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ON LIMITING DIRECTIONS OF JULIA SETS

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Abstract. We deal with the iteration of transcendental entire functions, and prove some properties on the Julia sets.

1. Introduction and main results

Let $f\colon \mathbf{C} \to \mathbf{C}$ be a transcendental entire function; we define the iterated sequence of f by $f^0(z) = z$, $f^{n+1}(z) = f \circ f^n(z)$, $n = 1, 2, \dots$. The Fatou set and the Julia set are defined by $N(f) = \{z \in \mathbf{C} \mid \{f^n\} \text{ is normal at } z\}$ and $J(f) = \mathbf{C} \setminus N(f)$ respectively. Qiao ([7]) proved that the Julia set of a transcendental entire function of finite order has infinitely many limiting directions; here a limiting direction of $J(f)$ means a limit of the set $\{\arg z_n \mid z_n \in J(f) \text{ is an unbounded sequence}\}$. The example in [1] shows that there exists an entire function of infinite order whose Julia set has only one limiting direction. In this note we shall prove

Theorem 1. *Let f be a transcendental entire function of lower order $\lambda < \infty$. Then there exists a closed interval $I \in \mathbf{R}$ such that all $\theta \in I$ are the common limiting directions of $J(f^{(n)})$, $n = 0, \pm 1, \pm 2, \cdots$, and $\operatorname{mes} I \geq \pi / \max(\frac{1}{2}, \lambda)$. Here $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n \geq 0$ or $n \leq 0$ respectively.*

We know that Mittag-Leffler's function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + \alpha n)}, \qquad 0 < \alpha < 2,$$

is a transcendental entire function of order $1/\alpha$. Put $\Omega(-\theta, \theta) = \{z \in \mathbf{C} \mid -\theta < \arg < \theta\}$. By the discussion used in [3] it is easy to verify that for any $E_\alpha(z)$, $0 < \alpha < 2$, there exists a constant $k > 0$, such that

$$f_{\alpha k}\left(\mathbf{C} \setminus \overline{\Omega}\left(-\frac{\alpha}{2\pi}, \frac{\alpha}{2\pi}\right)\right) \subset \mathbf{C} \setminus \overline{\Omega}\left(-\frac{\alpha}{2\pi}, \frac{\alpha}{2\pi}\right),$$

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here $f_{\alpha k} = E_\alpha(z) - k$. Hence $J(f_{\alpha k}) \subset \overline{\Omega}(-\alpha/2\pi, \alpha/2\pi)$. This shows that the estimate of the length of the closed interval I in Theorem 1 is sharp.

Liverpool ([5]) proved that: if $J(f)$ lies in the half-plane $\{z \in \mathbf{C} \mid \operatorname{Re} z \geq 0\}$ for a transcendental entire function f of order ≤ 1, then there exists a positive constant c such that for any horizontal strip region S with width c, $J(f) \cap S$ is unbounded; here a strip region means a region between two parallel straight lines. It is easy to see from Theorem 1 that f is of lower order ≥ 1 provided $J(f)$ lies in a half-plane. Therefore, Liverpool's result is valid only for entire functions with order and lower order one. We shall prove

Theorem 2. *Let f be a transcendental entire function of lower order 1, $J(f)$ lie in the half plane $\{z \in \mathbf{C} \mid \operatorname{Re} z \geq 0\}$. Then there exists a positive constant c such that all $J(f^{(n)}) \cap S$, $n = 0, \pm 1, \pm 2, \cdots$, are unbounded for any non-vertical strip region S with width c.*

Liverpool ([5]) has pointed out that

$$\left\{z \in \mathbf{C} \mid (2n + \tfrac{1}{2})\pi < \operatorname{Im} z < (2n + \tfrac{3}{2})\pi\right\} \cap J(e^z - 1) = \emptyset, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

But we shall prove this kind of "gap strips" will disappear for a class of entire functions of order > 1.

