The Annals of Probability
1974, Vol. 2, No. 3, 456–463

ON LEBESGUE-LIKE EXTENSIONS OF FINITELY ADDITIVE MEASURES[1]

BY LESTER E. DUBINS

University of California, Berkeley, California

Dedicated, with a deep sense of loss and wonder, to the memory of a searching, pioneering, open, totally honest, generous, courageous, controversial man, a good friend, my friend, my teacher, my colleague, Leonard Jimmie Savage.

Those measures that can be extended in the spirit of Lebesgue to all open subsets of some topological spaces include not only the usual countably additive measures, but also a large class of purely finitely additive measures that have arisen naturally in the theory of finitely additive stochastic processes.

0. Introduction. Let $F_1, F_2, \cdots$ be a countable number of discrete topological spaces and let H be their Cartesian product in the usual Cartesian-product topology. An open subset of H whose complement is also open is *finitary*, and a finitely additive probability measure defined on the field of finitary subsets of H is *finitary*.

To illustrate, if each F_i is a finite set, then H is compact, and a subset of H is finitary if, and only if, it is a finite dimensional cylinder set, which helps explain the terminology.

Finitary probability measures for compact H will be called *elementary*. As is easily verified, every elementary probability measure is countably additive on the field of finitary subsets of H. Hence they can be, and usually are, extended so as to be countably additive on the Baire subsets of H. What are the advantages of this extension over the elementary finitary measure? over other possible finitely additive extensions? and does the possibility of this extension really require that the finitary measure be countably additive?

For a moment, specialize still further, and let each F_i consist of two points 0 and 1, and let σ be a *fair coin*, that is, a probability measure on the finitary subsets of H that assigns to each sequence $f_1, \cdots, f_n$ (or rather to the finite dimensional cylinder set determined by $f_1, \cdots, f_n$) a probability that depends only on n (and hence is 2^{-n}). Let S_n be $f_1 + \cdots + f_n$. The usual law of large num-

Received March 30, 1973; revised December 5, 1973.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, USAF, under Grant No. AFOSR-1312-67. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

AMS 1970 *subject classifications*. Primary 28A35, 28A60; Secondary 28A05, 60B05.

Key words and phrases. Measures, finitely additive measures, extensions of measures, Lebesgue extensions, gambling, dynamic programming, control theory, optimization, topological measures, stochastic process, finitely additive stochastic process, limit laws, law of large numbers, martingales, conglomerative.

bers is:

(L) The event E that S_n/n converges to $\frac{1}{2}$ has probability 1 .

As is easily seen, E is not a finitary event, and (L) is not valid for every finitely additive extension of a fair coin σ. Indeed, for each p, $0 \leqq p \leqq 1$, there is an extension that assigns E probability p. As is very well known, among the extensions of σ for which (L) is valid is the countably additive extension λ of σ. Since the λ-probability of events such as E are determined by the σ-probability of the (denumerable number of) finitary events, (L) should be translatable into an assertion about the σ-probabilities of finitary events. Indeed, for λ, (L) is equivalent to

(L′) For every $\varepsilon > 0$, there is an integer N such that, for each positive integer k ,

$$\text{(0)} \qquad \text{Prob}\left[\exists n : N \leqq n \leqq N + k, \text{ and } \left|\frac{S_n}{n} - \tfrac{1}{2}\right| > \varepsilon\right] < \varepsilon \,.$$

For a moment, call any extension of the fair coin σ, a *Fair coin*. As (L′) does not depend on the probabilities of any events other than finitary ones, it is universally valid, that is, it is valid for *all* Fair coins, and not merely for the countably additive Fair coin λ, as was brought home to me by Bruno de Finetti through L. J. Savage. Moreover, the validity of (L′) is not impaired even if one does not share the unconfirmable belief that, for a real-world coin, S_n/n must converge to a limit. Furthermore, (L′) is finitistic and useful, especially if $N = N(\varepsilon)$ is given constructively, as it surely can be. Finally, (L′) is suggestive of further research, e.g. what is the best $N(\varepsilon)$? or which is a good, simple $N(\varepsilon)$?

