On Limitations to the Achievable Path Tracking Performance for Linear Multivariable Plants

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***Abstract*— In this paper we consider a problem termed "path tracking". This differs from the common problem of reference tracking, in that here we can adjust the speed at which we traverse the reference trajectory. We are interested in ascertaining the degree to which we can track a given trajectory, and in characterizing the class of paths for which we can generate an appropriate temporal specification so that the path can be tracked arbitrarily well in an L_2 sense.**

Keywords: path tracking, L_2 performance limitations, nonminimum phase systems.

I. Introduction

The problem of tracking - causing the output $y(t)$ of a dynamic system to follow a commanded trajectory $r(t)$ - is a classical problem in systems control. It was realized some time ago (see, for example, [13] for the linear case, or [15] for a nonlinear version of these results) that the lack of a stable inverse (or the existence of unstable zero dynamics) limits the achievable tracking performance. It is therefore of interest to investigate circumstances under which additional information or an alternate problem formulation can be used to enhance the tracking performance.

One means by which tracking performance restrictions might be relaxed is the use of "preview" control, wherein advance knowledge of the trajectory to be tracked permits improved performance, e.g. see [5] and [6]. Another means to enhance performance is to consider an alternate problem formulation: in some situations, as in the problem of steering an object (such as a ship, a robot, or a cutting tool), the primary objective is to follow a certain path, with the speed at which the path is traversed being of secondary importance. In this context one introduces a "path variable" (or "timing signal") $\theta(t)$, which creates an extra degree of freedom, so that the goal is to make $y(t)-r(\theta(t))$ small rather than $y(t)-r(t)$ small. Problems of this form have been considered for some time, with recent work including [16], [1], [8], [2], [3], and [7], wherein it is is shown that such *path tracking* problems permit significant improvement in the achievable performance. One fundamental objective is to ascertain sufficient conditions on r to guarantee that the tracking performance can be made as small as desired (by a suitable choice of control and path variable or timing signal) - the nonlinear case is considered in [8] and [3] with the linear case considered in [1], [2] and [7]; the approaches are constructive. In particular, in the linear case it is proven in [2] that it is sufficient that $r(t)$ be a finite sum of sinusoids (of non-zero frequency) and it is proven in [7] that (roughly speaking) it is sufficient that the path be repeatable and that the origin lie in the interior of its convex hull.

Here we consider the linear case and we are interested in answering two questions: (i) For which trajectories can we make the cost as small as desired? and (ii) How do we compute the (infimal) optimal cost when it is non-zero? Our focus is analysis, although synthesis is achievable with additional work. We provide conditions, weaker than those listed above, to ensure that the cost can be made as small as desired, and in a reasonably general situation we show how to compute the optimal cost by solving a finite dimensional convex optimization problem. Because of space constraints, all proofs will be omitted.

A. Preliminary Mathematics

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}^+$ denote the set of non-negative numbers. The norm of $x \in \mathbf{R}^n$ is the Holder 2-norm; with $A \in \mathbf{R}^{n\times m}$, the corresponding induced norm is denoted by $\|A\|$, and $\sigma_{min}(A)$ denotes the smallest singular value of A.

We let $PC(\mathbf{R}^n)$ denote the set of $\mathbf{R}^n-$valued piecewise continuous signals on $\mathbf{R}^+$, and with $k \in \mathbf{Z}^+$ we let $\mathcal{C}^k(\mathbf{R}^n)$ denote the set of $\mathbf{R}^n-$valued continuous signals on $\mathbf{R}^+$ which are k times continuously differentiable. We let $L_2(\mathbf{R}^n)$ denote the set of $\mathbf{R}^n$-valued, Lebesgue measureable, square integrable functions and use $||f||$ to denote the norm of $f \in L_2$. Henceforth we write $f \in PC$, $f \in \mathcal{C}^k$ and $f \in L_2$, as appropriate; the dimension will be clear from the context.