Theorem 3. *Let f be a transcendental entire function of order $\varrho > 1$, and all limiting directions of $J(f)$ belong to $(-\pi/\varrho, \pi/\varrho)$. Then all $J(f^{(n)}) \cap S$, $n = 0, \pm 1, \pm 2, \cdots$, are unbounded for an arbitrary horizontal strip region S.*

Theorem 4. *Let f be a transcendental entire function of order $\varrho > 1$, and lower order $\lambda > \frac{1}{2}$. If all limiting directions of $J(f)$ belong to $\left[-\pi/2\lambda, \pi/2\lambda\right]$, then all $J(f^{(n)}) \cap S$, $n = 0, \pm 1, \pm 2, \ldots$, are unbounded for an arbitrary strip region S which is parallel to $\theta \in (-\pi/2\lambda, \pi/2\lambda)$.*

2. Some lemmas

In order to prove the above results, we investigate the growth of f on its Fatou set. The following two lemmas are the improvements of the main results in [2] and [5] respectively. For $z_0 \in \mathbf{C}$ and $\theta, \delta \in \mathbf{R}$, put

$$\Omega(z_0, \theta, \delta) = \{z \in \mathbf{C} \mid |\arg(z - z_0) - \theta| < \delta\}.$$

We have

Lemma 1. *Let f be a transcendental entire function, and $\Omega(z_0, \theta, \delta) \subset N(f)$. Then*

$$|f(z)| = O(|z|)^{\pi/\delta}, \qquad z \in \Omega(z_0, \theta, \delta')$$

for arbitrary $\delta' \in (0, \delta)$.

Proof. Since $\Omega(z_0,\theta,\delta) \subset N(f)$, there is an unbounded component G_0 of $N(f)$ such that $\Omega(z_0,\theta,\delta) \subset G_0$. By [1] we know that every component of $N(f)$ is a simply connected hyperbolic domain. Let $f(G_0)$ belong to some component G of $N(f)$. It is easy to verify that the mapping

$$w = h(z) = \frac{(e^{-i\theta}z - e^{-i\theta}z_0)^{\pi/2\delta} - 1}{(e^{-i\theta}z - e^{-i\theta}z_0)^{\pi/2\delta} + 1}$$

maps $\Omega(z_0,\theta,\delta)$ conformally onto the unit disk $\{|w| < 1\}$. Put $h^{-1}(0) = a \in \Omega(z_0,\theta,\delta)$. By the Riemann theorem, there is a conformal mapping $w = g(z)\colon G \to \{|w| < 1\}$ satisfying $g\big(f(a)\big) = 0$ and $g'\big(f(a)\big) > 0$. Hence $F(w) = g \circ f \circ h^{-1}(w)$ is an analytic mapping from the unit disk to itself. By the Schwarz lemma,

$$|F(w)| \le |w|, \qquad |w| < 1. \tag{1}$$

Since g^{-1} is univalent on $\{|w| < 1\}$, by Koebe's distortion theorem we have

$$\big|\big(g^{-1}(w) - f(a)\big)g'\big(f(a)\big)\big| \le \frac{|w|}{(1-|w|)^2}, \qquad |w| < 1. \tag{2}$$

Since $f = g^{-1} \circ F \circ h$, it follows from (1) and (2) that

$$|f(z)| \le |f(a)| + \frac{1}{\big|g'\big(f(a)\big)\big|\big(1 - |h(z)|^2\big)}, \qquad z \in \Omega(z_0,\theta,\delta). \tag{3}$$

For arbitrary $z \in \Omega(z_0,\theta,\delta')$, put

$$\eta = z - z_0 = re^{i\alpha}, \qquad \sigma = \frac{\pi}{2\delta}, \qquad \lambda = \sin\frac{\delta'\pi}{2\delta} > 0.$$

Then

$$\begin{aligned} |h(z)|^2 &= \left|\frac{1 - \big(\cos\sigma(\alpha-\theta)\big)/r^\alpha + i\big(\sin\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)}{1 + \big(\cos\sigma(\alpha-\theta)\big)/r^\alpha - i\big(\sin\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)}\right|^2 \\ &= \frac{1 - 2\big(\cos\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)}{1 + 2\big(\cos\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)}. \end{aligned}$$

Thus

$$\begin{aligned} 1 - |h(z)| > \frac{1 - |h(z)|^2}{2} &= \frac{4\big(\cos\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)}{1 + 2\big(\cos\sigma(\alpha-\theta)\big)/r^\alpha + o(1/r^\alpha)} \\ &\ge \frac{4\lambda/r^\alpha + o(1/r^\alpha)}{1 + 2/r^\alpha + o(1/r^\alpha)}. \end{aligned}$$

By the above inequality and (3) we can easily deduce the result of Lemma 1. The proof of Lemma 1 is complete.

