{d-1})T/\beta)}(\Gamma_i) + \left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T} + 1 - \frac{\alpha}{\beta}\right)^{-1/\alpha} \mathbf{1}_{(\sigma(S^{d-1})T/\beta,\infty)}(\Gamma_i)\Bigg] V_i\mathbf{1}(T_i \leq t) + k_0 t,$$

where all the random sequences are the same as those appearing in $\{Y'_t : t \in [0,T]\}$ above. By Theorem 4.1, they are mutually absolutely continuous if and only if

$$k_0 - k_1 = \begin{cases} \frac{1}{\alpha-1}\int_{S^{d-1}} \xi\sigma_1(d\xi), & \text{if } \alpha \in (0,1)\cup(1,2), \\ 0, & \text{if } \alpha = 1. \end{cases}$$

We infer that the Lévy process $\{U_t : t \in [0,T]\}$ in the Radon-Nykodym derivative of Theorem 4.1 (iii), that is,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{U_t},$$

has a version given by

$$U'_t = -\frac{\alpha-\beta}{\alpha}\sum_{i=1}^{\infty} \ln\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)\mathbf{1}_{(0,\sigma(S^{d-1})T/\alpha]}(\Gamma_i)\mathbf{1}(T_i \leq t) - t\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

As a direct consequence, we have

$$\mathbb{P}(X \in B) = \mathbb{E}_{\mathbb{P}}[e^{U'_T}\mathbf{1}_B(Y')], \quad B \in \mathcal{B}(\mathbb{D}([0,T],\mathbb{R}^d)).$$

Moreover, in view of Theorem 33.2 of Sato [13],

$$\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} = e^{-U_t},$$

and so we can derive a version of $\{U_t : t \in [0,T]\}$ in terms of the jumps of the layered stable process as follows;

$$U_t'' = -\frac{\alpha-\beta}{\beta}\sum_{i=1}^{\infty} \ln\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)\mathbf{1}_{(0,\sigma(S^{d-1})T/\beta]}(\Gamma_i)\mathbf{1}(T_i \le t) - t\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

Similarly, we have

$$\mathbb{Q}(Y \in B) = \mathbb{E}_{\mathbb{Q}}[e^{-U_T''}\mathbf{1}_B(X')], \quad B \in \mathcal{B}(\mathbb{D}([0,T],\mathbb{R}^d)).$$

5 Concluding remarks

• The weak convergence towards a Brownian motion, proved in Proposition 3.3 (i), is interesting in the sense that a stable process with uniformly dependent components converges in law to standard Brownian motion, i.e., with independent components. It is also interesting to see how a stable process with independent components converges towards a Brownian motion. To this end, for $i = 1, \ldots, d$, let $a_i \in [0,\infty)$, let

$$b_{i+} := (0,\ldots,0,+1,0,\ldots,0), \quad b_{i-} := (0,\ldots,0,-1,0,\ldots,0),$$

where $+1$ and -1 are located in the i-th component, and finally set

$$\sigma(d\xi) := \sum_{i=1}^{d} \frac{2-\alpha}{2} a_i(\delta_{b_{i+}}(d\xi)+\delta_{b_{i-}}(d\xi)), \quad \xi \in S^{d-1},$$

where δ is the Dirac measure. Clearly, σ is a symmetric finite positive measure on S^{d-1}. Also, let $\{X_t^{(\alpha)} : t \ge 0\} \sim S_\alpha(\sigma;0)$. Then, if y_i is the i-th component of y, we have, using $\Gamma(x+1) = x\Gamma(x)$, $x > 0$,

$$\begin{aligned}
\mathbb{E}[e^{i\langle y, X_1^{(\alpha)}\rangle}] &= \exp\left[-\frac{\Gamma(1/2)\Gamma((2-\alpha)/2)}{2^\alpha\alpha\Gamma((1+\alpha)/2)}\int_{S^{d-1}} |\langle y,\xi\rangle|^\alpha\sigma(d\xi)\right] \\
&= \exp\left[-\frac{1}{2}\sum_{i=1}^{d}\frac{\Gamma(1/2)\Gamma(1+(2-\alpha)/2)}{2^{\alpha-2}\alpha\Gamma((1+\alpha)/2)}a_i|y_i|^\alpha\right] \\
&\to \exp\left[-\frac{1}{2}\sum_{i=1}^{d} a_i|y_i|^2\right], \quad \text{as } \alpha \uparrow 2.
\end{aligned}$$