II. The Setup

Here we consider a square plant model of the form

$$\dot{x}_p = A_p x_p + B_p u \quad (1)$$

$$y = C_p x_p, \quad (2)$$

with $x(t) \in \mathbf{R}^n$ and $u(t), y(t) \in \mathbf{R}^m$. We assume that (A_p, B_p) is controllable, (C_p, A_p) is observable, and the system has no transmission zeros (in the sense of [9]) on the imaginary axis, i.e.

$$\text{rank} \begin{bmatrix} A_p - j\omega I & B_p \\ C_p & 0 \end{bmatrix} = n + m, \;\; \omega \in \mathbf{R}.$$

A. Isolating the Zero Dynamics

Following a linear version of [11], under some modest assumptions we can transform the plant to isolate the zero dynamics. For simplicity, we derive this in the case of "uniform relative degree"; the more general vector relative degree case can be carried out with similar analysis to that which follows, but with substantially more complexity.

Definition 1: The transfer function $C(sI - A)^{-1}B$ is said to have *uniform relative degree* p if

$$CA^j B = 0, \;\; j = 0, 1, ..., p-2$$

and $CA^{p-1}B$ is nonsingular.

Standing Assumption 1: $C_p(sI - A_p)^{-1}B_p$ has uniform relative degree p.

With

$$v^T := \begin{bmatrix} y^T & (y^{(1)})^T & \cdots & (y^{(p-1)})^T \end{bmatrix}, \quad (3)$$

it can be shown that we can choose $x = \begin{bmatrix} x^+ \\ x^- \end{bmatrix} = Tx_p$ with $T \in \mathbf{R}^{(n-mp)\times n}$ so that the plant model can be rewritten in the form

$$\begin{aligned} \dot{x}^+ &= A^+ x^+ + B^+ y \\ \dot{x}^- &= A^- x^- + B^- y \\ \dot{v} &= A_v v + A_1 x^+ + A_2 x^- + B_v u \\ y &= Cv \end{aligned} \quad (4)$$

with A^- and $-A^+$ Hurwitz and the transmission zeros of (1)-(2) equal to the eigenvalues of A^- and A^+.

B. The Control Objective

In this paper the objective will be to control the system in order to follow a pre-specified path r, parametrized by the timing signal θ, while maintaining closed loop stability; the tracking error is defined by

$$e(t) := r(\theta(t)) - y(t).$$

The objective is different from the common one of reference tracking: we are allowed to choose $\theta(t)$ to follow the path as rapidly or as slowly as desired; indeed, in some cases it is permissible to reverse course, although we distinguish between the case where we must always go forward and that for which we need not. Given that the plant is strictly proper, we cannot expect to asymptotically track an arbitrary $r \in PC$; since p represents the uniform relative degree, it is natural to require that $r \in \mathcal{C}^p$. Furthermore, while in this context we would typically want to track bounded trajectories (often periodic), we will allow for trajectories which are unbounded as long as they diverge more slowly than exponentially. To this end, we define

$$\hat{L}_\infty := \{f \in PC : \text{ for every } \sigma > 0 \text{ we have } \sup_{t \geq 0} e^{-\sigma t}\|f(t)\| < \infty\}$$

and the class of admissible trajectories by

$$\mathcal{R}_p := \{r \in C^p : r^{(i)} \in \hat{L}_\infty, \;\; i = 0, 1, ..., p\}.$$

Now we consider the class of timing signals θ. Since the reference signal will be $r \circ \theta$, we will need θ to be sufficiently smooth - it must belong to $\mathcal{C}^p$. We will also insist that eventually it traverse the trajectory at the original speed. Now we define two different classes of timing signals.

Definition 2: Θ_p is the set of $\theta \in \mathcal{C}^p$ for which $\theta(0) = 0$, $\theta(\cdot) \geq 0$, and for which there exists a $T > 0$ so that

$$\dot{\theta}(t) = 1, \;\; t \geq T.$$

Θ_p^+ is the subset of those $\theta \in \Theta_p$ for which $\theta(t)$ is monotonically increasing.

Remark 1: The constraint on θ ensures that eventually we will be trying to track the unscaled trajectory. Allowing $\dot{\theta}$ to be negative is akin to allowing one to maneuver a car into a tight parking spot.

Remark 2: In practise there are a number of other constraints that may be important for path tracking problems, including a constraint on the maximum rate of change of θ, a constaint on the maximum T, or actuator constraints. While these features may be important in practice, they add substantial further complications to the analysis presented here.