To me, it is noteworthy that those who search for probabilistic limit laws that are general, that is, that are not restricted to countably additive, measurable situations, are led to formulations such as (L′) which are of the same nature as those to which constructionists are led [2].

What has just been illustrated by the example of the law of large numbers seems to hold for most, if not all, the usual limit laws, and for the same reasons. Thus, for the most important purposes of probability, there seems to be no important need to extend finitary probabilities beyond the field of finitary sets. Yet there may be some justification in doing so. For, countably additive formulations such as (L) have one advantage over their universally valid counterpart (L′). They seem to be linguistically simpler. There may therefore be some interest in seeing whether other finitary probabilities, (including those that are not countably additive) do not have, among their extensions, a special one, that in the particular case of countably additive probabilities, reduces to the usual countably additive extension.

Among the finitary probability measures there are some that are easier to work with. If σ_0 is a probability measure defined on (all) subsets of F_1 and, for each partial history $p = (f_1, \cdots, f_n)$, $\sigma(p)$ is a probability measure defined on all subsets of F_{n+1}, then σ_0, together with $\sigma(\cdot)$ determine a finitary probability measure under which f_1 is distributed according to σ_0, and $\sigma(p)$ is the conditional distribution of f_{n+1} given p. This measure is also designated by σ, and is called a *strategic* measure.

2. Genesis of this paper. I quote from unpublished notes, dated September 3, 1962, written jointly with L. J. Savage.

"We suspect that (strategic measures) σ can be so extended that usual limit theorems, such as the martingale limit theorem, become literally true for strategies, that is, there may be a limit function to which the martingale converges. As a step in this direction, we have fairly well convinced ourselves that the inner measure of open sets leads to a consistent extension of the integral to finite linear combinations of the indicator functions of open sets. The fact behind this seems to be that in the present case, a closed-open set contained in the union of two open sets is the union of two closed-open sets, one contained in each of the open sets. We suspect that there may be a recapitulation of the Baire theory of functions and, in particular, that the limit function of some, or possibly all, uniformly bounded functions of the past will be integrable with respect to σ, with integration interchangeable with the taking of limits."

Much of the conjecture quoted above has already been verified. Namely, if *the Lebesgue extension* λ of a finitary probability measure σ is defined as the restriction of the inner measure of σ to the open subsets of H, λ is indeed additive and can be extended in one, and only one, way to be a probability measure λ' on the smallest field of sets that includes the open sets, as will be shown below. Of course, λ', like any probability measure, can be completed by extending λ' to all those sets whose inner and outer λ'-measures are equal. One strong, though not the only, way in which the quoted conjecture might come to pass is for all Baire subsets of H to be in the completion of λ', which is known to be the case if each F_i is a finite set. Recently, Roger Purves and William Sudderth have taken the important and difficult step forward of showing that, for strategic measures, every G_δ is in this completion. Their success motivated me to develop the aforementioned joint notes with Jimmie Savage and offer this paper.

What is presented below is in somewhat greater generality than what is necessary for the particular application which is the focus of attention above.

3. An abstract Lebesgue-like extension theorem. This section gives a necessary and sufficient condition on a pair of lattices L and L' of subsets, including the emply subset, of a set Ω so that, for every finitely additive measure σ on L, the restriction to L' of the inner measure of σ is a measure on L'.

Since every distributive lattice is isomorphic to a lattice of sets ([1] page 140), the material of this section can plainly be carried over to that more general setting.

In this section:

(i) R is the real line, or, more generally, a *real-like group*, that is, a partially ordered, commutative group in which each set bounded from above has a least upper bound;

(ii) Ω is a nonempty set;

(iii) A *lattice* is a collection of subsets of Ω that contains the set-theoretic union and intersection of each two of its members.

(iv) L and L' are lattices that contain the empty set 0.