Therefore, we get $\{X_t^{(\alpha)} : t \ge 0\} \xrightarrow{d} \{W_t : t \ge 0\}$ as $\alpha \uparrow 2$, where $\{W_t : t \ge 0\}$ is a Brownian motion with covariance matrix

$$\begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_d \end{pmatrix}.$$

• By making use of the absolute continuity of Lévy measures, we can derive two more forms of series representations for a layered stable process induced by the Lévy measure (2.7), with $\alpha < \beta$. With the notations of Theorem 4.1, we get for $z \in \mathbb{R}_0^d$,

$$\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_\sigma^\alpha}(z) = \mathbf{1}_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta}\mathbf{1}_{(1,\infty)}(\|z\|) \le 1,$$

and
$$\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\beta}}(z) = \|z\|^{\beta-\alpha}\mathbf{1}_{(0,1]}(\|z\|) + \mathbf{1}_{(1,\infty)}(\|z\|) \le 1.$$
Then, by the rejection method of Rosiński [9], the summands $\{\overleftarrow{q}(\Gamma_i/T, V_i)V_i\}_{i\ge1}$ in (2.11) can be respectively replaced by
$$\left\{\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha}\mathbf{1}\left(\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\alpha}}\left(\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha}V_i\right)\ge U_i\right)V_i\right\}_{i\ge1},$$
and
$$\left\{\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}\mathbf{1}\left(\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\beta}}\left(\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}V_i\right)\ge U_i\right)V_i\right\}_{i\ge1},$$
where $\{U_i\}_{i\ge1}$ is a sequence of iid uniform random variables on $[0,1]$, independent of all the other random sequences.

• In similarity to the work presented in [5], it is possible to define a notion of fractional layered stable motion (fLSm). Then, as in [5], fLSm will, in short time, be close to fractional stable motion (with inner index α) while in long time it is close to either fractional Brownian motion (if $\beta > 2$) or to fractional stable motion (with index $\beta < 2$).

• Let us observe some sample paths of a layered stable process, generated via the series representation (2.11). By Theorem 3.1 and 3.2, the entire situation is exhausted by the following three cases;

(i) $\alpha < \beta < 2$,
(ii) $\beta \in (2,\infty)$,
(iii) $\alpha > \beta$ with $\beta \in (0,2)$.

Figure 1 corresponds to the case (i) and typical sample paths of a symmetric layered stable process with $(\alpha,\beta) = (1.3, 1.9)$ are drawn in short, regular, and long time span settings. For better comparison, we also drew its corresponding 1.3-stable and 1.9-stable processes. All these sample paths are generated via the series representation (2.11) for a layered stable process, or the one given in Lemma 1.1 for stable processes. Three sample paths within each figure are generated on a common probability space in the sense that a common set of random sequences $\{\Gamma_i\}_{i\ge1}$, $\{V_i\}_{i\ge1}$ and $\{T_i\}_{i\ge1}$ are used. The desired short and long time behaviors are apparent. In the top figure, the layered stable process and its short time limiting stable process are almost indistinguishable in a graphical sense (of course, not in a probabilistic sense).