Remark 3: It is routine to confirm that $\theta \in \Theta_p$ and $r \in \mathcal{R}_p$ implies that $r \circ \theta \in \mathcal{R}_p$.

Given the problem setting, we allow feed-forward control and perfect prior knowledge of r. In this context, stability can be expressed in terms of how fast the control signal and state are allowed to grow.

Definition 3: $(u, \theta) \in PC \times \Theta_p$ is *stabilizing* if $u, x \in \hat{L}_\infty$, $\lim_{t\to\infty}[y(t) - r(\theta(t))] = 0$, and $J(r, \theta, u) := \|y - r(\theta)\|_2 < \infty$, in which case we write $(u, \theta) \in S(r)$; we define $S^+(r)$ in an analogous way.

Our control problems are two-fold:

(i) Given $r \in \mathcal{R}_p$, compute the following two quantities:

$$J_{opt}(r) := \inf_{(u,\theta)\in S(r)} J(r,\theta,u),$$

$$J^+_{opt}(r) := \inf_{(u,\theta)\in S^+(r)} J(r,\theta,u).$$

(ii) Second, characterize the subset of $r \in \mathcal{C}^p$ for which $J_{opt}(r)$ and $J^+_{opt}(r)$ are zero, i.e. those trajectories for which we can obtain near optimal path-following.

We will be able to convert the first problem into an unconstrained convex optimization problem, at least to compute $J_{opt}(r)$; we will also prove that $J_{opt}(r) = J^+_{opt}(r)$ if r is periodic, as it often is. We have a partial solution to the second problem.

The next step is to change the optimization problem from one in terms of both u and θ to one solely in terms of θ. There are two possible approaches. One is to convert the problem into "error coordinate form", as in [7], while the other is to use the original representation, which is what we do here.

Standing Assumption 2: $r \in \mathcal{R}_p$.

III. Simplifying the Problem

Here we show how to simplify the optimization problem - we eliminate u so that we end up with θ as the only free variable. To proceed, define the controllability Grammian:

$$W := \int_0^\infty e^{-A^+\tau} B^+ (B^+)^T e^{-(A^+)^T\tau}\, d\tau.$$

Proposition 1:

$$J_{opt}(r) = \inf_{\theta\in\Theta_p} \|W^{-1/2} \int_0^\infty e^{-A^+\tau} B^+ r(\theta(\tau))\, d\tau\|.$$

$$J^+_{opt}(r) = \inf_{\theta\in\Theta_p{}^+} \|W^{-1/2} \int_0^\infty e^{-A^+\tau} B^+ r(\theta(\tau))\, d\tau\|.$$

Remark 4: Motivated by Proposition 1, with $T \geq 0$ we define $\rho, \rho_T : \hat{L}_\infty \to \mathbf{R}^+$ by

$$\rho(f) := \|W^{-1/2} \int_0^\infty e^{-A^+\tau} B^+ f(\tau)\, d\tau\|,$$

$$\rho_T(f) := \|W^{-1/2} \int_0^T e^{-A^+\tau} B^+ f(\tau)\, d\tau\|.$$

These are pseudo-norms: they satisfy the triangular inequality and the scaling property but do not have a unique zero.

At this point we have eliminated u from the optimization problem. We now show that the optimal cost is unchanged if we restrict our attention to piecewise constant θ. To this end, define

$$\hat{J}_{opt}(r) = \inf\{\rho(r\circ\theta) : \text{ there exists a } T > 0 \text{ so that } \theta \text{ is piecewise constant on } [0,T] \text{ and satisfies } \dot{\theta}(t) = 1 \text{ for } t > T\ \},$$

$$\hat{J}^+_{opt}(r) = \inf\{\rho(r\circ\theta) : \text{ there exists a } T > 0 \text{ so that } \theta \text{ is piecewise constant on } [0,T], \text{ is monotonically increasing, and } \dot{\theta}(t) = 1 \text{ for } t > T\}.$$

Proposition 2:
(i) $\hat{J}_{opt}(r) = J_{opt}(r)$.
(ii) $\hat{J}^+_{opt}(r) = J^+_{opt}(r)$.

