ON LIMIT ANALYSIS OF PLATES*

BY

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Summary. The bending of thin perfectly plastic plates of arbitrary shape under transverse load is studied. The collapse load for a concentrated force on such a plate is found to be 2π times the yield moment.

1. Introduction. In the following an attempt is made to generalize the work of Hopkins and Prager [1]† on the load-carrying capacities of circular plates to non-symmetrical cases. The basic equations for this purpose have been given previously by Hopkins [2] for a plate composed of a material obeying Tresca's yield criterion. We now discuss in detail the different types of plastic regimes that can arise. After a brief consideration of the discontinuities in the moment field, the technique of limit analysis [3] is used to determine the limit load for a concentrated force. Upper bounds on the limit loads in non-symmetrical cases have also been obtained by Rzhanitsyn [4]. Only in very artificial examples has it been found possible to obtain the actual moment distribution and deformation mode during collapse.

2. Equations of equilibrium in curvilinear coordinates. Let M_1, M_2 be the principal moments; Q_1, Q_2 the shearing forces; p the load per unit area; ρ_1, ρ_2 the radii of curvature and s_1, s_2 the arc-lengths of the principal stress trajectories (Fig. 1). Then the equations of equilibrium are

$$\frac{\partial M_1}{\partial s_1} + \frac{M_1 - M_2}{\rho_2} - Q_1 = 0,$$
$$\frac{\partial M_2}{\partial s_2} + \frac{M_1 - M_2}{\rho_1} - Q_2 = 0, \tag{1}$$
$$\frac{\partial Q_1}{\partial s_1} + \frac{\partial Q_2}{\partial s_2} + \frac{Q_1}{\rho_2} - \frac{Q_2}{\rho_1} = -p.$$

We now introduce orthogonal curvilinear coordinates u, v such that the principal stress trajectories are given by u = const. and v = const. Let

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{2}$$

be the first fundamental form of the net of stress trajectories.** We have then

$$F = 0, \qquad \frac{\partial}{\partial s_1} = \frac{1}{E^{1/2}}\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial s_2} = \frac{1}{G^{1/2}}\frac{\partial}{\partial v},$$
$$\frac{1}{\rho_1} = \frac{\partial \varphi}{\partial s_1} = \frac{\varphi_u}{E^{1/2}} = \frac{-E_v}{2E\,G^{1/2}}, \tag{3}$$
$$\frac{1}{\rho_2} = \frac{\partial \varphi}{\partial s_2} = \frac{\varphi_v}{G^{1/2}} = \frac{G_u}{2G\,E^{1/2}},$$

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†Numbers in square brackets refer to the Bibliography at the end of the paper.

**Because $F = 0$, it is not necessary to use here the tensor notation g_{ij}.

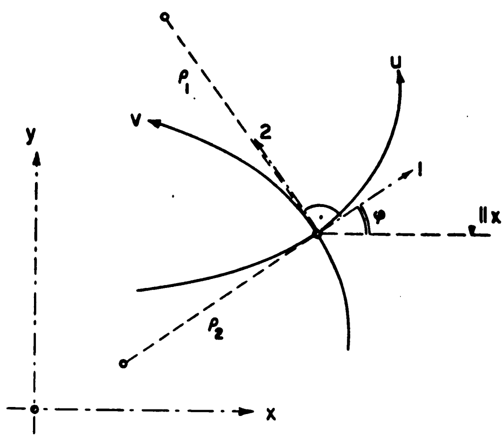

FIG. 1.

where φ is the angle of inclination of the curves v = const. to the x-axis, and subscripts denote partial differentiation. Further, it is convenient to introduce the quantities $\omega = (M_1 + M_2)/2$ and $\delta = (M_1 - M_2)/2$. The first two of Eqs. (1) are then

$$\frac{(\omega + \delta)_u}{E^{1/2}} + \frac{2\delta \cdot \varphi_v}{G^{1/2}} - Q_1 = 0, \qquad \frac{(\omega - \delta)_v}{G^{1/2}} + \frac{2\delta \cdot \varphi_u}{E^{1/2}} - Q_2 = 0, \tag{4}$$

and the third equation of equilibrium becomes

$$\frac{(Q_1)_u}{E^{1/2}} + \frac{(Q_2)_v}{G^{1/2}} + Q_1 \frac{G_u}{2G\, E^{1/2}} + Q_2 \frac{E_v}{2E\, G^{1/2}} = -p. \tag{5}$$