For the case (ii), we drew in Figure 2 typical sample paths of a symmetric layered stable process with $(\alpha,\beta) = (1.1, 2.5)$, along with its corresponding 1.1-stable process and a Brownian motion with a suitable variance. The layered stable process and the 1.1-stable process are generated dependently as before, while the Brownian motion is independent of the others. As expected, the long time Gaussian type behavior (Theorem 3.2 (ii)) is clearly apparent. These stable type short time and Gaussian type long time behaviors have long been considered to be very appealing in applications. Such a study for asset price modeling will be presented elsewhere [6].

Finally, for the case (iii), we give in Figure 3 typical sample paths of a symmetric layered stable process with $(\alpha,\beta) = (1.9, 1.3)$, along with its corresponding 1.9-stable and 1.3-stable processes. Unlike the sample path behaviors observed in Figure 1, the path of the layered stable processes behaves more continuously (like a 1.9-stable) in short time, while more discontinuously in long time (like a 1.3-stable). In the short time figure, the layered stable and the 1.9-stable are graphically indistinguishable.

• To finish this study, we briefly introduce another generalization of stable processes. Again, let μ be an infinitely divisible probability measure on $\mathbb{R}^d$ and without Gaussian component. Then, μ is *mixed stable*

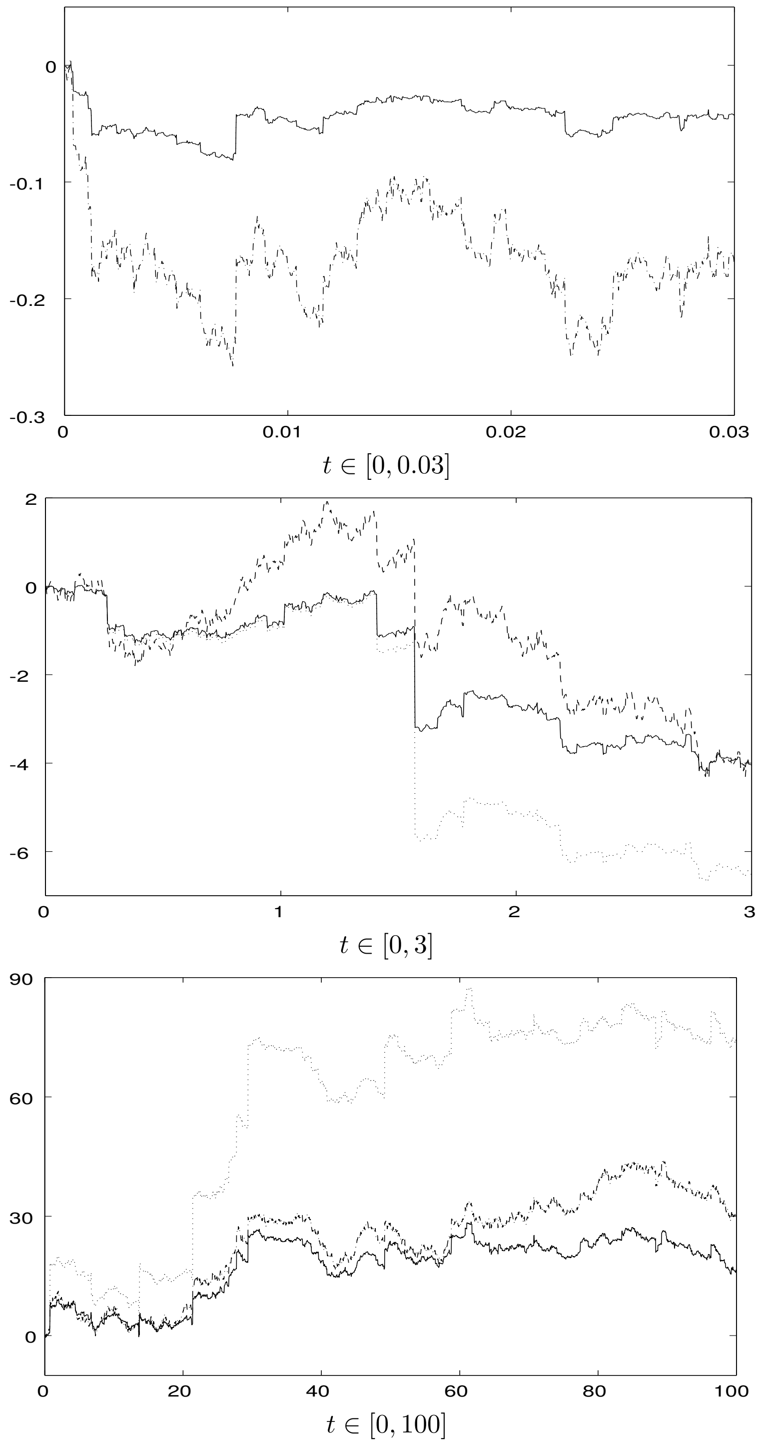

Figure 1: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.3, 1.9)$, 1.3-stable process $(\cdots)$, and 1.9-stable process (-·-)