We now demonstrate that we can simplify the problem even further by eliminating θ from the optimization problem and replacing it with something simpler. To this end, the following two sets prove useful:

$$CIM(r) := \text{ closure of } \{r(t) : t \geq 0\},$$

$$CIM^+(r) := \text{closure of the positive limit set of } r.$$

We now define

$$\tilde{J}_{opt}(r) := \inf\{\rho(f) : f \in \hat{L}_\infty \text{ and } f(\cdot) \in CIM(r)\},$$

$$\tilde{J}^+_{opt}(r) := \inf\{\rho(f) : f \in \hat{L}_\infty \text{ and } f(\cdot) \in CIM^+(r)\}.$$

Remark 5: In the definition of $\hat{J}_{opt}(r)$, let θ be an admissible function over which the optimization is carried out, and define $f(t) := r(\theta(t))$. Then it follows immediately that f is piecewise continuous (indeed, it is piecewise constant for the first interval of time, and continuous thereafter) and takes values in $CIM(r)$; furthermore, it is easy to see that $f \in \hat{L}_\infty$. Using Proposition 2, this means that

$$\tilde{J}_{opt}(r) \leq \hat{J}_{opt}(r) = J_{opt}(r).$$

The same is not true of $\tilde{J}^+_{opt}(r)$ and $J^+_{opt}(r)$ unless there is additional structure on r, such as periodicity.

Proposition 3:
(i) $\tilde{J}_{opt}(r) = J_{opt}(r)$.
(ii) $\tilde{J}^+_{opt}(r) \geq J^+_{opt}(r)$.

It turns out that if r is periodic, then the above result can be used to prove

Proposition 4: If $r \in R_p$ is periodic then

$$J^+_{opt}(r) = \tilde{J}^+_{opt}(r) = \tilde{J}_{opt}(r) = J_{opt}(r).$$

IV. Some Initial Bounds

In this section we prove some preliminary results. To proceed, we choose $c_0 > 0$ and $\lambda_0 < 0$ so that

$$\|e^{-A^+t}\| \leq c_0 e^{\lambda_0 t}, \ \ t \geq 0.$$

Theorem 1:
(i) $J_{opt}(r) \leq \frac{c_0}{|\lambda_0|} \|W^{-1/2}\| \inf_{t\geq 0} \|B^+ r(t)\|$.
(ii) $J^+_{opt}(r) \leq \frac{c_0}{|\lambda_0|} \|W^{-1/2}\| \inf_{t\geq 0} \|B^+ r(t)\|$.
(iii) $J_{opt}(r) \leq \frac{c_0}{|\lambda_0|} \|W^{-1/2}\| \inf\{\|B^+ w\| : w \in CIM(r)\}$.
(iv) $J^+_{opt}(r) \leq \frac{c_0}{|\lambda_0|} \|W^{-1/2}\| \inf\{\|B^+ w\| : w \in CIM^+(r)\}$.

It turns out that we can have $J_{opt}(r) = 0$ even if $0 \notin B^+ \times CIM(r)$, with analogous results for the case of $J^+_{opt}(r)$.

Theorem 2: (i) If there exist $w_1, w_2 \in CIM(r)$ and $T \geq 0$ so that $B^+ w_1 + e^{A^+ T} B^+ w_2 = 0$, then $J_{opt}(r) = 0$.
(ii) If there exist $w_1, w_2 \in CIM^+(r)$ and $T \geq 0$ so that $B^+ w_1 + e^{A^+ T} B^+ w_2 = 0$, then $J^+_{opt}(r) = 0$.

Remark 6: There are cases in which the bounds provided by Theorem 1 are not tight. To see this, suppose that

$$A^+ = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$r(t) = \begin{bmatrix} 1/6 \\ 5/18 \end{bmatrix} + sin(t) \begin{bmatrix} 5/6 \\ 13/18 \end{bmatrix},$$

i.e., $r(t)$ oscillates in a straight line between $w_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 := \begin{bmatrix} -2/3 \\ -4/9 \end{bmatrix}$, which means that $0 \notin CIM(r) = B^+ \times CIM(r)$, so the upper bound provided by Theorem 1 is positive. If we choose $T = \ln 1.5$, it is easy to see that

$$B^+ w_1 + e^{A^+ T} B^+ w_2 = 0,$$

so Theorem 2 says that $J_{opt}(r) = 0$. This shows that we do not need $0 \in CIM(r)$ or $0 \in B^+ \times CIM(r)$ for $J_{opt}(r) = 0$.