To eliminate Q_1, Q_2 we introduce (4) in (5) and get

$$\frac{\omega_{uu}}{E} + \frac{\omega_{vv}}{G} + \frac{1}{(EG)^{1/2}}\left[\left(\frac{G}{(EG)^{1/2}}\right)_u \omega_u + \left(\frac{E}{(EG)^{1/2}}\right)_v \omega_v\right] + \frac{\delta_{uu}}{E} - \frac{\delta_{vv}}{G} + \frac{1}{(EG)^{1/2}}\left[\left(\frac{G}{(EG)^{1/2}}\right)_u \delta_u - \left(\frac{E}{(EG)^{1/2}}\right)_v \delta_v\right] + 4\frac{\delta}{(EG)^{1/2}}\varphi_{uv} + 2\left[\frac{\varphi_v}{(EG)^{1/2}}\delta_u + \frac{\varphi_u}{(EG)^{1/2}}\delta_v\right] = -p. \tag{6}$$

3. The yield condition. Tresca's yield condition is shown in Fig. 2. For this condition only four essentially different types of regimes occur, namely two stress determined regimes, represented for example by the points A and B, and two kinematically determined regimes, represented for example by the sides AB and BC of the yield hexagon.

Regime A. As $\delta = 0$, the angle φ is indeterminate. This regime can occur in a finite region only if the pressure p is zero (see also Prager [5]).

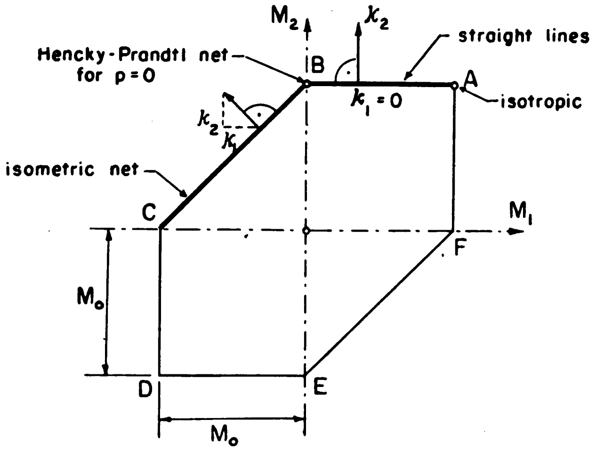

FIG. 2.

Regime B. In this regime $\omega = M_0/2$ and $\delta = -M_0/2$, where M_0 is the yield moment. Equation (6) becomes

$$2M_0\varphi_{uv} = p\,(EG)^{1/2}. \tag{7}$$

For the special case $p = 0$, we have

$$\varphi_{uv} = 0. \tag{8}$$

This is the condition that the stress trajectories form a Hencky-Prandtl-net [6].

Regime AB. For this regime it is necessary to consider the velocity field. The principal curvature-rates are denoted by κ_1 and κ_2 . For moment states represented by points on the side AB, the flow rule requires $\kappa_1 = 0$. Thus the surface into which the plate is deforming has *parabolic* curvature, and therefore one set of the principal lines of curvature is *straight* (see for example, Blaschke [7], p. 112). Because of the isotropy condition it follows that one family of the principal stress trajectories consists of straight lines. As $\omega - \delta = M_2 = M_0$ the second of Eqs. (1) becomes

$$Q_2 = 0. \tag{9}$$

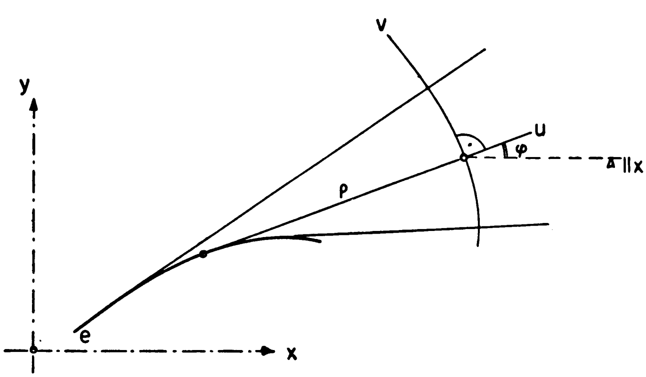

FIG. 3.