For any real numbers $a>0$ and $A>0$, put

$$H(a,A)=\{z\in\mathbf{C}\mid \operatorname{Re} z>a,\ |\operatorname{Im} z|<A\}.$$

We have

Lemma 2. *Let f be a transcendental entire function, and $H(a,A)\subset N(f)$, then*

$$|f(z)|=O\left(\exp\frac{\pi}{A}|z|\right),\qquad z\in H(a,A')$$

for arbitrary $A'\in(0,A)$.

Proof. Let G_0, G be two components of $N(f)$ such that $H(a,A)\subset G_0$, $f(G_0)\subset G$. It is easy to verify that

$$w=h_1(z)=\exp\left(\frac{\pi}{2A}z-\frac{\alpha\pi}{2A}\right)$$

maps $H(a,A)$ conformally onto $\{\operatorname{Re} w>0\}\backslash\{|w|<1\}$, and $w=h_2(z)=(z-2)/z$ maps $\{\operatorname{Re} z>1\}$ conformally onto $\{|w|<1\}$. By the Riemann theorem, there exists an univalent analytic function $g(z)$ which maps G onto $\{|w|<1\}$. Hence $F(w)=g\circ f\circ h_1^{-1}\circ h_2^{-1}(w)$ is an analytic mapping from the unit disk to itself. As in the proof of Lemma 1 we obtain

$$|g^{-1}\circ F(w)|=O\left(\frac{|w|}{(1-|w|)^2}\right),\qquad |w|<1. \tag{4}$$

Obviously, $f=g^{-1}\circ F\circ h_2\circ h_1$ and $h_1(z)\in\{\operatorname{Re} w>1\}$ for $z\in H(a,A')$ and sufficiently large $|z|$. By similar calculations as in the proof of Lemma 1, we can deduce the result of Lemma 2. The proof of Lemma 2 is complete.

Below we shall use the fundamental concepts and basic notations of Nevanlinna's theory ([4]).

Lemma 3 ([7]). *Let f be a transcendental entire function satisfying*

$$\varliminf_{r\to\infty}\frac{T(r,f)}{r^m}=0$$

for some fixed natural number m. Then for arbitrary $\alpha\in[0,2\pi)$, the set

$$J(f)\cap\left[\bigcup_{k=1}^{m}\left\{z\in\mathbf{C}\;\middle|\;\frac{2k-1}{m}\pi+\alpha<\arg z<\frac{2k}{m}\pi+\alpha\right\}\right]$$

is unbounded.

Lemma 4 ([4]). *Let f be a transcendental entire function. Then*

$$m\left(r,\frac{f'}{f}\right)=O\big(\log rT(r,f)\big),\qquad r\to\infty,$$

at most with an exceptional set of r whose linear measure is finite.

3. The proofs of the theorems

The proof of Theorem 1. We distinguish the following two cases:

(A) Suppose f is of lower order $\lambda < \frac{1}{2}$. We shall prove that, all $\theta \in [0, 2\pi)$ are the limiting directions of all $J(f^{(n)})$, $n = 0, \pm 1, \pm 2, \cdots$. Assume this statement is not true, then there exist $\theta \in [0, 2\pi)$ and an integer n_0 such that θ is not a limiting direction of $J(f^{n_0})$. Therefore $J(f^{n_0}) \cap \Omega(0, \theta, \delta)$ is bounded for some constant $\delta > 0$. By Lemma 1,

$$|f^{(n_0)}(z)| = O(|z|^k), \qquad \arg z = \theta; \tag{5}$$

here k is a positive constant. Since the lower order of $f^{n_0}(z)$ is less than $\frac{1}{2}$, by (5) and Wiman's theorem on minimum modulus (see [4]) we get a contradiction.

(B) Suppose f is of lower order $\lambda \geq \frac{1}{2}$, put

$$E_n = \{e^{i\theta} \mid \theta \text{ is a limiting direction of } J(f^{(n)})\}.$$

Obviously, E_n is a closed set on the unit circle Γ. Denote $E = \bigcap_{n \in \mathbf{Z}} E_n$, here $\mathbf{Z}$ is the set of integers. It is easy to see that the arguments of the points in E are the common limiting directions of all $J(f^{(n)})$, and the components of E are closed arcs on Γ. Put

$$\gamma = \left\{\alpha \;\middle|\; \alpha \text{ is an open arc on } \Gamma \text{ with length } < \frac{\pi}{\lambda}, \text{ and its endpoints are not in } E\right\}.$$