(v) An *R-measure* or, more briefly, a *measure*, on a lattice L is a function σ defined on L with values in R which satisfies these three conditions

$$\sigma(A) + \sigma(B) = \sigma(A \cup B) + \sigma(A \cap B)\,; \tag{1}$$

$$A \subset B \quad \text{implies} \quad \sigma(A) \leqq \sigma(B)\,; \tag{2}$$

and

$$\sigma(0) = 0\,, \qquad \text{and,} \quad \text{for some} \quad x \in R\,, \qquad \sigma(A) \leqq x \quad \text{for all} \quad A\,. \tag{3}$$

(vi) The *inner measure* of σ is, as usual, that function σ_* defined for all subsets E of Ω, thus

$$\sigma_*(E) = \sup[\sigma(A) : A \leqq E,\, A \in L]\,. \tag{4}$$

(The assumptions (iv) that L contain the empty set and (3), that σ be bounded and vanish on the empty set, were made only so as to avoid unimportant complications in the definition of σ_*.)

If $L \subset L'$, there are usually many ways to extend a measure σ so as to be a measure λ on L'. For each such λ, $\lambda \geqq \sigma_*$ on L', as is evident. If σ_* restricted to L' is a measure, then it is the least such λ. Thus arises an interest in determining the circumstances under which σ_* is itself a measure on L'. Plainly, σ_* is *isotone*, that is, satisfies (2), and is *bounded*, that is, satisfies (3), even with the same upper bound x as does σ.

Say that (L, L') has the *Lebesgue-extension* property if, for every measure σ defined on L, the restriction of σ_* to L' is a measure on L'.

If, for every K, A_1, A_2 with $K \in L$, $A_i \in L'$, $i = 1, 2$, and $K \subset A_1 \cup A_2$, there exist $K_i \in L$, $i = 1, 2$ such that $K_i \subset A_i$ and $K = K_1 \cup K_2$, then (L, L') has the *allocation* property.

THEOREM 1. *A necessary and sufficient condition for a pair of lattices (L, L') that contains the empty set to have the Lebesgue extension property is that it have the allocation property.*

PROOF OF THEOREM. Suppose (L, L') has the allocation property. Let σ be a measure defined on L, and let λ be the restriction of σ_* to L'. Plainly, λ is isotone, that is, λ satisfies (2) on L', and λ is bounded from above, that is, λ satisfies (3) on L'. What must be seen is that λ is a *valuation*, that is, that it satisfies (1) on L'.

Let A and B be in L'. To see that

$$\lambda(A \cup B) + \lambda(A \cap B) \leqq \lambda(A) + \lambda(B)\,, \tag{5}$$

it plainly suffices to show:

$$\sigma(K) + \sigma(J) \leqq \lambda(A) + \lambda(B) \tag{6}$$

for all K and J in L such that

$$K \subset A \cup B \quad \text{and} \quad J \subset A \cap B \tag{7}$$

Since (L, L') has the allocation property, there are elements K_1 and K_2 of L whose union is $K \cup J$ for which $K_1 \subset A$ and $K_2 \subset B$. By replacing K_i by $K_i \cup J$, it may be assumed that $J \subset K_1 \cap K_2$. Therefore,

$$\begin{aligned} \sigma(K) + \sigma(J) &\leqq \sigma(K \cup J) + \sigma(J) \\ &\leqq \sigma(K_1 \cup K_2) + \sigma(K_1 \cap K_2) \\ &\leqq \sigma(K_1) + \sigma(K_2) \\ &\leqq \lambda(A) + \lambda(B) . \end{aligned} \tag{8}$$

where the penultimate inequality holds because of the assumption that σ is a valuation. To show that

$$\lambda(A \cup B) + \lambda(A \cap B) \geqq \lambda(A) + \lambda(B) , \tag{9}$$

it suffices to see that

$$\lambda(A \cup B) + \lambda(A \cap B) \geqq \sigma(K) + \sigma(J) \tag{10}$$

for all K and J in L for which $K \subset A$ and $J \subset B$. Plainly, the right-hand side of (10) equals $\sigma(K \cup J) + \sigma(K \cap J)$, which is clearly majorized by the left-hand side of (10). This completes the proof that λ is a measure.