If we introduce $\rho = \rho_2$ and φ as new coordinates (Fig. 3) the first of Eqs. (1) gives

$$\frac{\partial(\rho M_1)}{\partial\rho} - M_0 = \rho Q_1 , \tag{10}$$

and the third of Eqs. (1) becomes

$$\frac{\partial(\rho Q_1)}{\partial\rho} + p\rho = 0. \tag{11}$$

This is taken from a manuscript by R. T. Shield. The integration of (10) and (11) leads to the expressions

$$M_1 = M_0 + A(\varphi) + \frac{B(\varphi)}{\rho} + \frac{1}{\rho}\int \rho^2 p \, d\rho - \int p\rho \, d\rho, \tag{12}$$

$$Q_1 = \frac{A(\varphi)}{\rho} - \frac{1}{\rho}\int p\rho \, d\rho, \tag{13}$$

where A and B are functions of φ only. The evolute e and the functions A and B are to be determined from the conditions of the problem, such as joining a region in regime AB to regions in other plastic regimes.

Regime BC. In this case we have $\kappa_1 = -\kappa_2$ which is the definition of a minimal surface. On such a surface the lines of principal curvature form an ***isometric net*** (Weatherburn [8]) and isometric coordinates u, v may be introduced for which $E = G$. If x, y are rectangular Cartesian coordinates, then $z = x + iy$ is an *analytic function* of $\chi = u + iv$ (see, for example [7] p. 179). Also $\varphi = \arg (dz/d\chi) = \arg z' = \text{Im} (\log z')$ is a harmonic function (Fig. 4).

The equilibrium equation (6) becomes with $E = G$ and $\delta = -M_0/2$ (from the yield condition)

$$\Delta\omega - 2M_0\varphi_{uv} = -pE, \tag{14}$$

where Δ is the Laplace operator referred to u, v. As $E = x_u^2 + y_u^2 = | z' |^2$, and as φ_{uv} is the real part of the analytic function

$$(\log z')'' = \frac{z'z''' - (z'')^2}{(z')^2} ,$$

Eq. (14) may be written

$$\Delta\omega - 2M_0 \,\text{Re}\left[\frac{z'z''' - (z'')^2}{(z')^2}\right] = -p \,| z' |^2. \tag{15}$$

We remark that instead of φ the conjugated function ψ may be used. Equation (14) may be easily transformed to the principal shear-lines, which also form an isometric net.

For the special case $p = 0$ one can find immediately two particular solutions of (14),

$$\begin{aligned} \omega_1 &= u\varphi_v M_0 = uM_0 \cdot \text{Re}\left[\frac{z''}{z'}\right], \\ \omega_2 &= v\varphi_u M_0 = vM_0 \cdot \text{Im}\left[\frac{z''}{z'}\right]. \end{aligned} \tag{16}$$

The general solution is therefore

$$\omega = \omega_1 + \omega_0 , \quad \text{or} \quad \omega = \omega_2 + \omega_0 , \tag{17}$$

where ω_0 is any harmonic function.

4. Boundary conditions. We suppose that the plate is at yield in bending near the boundary and discuss the three following types of boundary conditions: *a*) simply supported edge, *b*) clamped edge, *c*) free edge.

a) *The simply supported edge.* For the velocity field we have $W = 0$ at the edge, where W denotes the rate of normal deflection of the plate. For the moment field the moment M_n , acting in the direction of the normal n to the boundary, vanishes. This shows that at the edge either the two principal moments have opposite signs, or at least one of them is zero. We are therefore in regime B when the boundary is a principal stress trajectory, or in regime BC when the trajectories meet the boundary at an angle. In the latter case $\omega = \frac{1}{2}M_0 \cos 2\alpha$, α being the angle between the direction 1 and the outwards-drawn normal.

b) *Clamped edge.* Along a clamped edge where deformation of the plate occurs the edge is a principal stress trajectory.