We can also find a lower bound on the performance in some cases.

Theorem 3: If A^+ and B^+ are of the form

$$A^+ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & * \end{bmatrix}, \quad B^+ = \begin{bmatrix} B_1^+ \\ * \end{bmatrix},$$

with $\lambda_1 \in \mathbf{R}$ and $\inf_{t \geq 0} \|B_1^+ r(t)\| > 0$, then

$$J_{opt}(r) \geq \sigma_{min}(W^{-1/2}) \frac{1}{\lambda_1} \inf_{t \geq 0} \|B_1^+ r(t)\| > 0.$$

V. The Single-Input Case

In the single-input case we can obtain conditions ensuring that $J_{opt}(r) = 0$ (or $J^+_{opt}(r) = 0$) which are crisper than in the general case. We consider two situations.

A. At Least One Real Positive Zero

Theorem 4: If A^+ has at least one real eigenvalue, then

$$J_{opt}(r) = 0 \text{ iff } \inf_{t \geq 0} |r(t)| = 0$$

and

$$J^+_{opt}(r) = 0 \text{ iff } \inf_{t \geq 0} |r(t)| = 0.$$

B. Two Complex Zeros

It turns out that if A^+ has exactly two complex eigenvalues then we no longer need $\inf_{t \geq 0} |r(t)| = 0$ to have $J_{opt}(r) = 0$. We begin with an illustrative example.

Example 1: Consider the case of

$$A^+ = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad B^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with a reference trajectory periodic of period $T := \frac{2\pi}{\beta}$ given by

$$r(t) = e^{\alpha t}, \quad t \in [0, T).$$

It follows that

$$\int_0^\infty e^{-A^+ \tau} B^+ r(\tau) d\tau = 0.$$

While r is discontinuous, it can be approximated arbitrarily well (in L_2) by a function which belongs to C^∞.

Now the question is: in this situation, what is a necessary and sufficient condition for $J_{opt}(r) = 0$? It is clearly not the same as in the case of having at least one real zero; results on the structure of an associated controllability subset derived in [10] can be used to obtain bounds. To proceed, we write

$$[\underline{b}, \bar{b}] := CIM(r) \supset CIM^+(r) =: [\underline{b}^+, \bar{b}^+].$$

Theorem 5: Suppose that $A^+ \in \mathbf{R}^{2 \times 2}$ and $sp(A^+) = \alpha \pm j\beta$. Then
(i) $J_{opt}(r) = 0$ iff $0 \in [\underline{b}, \bar{b}]$ or $e^{\pi \frac{\alpha}{\beta}} \leq \frac{\bar{b}}{\underline{b}}$.
(ii) $J^+_{opt}(r) = 0$ if $0 \in [\underline{b}^+, \bar{b}^+]$ or $e^{\pi \frac{\alpha}{\beta}} \leq \frac{\bar{b}^+}{\underline{b}^+}$.

VI. A Computational Approach

Now we turn to the problem of computing $J_{opt}(r)$ and $\tilde{J}^+_{opt}(r)$ in the general case. In the forth-going, we focus on $J_{opt}(r)$ (which equals $J^+_{opt}(r)$ if r is periodic), but computing $\tilde{J}^+_{opt}(r)$ is similar, with a simple replacement of $CIM(r)$ with $CIM^+(r)$. To proceed, we impose

Standing Assumption 3: $r \in \mathcal{R}_p$ is bounded and $CIM(r)$ is strictly convex.

In the general case, we do not have any closed form results. Making use of Proposition 3, we end up with the technical problem of computing

$$J_{opt}(r) = \inf\{\|W^{-1/2} \int_0^\infty e^{-A^+ \tau} B^+ f(\tau) d\tau\| : f \in PC \text{ and } f(\cdot) \in CIM(r)\}$$

and its approximation:

$$J^T_{opt}(r) := \inf\{\|W^{-1/2} \int_0^T e^{-A^+ \tau} B^+ f(\tau) d\tau\| : f \in PC \text{ and } f(\cdot) \in CIM(r)\}.$$

It is clear that $\lim_{T\to\infty} J^T_{opt}(r) = J_{opt}(r)$. The following easily proven result provides a bound on the speed of convergence.