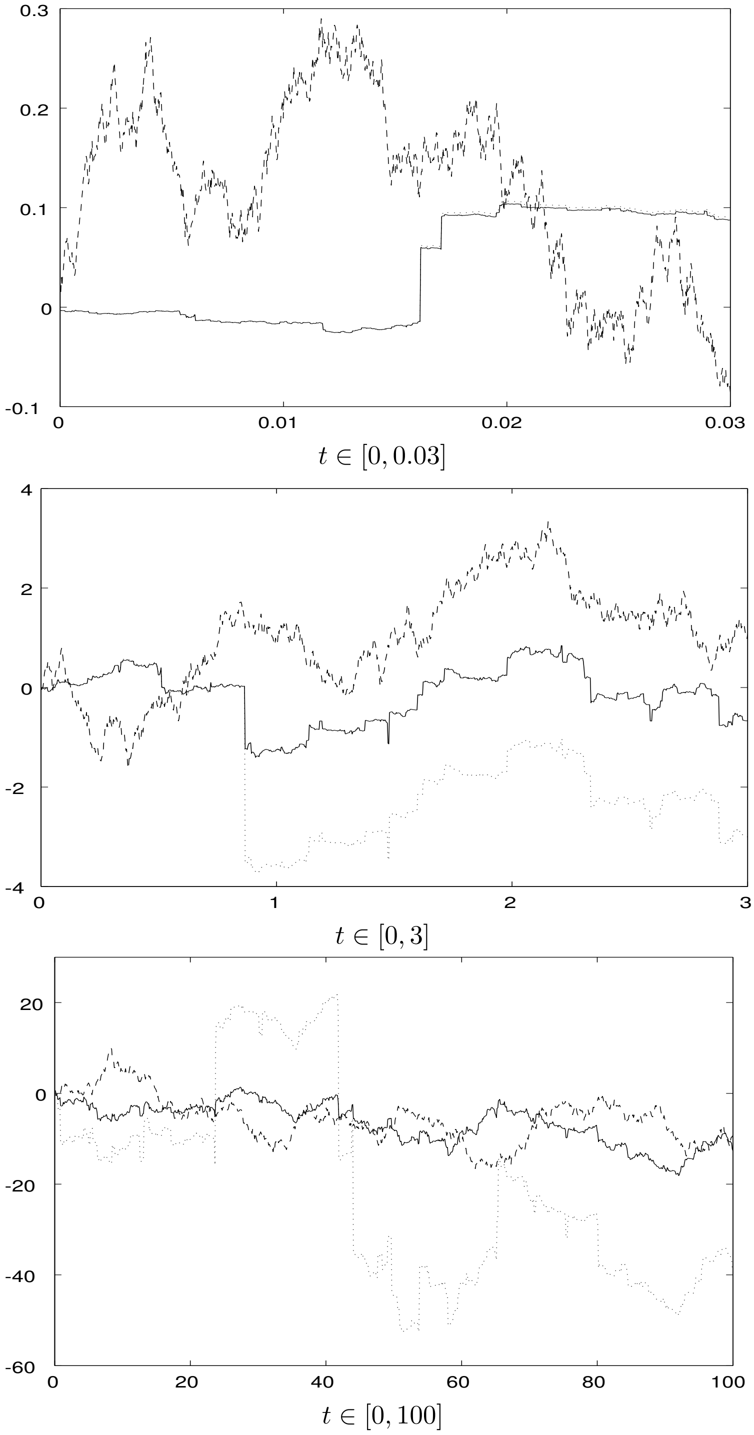

Figure 2: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.1, 2.5)$, 1.1-stable process $(\cdots)$, and a Brownian motion (-·-)