To prove this let us suppose first that no hinge occurs along the edge. If u, v are orthogonal parameters such that at the edge $u = 0$, and if

$$\text{II} = L\, du^2 + 2M\, du\, dv + N\, dv^2 \tag{18}$$

is the second fundamental form of the surface into which the plate is deforming, we conclude from $W_u = W_v = W_{vv} = 0$ along the boundary $u = 0$, that

$$M = N = 0. \tag{19}$$

This shows that the edge is a line of principal curvature and therefore a line of principal stress. In the case $L = 0$ it follows from the fact that as $u \to 0$, $N/L \to 0$. If a hinge occurs, then $| M_n | = M_0$ is necessarily a principal moment. If the plate is bent downwards, then in the latter case $\omega = - M_0/2$.

c) *Free edge.* As for the simply supported edge we have the condition $\omega = (M_0/2) \cos 2\alpha$. In addition the shearing force vanishes at the edge,

$$\left(Q_n - \frac{\partial M_{nt}}{\partial t}\right)_{\text{edge}} = 0,$$

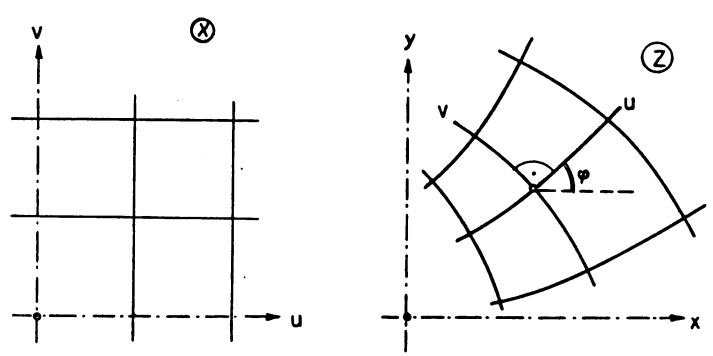

FIG. 4.

or

$$Q_n = -\frac{M_0}{2}\frac{\partial}{\partial t}(\sin 2\alpha) = -2\omega\frac{\partial \alpha}{\partial t}, \tag{20}$$

where Q_n can be calculated from (4), and where $\partial/\partial t$ denotes differentiation along the edge.

5. Discontinuities. The question of discontinuities in the moment field has been discussed previously by Prager [9] for the von Mises yield condition. In the following we investigate a discontinuity separating two regions which may or may not be in the same plastic regime. Figure 5 shows the plane of Mohr's circles and the image of Tresca's yield condition which is an oval of diameters M_0 and $2M_0$. Looking at the possible intersection of two circles, we see that three essentially different possibilities of strong discontinuities can arise, namely a jump of type "$BC \to BC$", which can be $BC \to BC$, $BC \to EF$, $EF \to EF$ (see notation of Fig. 1), or a jump of type "$BC \to AB$", or finally a jump of type "$BC \to B$".

a) Jump of type "$BC \to BC$". Denoting by M_n, M_{nt} the moments acting on an element of the line of discontinuity and by α_a, α_i the angles of the stress trajectories on both sides with respect to the normal, we conclude from Fig. 5 that

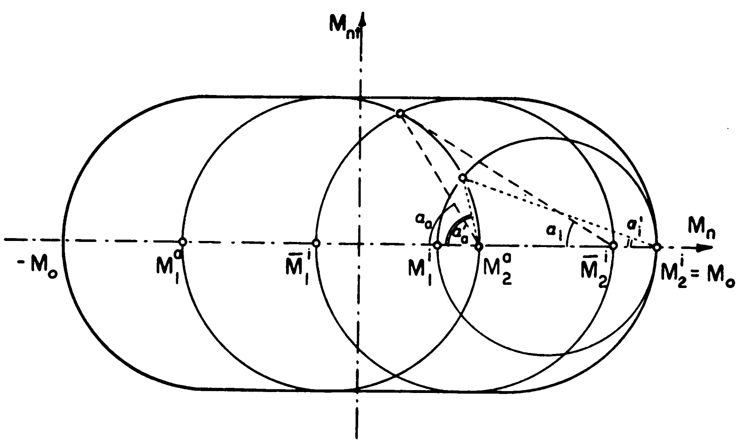

FIG. 5.