Proposition 5:

$$|J_{opt}(r) - J^T_{opt}(r)| \leq$$
$$e^{\lambda_0 T}\frac{c_0}{|\lambda_0|}\|W^{-1/2}\| \times \|B^+\| \times \|CIM(r)\|.$$

Hence, it follows that to compute $J_{opt}(r)$ to within a prescribed bound, it is enough to compute $J^T_{opt}(r)$. Therefore, we turn our attention to this latter problem, which we can rewrite as a classical control problem. To minimize clutter we define

$$U := CIM(r).$$

Optimization Problem 1 (OPT-1)
Minimize

$$g(x(T)) := \|W^{-1/2}e^{-A^+T}x(T)\|^2$$

subject to

$$\begin{aligned} \dot{x} &= A^+x + B^+u, \ \ x(0) = 0 \\ u(\cdot) &\in U. \end{aligned}$$

This is simply a classical control problem with end constraints. To proceed, we define two associated functions:

$$\begin{aligned} \psi : \mathbf{R}^m &\to \mathbf{R} \\ \xi &\longmapsto \max_{u\in U} \xi^T u \end{aligned}$$

and

$$\begin{aligned} \phi : \mathbf{R}^m &\to \mathbf{R} \\ \xi &\longmapsto \text{argmax}_{u\in U} \xi^T u. \end{aligned}$$

The first is well-defined because of the compactness of U; indeed, it is convex. The second is well-defined since U is strictly convex. To illustrate these definitions an example is in order.

Example 2: Consider the case of U being an ellipsoid: with $J > 0$ positive definite and symmetric and $u_0 \in \mathbf{R}^m$, consider

$$U = \{u \in \mathbf{R}^m : \|J(u - u_0)\|_2 = 1\}.$$

Using a Lagrange Multiplier approach, it is easy to verify that

$$\psi(\xi) = \max_{u\in U} \xi^T u = \xi^T u_0 + \frac{\xi^T J^{-2}\xi}{\|J^{-1}\xi\|},$$

$$\phi(\xi) = \text{argmax}_{u\in U} \xi^T u = u_0 + \frac{1}{\|J^{-1}\xi\|}J^{-2}\xi.$$

It turns that OPT-1 is a special case of the classical Meyer Problem. A detailed study of this problem is given in the MASc thesis [12], which is based in large part on the results of Rockafellar [14]. In the following we provide a summary of the main steps.

Proposition 6: (Theorem 1.1 of [12]) Every optimal control u^* of OPT-1 is of the form

$$u^*(t) = \phi((B^+)^T\rho(t))$$

subject to

$$\begin{aligned} \dot{x}^* &= A^+x^* + B^+u^*, \ \ x^*(0) = 0 \\ \dot{\rho} &= -(A^+)^T\rho, \ \ \rho(T) = -(\partial_x g)(x^*(T)). \end{aligned}$$

So to obtain the optimal cost it is enough to find all triples (u^*, x^*, ρ) which satisfy Proposition 6 and then find the one which is optimal. This constrained optimization problem is difficult so we turn to a dual problem.

Optimization Problem 2 (OPT-2) Minimize

$$h(\alpha) := \frac{1}{4}\alpha^T W\alpha + \int_0^T \psi((B^+)^T e^{-(A^+)^T t}\alpha)\, dt$$

subject to $\alpha \in \mathbf{R}^n$.

Since h is strictly convex, a broad range of software tools can be used to minimize it (assuming that ψ can be easily computed, as it will be in several special cases), yielding the unique minimum; since $\psi(0) = 0$, this minimum is at most zero. It turns out that the solutions of OPT-1 and OPT-2 are related.