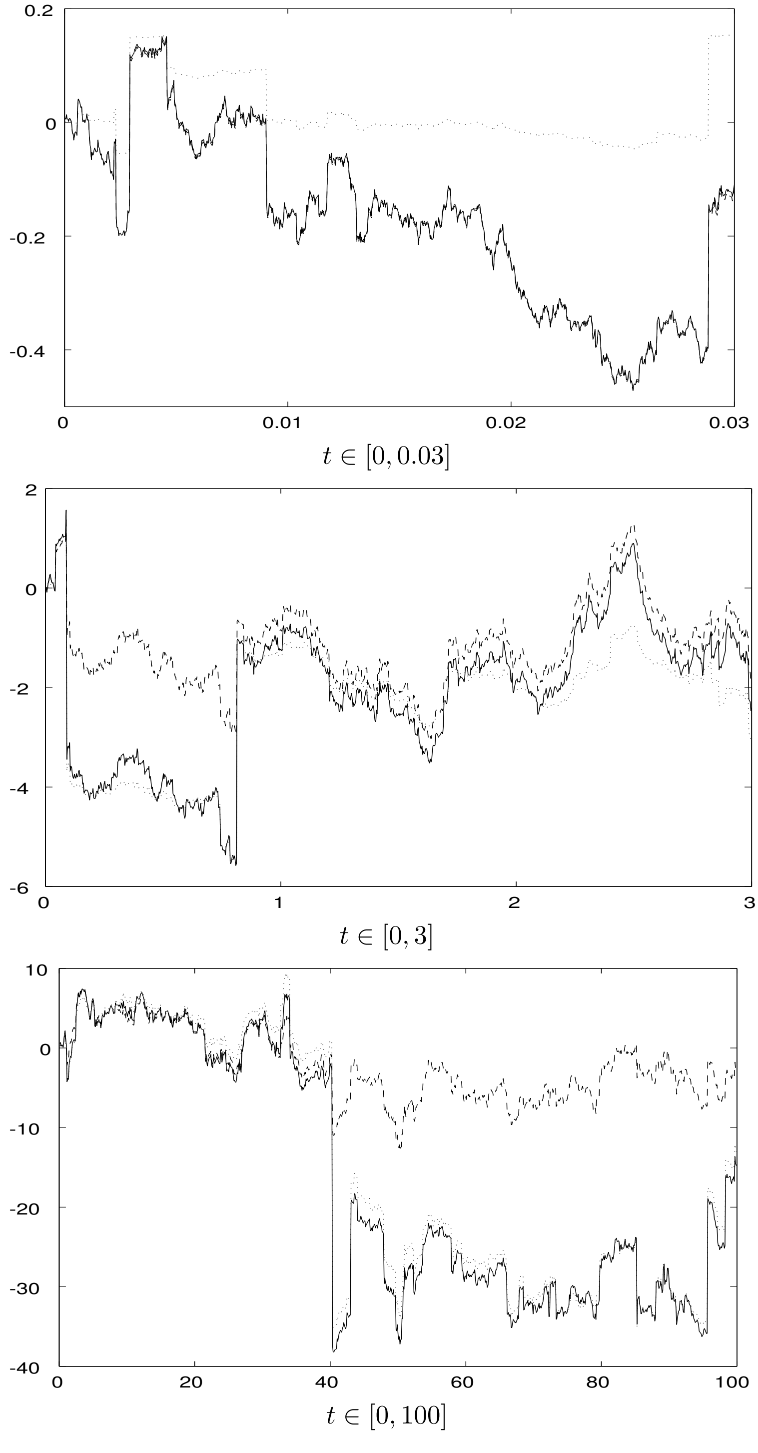

Figure 3: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.9, 1.3)$, 1.9-stable process (-·-), and 1.3-stable process $(\cdots)$

if its Lévy measure is given by

$$\nu(B) = \int_{(0,2)} \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{dr}{r^{\alpha+1}} \varphi(d\alpha), \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{5.1}$$

where φ is a probability measure on $(0, 2)$ such that

$$\int_{(0,2)} \frac{1}{\alpha(2-\alpha)} \varphi(d\alpha) < \infty.$$

Its characteristic function is given by

$$\begin{aligned} \widehat{\mu}(y) &= \exp\Bigg[i\langle y, \eta\rangle - \int_{(0,2)} c_\alpha \int_{S^{d-1}} |\langle y, \xi\rangle|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \mathrm{sgn}\langle y, \xi\rangle) \sigma(d\xi) \varphi(d\alpha) \\ &\qquad - \varphi(\{1\}) c_1 \int_{S^{d-1}} (|\langle y, \xi\rangle| + i \frac{2}{\pi} \langle y, \xi\rangle \ln |\langle y, \xi\rangle|) \sigma(d\xi) \Bigg], \end{aligned}$$

for some $\eta \in \mathbb{R}^d$, and where $c_\alpha = |\Gamma(-\alpha) \cos \frac{\pi\alpha}{2}|$ when $\alpha \neq 1$ while $c_1 = \pi/2$. Recall that in Example 2.2 we defined the classes L_m, $m = 0, 1, \ldots$ Let also $L_\infty := \cap_{m=0}^\infty L_m$. It is proved in Sato [12] that an infinitely divisible probability measure without Gaussian component is in L_∞ if and only if its Lévy measure has the form (5.1). We have seen in Example 2.2 that an infinitely divisible probability measure μ is in L_0 if and only if the Lévy measure of μ has the form