$$\alpha_i = \frac{\pi}{2} - \alpha_a \text{ (or } \alpha_i = \pi - \alpha_a), \tag{21}$$

because M_n, M_{nt} are continuous. When there is deformation of the plate near the discontinuity and assuming that $\alpha_a \neq 0, \pi/2$, both of the curvature-rates κ_t and κ_{nt} must be continuous. This requires

$$\left.\begin{aligned} \kappa_1^i \cos 2\alpha_i &= \kappa_1^a \cos 2\alpha_a, \\ \kappa_1^i \sin 2\alpha_i &= \kappa_1^a \sin 2\alpha_a, \end{aligned}\right\} \tag{22}$$

because $\kappa_1 = -\kappa_2$ on both sides of the line. From this and from (21) it follows that $\kappa_1 = \kappa_2 = 0$ on the line of discontinuity which would therefore be a rigid fiber and a line of principal curvature. Thus α_i, $\alpha_a = 0, \pi/2$, but then κ_1, κ_2 can have any value.

A moment discontinuity between two regions which are both in the plastic regime BC is not permissible, but can take place between BC and EF.

b) Jump of type "$BC \to AB$". It is sufficient to consider the two cases of a jump from BC to AB and the case of a jump from BC to AF, all other cases being similar. Let us start with the first one.

The continuity of M_n and M_{nt} requires

$$\left.\begin{aligned} M_1^i \cos^2 \alpha_i' + M_0 \sin^2 \alpha_i' &= M_1^a + M_0 \sin^2 \alpha_a' , \\ (M_0 - M_1^i) \sin 2\alpha_i' &= M_0 \sin 2\alpha_a' , \end{aligned}\right\} \quad (23)$$

index i belonging to AB, index a to BC. The velocity field satisfies the conditions

$$\left.\begin{aligned} \kappa_2^i \cos^2 \alpha_i' &= \kappa_2^a \cos 2\alpha_a' , \\ \kappa_2^i \sin 2\alpha_i' &= 2\kappa_2^a \sin 2\alpha_a' . \end{aligned}\right\} \quad (24)$$

From (24) we get, κ_2^i , κ_2^a supposed non-zero,

$$\left.\begin{aligned} \operatorname{tg} 2\alpha_a' &= \operatorname{tg} \alpha_i' , \\ 2\alpha_a' &= \alpha_i' + m\pi, \end{aligned}\right\} \quad (25)$$

where m is 0 or 1. Equations (23) are now

$$\left.\begin{aligned} M_1^i \cos^2 \alpha_i' + M_0 \sin^2 \alpha_i' &= M_1^a + \frac{M_0}{2} (1 \mp \cos \alpha_i'), \\ 2(M_0 - M_1^i) \cos \alpha_i' &= \pm M_0 , \end{aligned}\right\} \quad (26)$$

where the upper sign refers to the case $m = 0$. Taking either sign we get from (26) $M_1^a = M_0/2$, which is not true. The assumption κ_2^i , $\kappa_2^a \neq 0$, is therefore false. Again we conclude that α_i' , $\alpha_a' = 0, \pi/2$. Similar considerations for the angles can be made for jumps between BC and AF.

From regime BC we pass to AB over a line of principal curvature which is in regime B. If this line is not straight it must be a line of the set 2 of the stress trajectories and $M_n = M_1 = 0$. The discontinuity is then a weak discontinuity in the terminology of

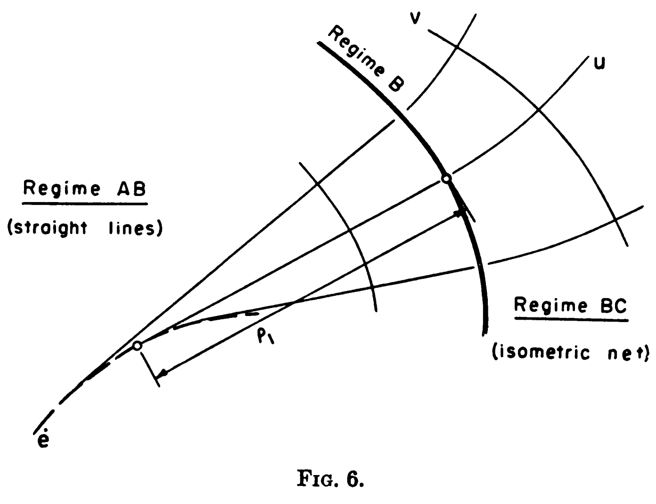

FIG. 6.