Theorem 6: (Theorem 4.3 of [12]) The optimal solution α^* of OPT-2 yields the optimal solution of OPT-1:

$$J^T_{opt}(r) = [-h(\alpha^*)]^{1/2}.$$

VII. An Example

Here we consider the example of [2]: it is a vehicle with mass M moving in the plane, on top of which is a mass m, modelled by

$$\begin{aligned} M\ddot{y} &= D(\dot{z} - \dot{y}) + u \\ m\ddot{z} &= D(\dot{y} - \dot{z}) + G(z - y); \end{aligned}$$

here $D = \text{diag}(d_1, d_2) > 0$ is associated with the viscous friction, $G = \text{diag}(g_1, g_2) > 0$ is associated with gravity, $u(t) \in \mathbf{R}^2$ is the force, while $y(t) \in \mathbf{R}^2$ and $z(t) \in \mathbf{R}^2$ are the positions of the vehicle and mass, respectively. It turns out that the system has two non-minimum phase zeros. For the choice of

$$M = 1, m = 0.1, d_1 = 1, d_2 = 2, g_1 = 1.5, g_2 = 1,$$

these zeros are 0.488 and 1.324; the unstable zero dynamics associated with x^+ are given by

$$\dot{x}^+ = \underbrace{\begin{bmatrix} 0.488 & 0 \\ 0 & 1.324 \end{bmatrix}}_{=:A^+} x^+ + \underbrace{\begin{bmatrix} 0 & .53 \\ -1.97 & 0 \end{bmatrix}}_{=:B^+} y.$$

In [2] it is shown that the infimal L_2 tracking performance can be made as small as desired for all paths arising from a reference trajectory which is a finite sum of sinusoids (with non-zero frequency); this

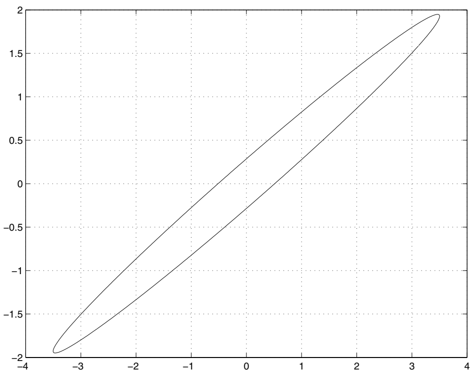

Fig. 1. The surface of $CIM(r_0)$.

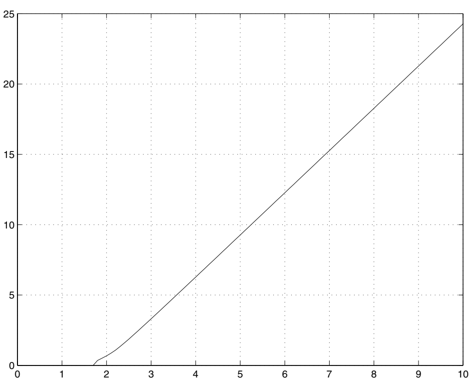

Fig. 2. A plot of $J^T_{opt}(r_a)$ as a function of a.

means, in particular, that 0 is contained in $CIM(r)$. As demonstrated in Remark 6, 0 need not lie in $CIM(r)$ for the infimal cost to be zero. To this end, consider the periodic reference signal

$$r_a(t) = \begin{bmatrix} 3.1216 & 1.5884 \\ 1.5884 & 1.2861 \end{bmatrix} [\begin{bmatrix} 0 \\ a \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}];$$

the surface of $CIM(r_0)$ is given in Figure 1; clearly $CIM(r_a)$ can be obtained by shifting $CIM(r_0)$ by $\begin{bmatrix} 1.5884 \\ 1.2861 \end{bmatrix} a$. From the formula for r_a, it is easy to see that

$$0 \in CIM(r_a) \quad \Leftrightarrow \quad a \in [-1, 1],$$

so it is immediate from Theorem 1 that

$$J_{opt}(r_a) = 0, \;\; a \in [-1, 1].$$

Using Theorem 6 we can compute $J^T_{opt}(r_a)$ as a function of a - see Figure 2 (we choose $T = 20$ here). Observe, in particular, that $J^T_{opt}(r_a) = 0$ for $a \in [0, 1.8]$.

VIII. Conclusions

In this paper we have considered path tracking performance limitations for multivariable non-minimum phase systems. In particular, we have studied problems related to infimal L_2 performance. By using the zero dynamic form of a system we are able to derive a number of results, including sufficient conditions for the infimal cost to be zero and other sufficient conditions for it to be strictly greater than zero. In a reasonably general situation, we show that the infimal cost can be computed by solving a finite dimensional convex optimization problem.

IX. Acknowledgements

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