$$\int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) k_\xi(r) \frac{dr}{r}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and where $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. Recently, Barndorff-Nielsen et al.[1] defined the class T by further requiring that the function $k_\xi(r)$ be completely monotone in r for σ-a.e. ξ. Mixed stable distributions are indeed in the class T since $\int_{(0,2)} r^{-\alpha} \varphi(d\alpha)$ is completely monotone.

Finally, note that the associated Lévy process that we call a *mixed stable process* possesses an interesting series representation. For simplicity, assume that σ in (5.1) is symmetric. Let $\{\Gamma_i\}_{i\geq 1}$, $\{T_i\}_{i\geq 1}$ and $\{V_i\}_{i\geq 1}$ be random sequences defined as before. In addition, let $\{\alpha_i\}_{i\geq 1}$ be a sequence of iid random variables with common distribution φ. Assume moreover that all these random sequences are mutually independent. Then, with the help of the generalized shot noise method of Rosiński [9], it can be shown that the stochastic process

$$\left\{ \sum_{i=1}^\infty \left(\frac{\alpha_i \Gamma_i}{\sigma(S^{d-1}) T} \right)^{-1/\alpha_i} V_i \mathbf{1}(T_i \leq t) : t \in [0, T] \right\},$$

converges almost surely uniformly in t to a mixed stable process whose marginal law at time 1 is mixed stable with the Lévy measure (5.1). Comparing this result with the series representation of a stable process given in Lemma 1.1, a mixed stable process can be thought of as a stable process with each of its jumps obeying a randomly chosen stability index.

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