Prager [9]. Figure 6 shows the behavior of the fields when we suppose that regime B does not hold throughout a finite region.

The evolute e is connected with the regime-B-line by the condition

$$\frac{1}{\rho_1} = \frac{E_u}{2\,E^{3/2}}\,, \tag{27}$$

where E is the scale factor of the isometric net at the regime-B-line.

For the moments we have the conditions

$$M_0 + A + \frac{B}{\rho_1} + \frac{1}{\rho_1}\int \rho_1^2 p\, d\rho_1 - \int \rho_1 p\, d\rho_1 = 0, \qquad \omega = \frac{M_0}{2}\,, \tag{28}$$

and for the normal component of the shear-force to be continuous,

$$\frac{1}{\rho_1}\left[A - \int p\rho_1\, d\rho_1\right] = \frac{\omega_u^2}{E^{1/2}} - \frac{M_0 E_u}{2E^{3/2}}. \tag{29}$$

Until now, no non-trivial examples of such a weak discontinuity have been found.

Further the jumps of type "$BC - B$" have not yet been studied.

6. Concentrated loads. The following consideration is not a direct application, but shows how useful the discussed fields are for the determination of the limit load. Let us take a plate with a convex boundary which is *simply supported* at its edge and which is loaded by a concentrated force (Fig. 7).

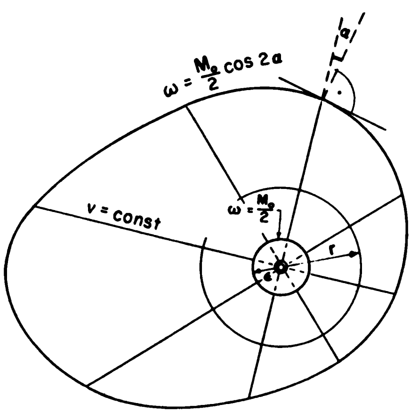

FIG. 7.

a) In order to get a *statically admissible stress field* [3] let us cut out a small circular region of radius ϵ around the force and consider the isometric net indicated in the figure. The corresponding isometric coordinates u, v are chosen such that $v = \varphi =$ const. are straight lines, and $u = \log r$, r being a measure of distance from the point of application of the load P. At the simply supported edge $\omega = \frac{1}{2}M_0 \cos 2\alpha$, and at the small circle $\omega = \frac{1}{2}M_0$. If we assume that the plate is in regime BC then Eq. (14) becomes

$$\Delta\omega = 0. \tag{30}$$

As $-M_0/2 \leq \omega \leq M_0/2$ on both boundaries, we conclude from a principal theorem on harmonic functions (Courant and Hilbert [10]) that ω satisfies the inequality

$$-\frac{M_0}{2} \leq \omega \leq \frac{M_0}{2} \tag{31}$$

in the region between the boundary and the small circle. The stress point therefore lies between the two points B, C on the yield curve, and as the harmonic function ω exists we have found a statically admissible stress field.

Let us now calculate the shearing force across the small circle. From (4) we get

$$E^{1/2} Q_1 = \omega_u - \varphi_v M_0 , \quad \varphi_v = 1, \quad -P = \int_0^{2\pi} E^{1/2} Q_1 \, dv = \int_0^{2\pi} \omega_u \, dv - 2\pi M_0 . \tag{32}$$

As ω_u is of the order $1/\log \epsilon$ the limit of this equation, when $\epsilon \to 0$, will be

$$\underline{P = 2\pi M_0 ,} \tag{33}$$

which is a lower bound for the limit load.

b) In order to find a *kinematically admissible* velocity field we take the isometric net of Fig. 8 which is given by the analytic function

$$z = f(e^{\chi}). \tag{34}$$

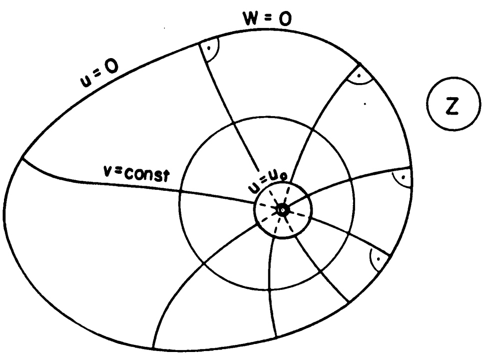

FIG. 8.