Now suppose that (L, L') is a pair of lattices which does not enjoy the allocation property. Then there exist K, A_1, A_2 with $K \in L$, $A_i \in L'$, and $K \subset A_1 \cup A_2$ such that if $K_i \in L$, $i = 1, 2$ with $K_i \subset A_i$, then $K_1 \cup K_2$ is not K. What must be seen is the existence of a measure σ on L for which the restriction of σ_* to L' is not a measure. For this, the following preliminaries are useful.

Preliminaries to the remainder of the proof of Theorem 1. As usual, an *ideal* in a lattice L is a subset I of L such that both $A \cup B$ and $A \cap C$ are in I whenever A and B are in I and $C \in L$. The following two simple lemmas are perhaps well known, but knowing of no reference, I give their proofs.

LEMMA 1. *If M is a maximal ideal in a distributive lattice L, $A \in L$, $B \in L$, $A \cap B \in M$, then A or B is in M.*

PROOF. Suppose A is not in M, and let J be the ideal generated by M and A. Then $B \in J$, since J is L. So $B \leqq E \cup A$ for some $E \in M$. Hence, $B \leqq E \cup (A \cap B)$. But $E \cup (A \cap B)$ is in M, hence so is B.

A *zero-one* measure on L is a measure that assumes both 0 and 1 as values, and no value other than 0 or 1.

Plainly, the set of elements of measure 0 is an ideal in L. As contrasts with the special case in which L is a Boolean algebra, even if the measure is a zero-one measure, this ideal need not be maximal, as easy examples show. However, the converse does hold, namely:

LEMMA 2. *If M is a maximal ideal in L, a lattice of sets, then there is one, and only one, zero-one measure which has M for its ideal of sets of measure zero.*

PROOF. Let $\sigma(A) = 0$ or 1 according as $A \in M$ or $A \in L - M$. That σ is a measure is immediate from Lemma 1. That there is no other zero-one measure σ with M for its null sets is obvious.

Proof of Theorem 1 *continued.* For the existence of the required σ, suppose first that every element of L is a subset of K. Then the elements of L that are subsets of A_1 together with the elements of L that are subsets of A_2 generate a proper ideal I of L. By one of the usual transfinite arguments, among the proper ideals of L that include I there is a maximal one, say M. As is easily verified, M is a maximal ideal of L. For $A \in L$, let $\sigma(A)$ be 0 or 1 according as $A \in M$ or not. As accords with Lemma 2, σ is a measure. As is evident, $\sigma(K') = 0$ for every K' which is a subset of A_1 or of A_2. Hence $\sigma_*(A_1) = \sigma_*(A_2) = 0$. On the other hand, K is not in M, so $\sigma(K) = 1$ which implies that $\sigma_*(A_1 \cup A_2) = 1$. Hence σ_* restricted to L' violates (1) and is, therefore, not a measure. The necessity of the condition is thus established in the special case that every element of L is a subset of K.

The general case is easily reduced to the special case, thus. Let L^* be the ideal in L consisting of all elements of L that are subsets of K. On L^* there exists, as has just been shown, a zero-one measure σ that is 1 on K and 0 on every element of L^* that is a subset of A_1 or of A_2. Extend σ to arbitrary $E \in L$ by letting $\sigma(E)$ be $\sigma(E \cap K\}$. As is now trivial to verify, σ is a measure on L, but σ restricted to L' is not a measure on L'. This completes the proof of Theorem 1.

When Ω is a topological space and L' is the lattice of open subsets of Ω, the restriction to L' of the inner measure σ_* of a measure σ on L is *the Lebesgue-like extension* of σ.