The function f is such that the transformation $\zeta = f^{-1}(z)$ maps the boundary of the plate into the unit circle and the point of application of the load into the origin $\zeta = 0$. The parameter u is then zero on the boundary and negative inside, and u has a logarithmic singularity at the loaded point. The velocity field $W = -uW_0$ satisfies the boundary conditions and the conditions of regime BC and is therefore a kinematically admissible velocity field. As the rate of work done by the concentrated load would be infinite, we replace the area inside the small "circle" $u = u_0 =$ const. by a flat area, for which the rate of dissipation is zero. It remains to calculate the dissipation outside this "circle" and at the hinge on $u = u_0$. As

$$\kappa_1 = \frac{-(L^2 + M^2)^{1/2}}{E} = \frac{-W_0(\varphi_v^2 + \varphi_u^2)^{1/2}}{E} ,$$

(this net does not represent the lines of curvature),

$$(D_{\text{total}})_{\text{outside}} = M_0 \cdot W_0 \int_0^{2\pi} dv \int_{u_0}^{0} (\varphi_u^2 + \varphi_v^2)^{1/2} \, du$$
$$\leq 2\pi M_0 W_0 \mid u_0 \mid + M_0 W_0 \int_0^{2\pi} dv \int_{u_0}^{0} \mid \varphi_u \mid du. \tag{35}$$

At the hinge Fig. 9 shows that

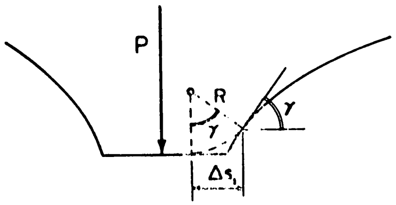

Fig. 9.

$$\operatorname{tg} \gamma = \frac{-dW}{ds_1} = \frac{-W_u}{E^{1/2}} = \frac{W_0}{E^{1/2}},$$

$$R = \frac{\Delta s_1}{\operatorname{tg} \gamma} = \frac{\Delta s_1 E^{1/2}}{W_0},$$

$$(D_{\text{total}})_{\text{hinge}} = M_0 \frac{W_0 \Delta s_1}{\Delta s_1 E^{1/2}} \cdot 2\pi E^{1/2} = 2\pi M_0 W_0 . \tag{36}$$

The second fundamental theorem of limit analysis gives therefore

$$PW_0 \leq 2\pi M_0 W_0 + \frac{2\pi M_0 W_0}{\mid u_0 \mid} + \frac{M_0 W_0}{\mid u_0 \mid} \int_0^{2\pi} dv \int_{u_0}^{0} \mid \varphi_u \mid du, \tag{37}$$

and in the limit

$$\underline{P \leq 2\pi M_0 ,} \tag{38}$$

which is an upper bound on the limit load. The exact value of the limit load is therefore $2\pi M_0$.

We remark that this result is also true for a concentrated load on a clamped plate. The same statically admissible stress field can be used, and for the velocity field we must only take into account the dissipation at the hinge produced at the clamped edge. This gives in Eq. (37) an additional term, which is of order $1/\mid u_0 \mid$ and which is zero in the limit. However as R. M. Haythornthwaite pointed out, the statement can be proved in this case in a more elementary way.

7. Saint-Venant's principle in plasticity. The application of the concentrated load is only of theoretical value, because in practice the load will always be distributed over a small area. However the examples of circular plates given by Prager and Hopkins show that the following conjecture may be true:

If the diameter of the area, where the load acts, is small in comparison with the distance from the edge, the limit load will be near the value $2\pi M_0$, and $2\pi M_0$ is the lowest limit load for the plate.

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