Incidentally, as is not difficult to verify, if L is the lattice of all finite unions of compact intervals and L' is the lattice of open subsets of the real line (or of any finite dimensional Euclidean space) then (L, L') has the allocation property. Hence, in view of Theorem 1, *the Lebesgue-like extension of every finitely additive measure on L is a measure.*

What motivated this note, however, is a different example, which is the concern of the next section.

4. Extensions of finitary measures. The role of Ω in this section is assumed by H, the Cartesian product of a countable number of discrete topological spaces $F_1, F_2, \cdots$. A subset of a topological space is *clopen* if it and its complement are open.

PROPOSITION 1. *Every clopen subset of H included in the union of two open subsets A_1 and A_2 of H is the union of two clopen sets K_1 and K_2 with $K_1 \subset A_1$ and $K_2 \subset A_2$. Moreover, there exist K_1 and K_2 with the additional property of having a void intersection.*

PROOF OF PROPOSITION 1. Let $F_1, F_2, \cdots$ be the countable number of discrete factors of H with elements $f_1, f_2, \cdots$. Thus $h \in H$ if, and only if, $h = (f_1, f_2, \cdots)$ with $f_i \in F_i$. For $h \in H$ and $h' \in H$, and any positive integer n, write $h = h'(\text{mod}\, n)$ if $f_i = f_i'$, for all i, $1 \leqq i \leqq n$. For each subset A of H, define the *stop rule* t_A thus. For $h \in H$, $t_A(h)$ is the least positive integer n, if any, such that, for all h' for which $h = h'(\text{mod}\, n)$, $h' \in A$; and let $t_A(h) = \infty$ if there is no such n.

Now let A and B be open subsets of H, and let K be a clopen subset of H included in the union of A and B. Let K_1 be all $h \in K$ such that $h \in A$ and $t_A(h) \leqq t_B(h)$, and let K_2 be the complement of K_1 in K. Verify that K_1 and K_2 are clopen. Plainly, they are disjoint and their union is K. □

When H is the Cartesian product of discrete spaces, the term "finitary" will now be used in lieu of clopen. And a *finitary measure* is a nonnegative, finitely additive measure on the lattice of finitary subsets of H with values in any real-like group.

THEOREM 2. *The Lebesgue-like extension of every finitary measure on H is a measure on the lattice of open sets.*

PROOF. Immediate from Proposition 1 and Theorem 1.

In the special case in which every factor space F_i of H is finite and the values of the measure σ are real numbers, the Lebesgue-like extension of σ, defined above, is precisely the usual countably additive extension of σ restricted to the open subsets of H.

The Lebesgue-like extension λ of any finitary measure possesses one, and only one, finitely additive extension to the ring generated by the open sets, which is automatically nonnegative, and so can be completed, as follows from work of Horn and Tarski [4], and possibly from earlier work of de Finetti and from later work of Pettis [5].

Whether, for all strategic measures, the completion includes all Baire sets, as it obviously does in the special case of elementary measures, was an open question that has been settled in the affirmative by Purves and Sudderth [6] while this paper was in print.

REFERENCES

[1] BIRKHOFF, GARRETT (1948). *Lattice Theory*. Amer. Math Soc., New York.

[2] BISHOP, ERRETT (1967). *Constructive Analysis*. McGraw-Hill, New York.

[3] DUBINS, L. E. and SAVAGE, L. J. (1965). *How to Gamble if you Must*. McGraw-Hill, New York, out of print. Available, from University microfilms, Ann Arbor, Mich., and, probably soon, from Dover, N.Y.

[4] HORN, A. and TARSKI, A. (1948). Measures in Boolean algebras. *Trans. Amer. Math. Soc.* **64** 467–497.

[5] PETTIS, B. J. (1951). On the extension of measures. *Ann. of Math.* **54** 186–197.
[6] PURVES, ROGER and SUDDERTH, WILLIAM ( ). Some finitely additive probability. Technical Report No. 220, Univ. of Minnesota.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720