Assume the maximum component of E is of length $< \pi/\lambda$, then the set γ covers Γ. So there exist finitely many $\alpha_1, \alpha_2, \dots, \alpha_p \in \gamma$ such that $\bigcup_{j=1}^{p} \alpha_j \supset \Gamma$. Denote the arguments of two endpoints of α_j by $\theta_{j_1}, \theta_{j_2}$, $\theta_{j_1} < \theta_{j_2}$, respectively, and suppose θ_{j_1} is not the limiting direction of $J(f^{(n_{j_1})})$, θ_{j_2} is not the limiting direction of $J(f^{(n_{j_2})})$. By Lemma 1,

$$|f^{(n_{j_1})}(z)| = O(|z|^{k_{j_1}}), \qquad \arg z = \theta_{j_1}, \tag{6}$$

$$|f^{(n_{j_2})}(z)| = O(|z|^{k_{j_2}}), \qquad \arg z = \theta_{j_2}. \tag{7}$$

Here k_{j_1}, k_{j_2} are two positive constants.

Put $m = \min_{1 \leq j \leq p}(n_{j_1}, n_{j_2})$. Note

$$f^{(n_{j_1}-1)}(z) = \int_0^z f^{(n_{j_1})}(\eta)\, d\eta + c,$$

where c is a constant, and the above integral path is the segment of a straight line from 0 to z. From the above equality and (10) we deduce

$$|f^{(n_{j_1}-1)}(z)| = O(|z|^{k_{j_1}+1}), \qquad \arg z = \theta_{j_1}.$$

Repeating the above discussion, we can obtain

$$|f^{(m)}(z)| = O(|z|^{k_1}), \qquad \arg z = \theta_{j_1},$$

where k_1 is a positive constant. By the same method we have

$$|f^{(m)}(z)| = O(|z|^{k_2}), \qquad \arg z = \theta_{j_2},$$

where k_2 is a positive constant. Note that $\theta_{j_2} - \theta_{j_1} < \pi/\lambda$. By the Phragmén–Lindelöf principle we have

$$|f^{(m)}(z)| = O(|z|^{k}), \qquad \theta_{j_1} \le \arg z \le \theta_{j_2},$$

where $k = \max(k_1, k_2)$. Since $\alpha_1, \alpha_2, \dots, \alpha_p$ cover Γ, it follows that f is a polynomial. This contradicts the transcendence of f. The proof of Theorem 1 is complete.

The proof of Theorem 2. Assume the conclusion of this theorem is not true; then there exists a sequence of non-vertical strip regions S_j with width $c_j \to \infty$, and a sequence of $J(f^{(n_j)})$ such that $J(f^{(n_j)}) \cap S_j$ is bounded. Let S_j be parallel to the ray $\arg z = \theta_j \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Choose points $z_j \in S_j$, $j = 1, 2, \dots$. Then the ray $L_j : z = z_j + te^{i\theta_j}$, $t > 0$, lies on S_j. By Lemma 2,

$$|f^{(n_j)}(z)| = O\biggl(\exp \frac{2\pi}{c_j}|z|\biggr), \qquad z \in L_j. \tag{8}$$

We distinguish two cases:

(A) Suppose there are infinitely many $n_j > 0$ such that (8) holds. Integrating $f^{(n_j)}(z)$, by (8) we easily obtain

$$|f^{(n_j-1)}(z)| = O\biggl(|z| \exp \frac{2\pi}{c_j}|z|\biggr), \qquad z \in L_j.$$

Repeating this procedure we can get

$$|f(z)| = O\biggl(|z|^{n_j} \exp \frac{2\pi}{c_j}|z|\biggr), \qquad z \in L_j. \tag{9}$$

Since L_j is not vertical, we can draw two rays:

$$L'_j : z = z_j + te^{i\alpha_j}, \quad t > 0, \qquad L''_j : z = z_j + te^{i\beta_j}, \quad t > 0,$$

satisfying $\alpha_j < \beta_j$, $\alpha_j, \beta_j \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$. The angle from L_j to L'_j and the angle from L''_j to L_j are both less than π. Since there are no points of $J(f)$ in the left half-plane, by Lemma 1

$$|f(z)| = O(|z|^2), \qquad z \in L'_j \text{ or } L''_j. \tag{10}$$

By (9), (10) and the Phragmén–Lindelöf principle we have

$$|f(z)| = O\left(|z|^{n_j} \exp \frac{2\pi}{c_j}|z|\right), \qquad z \in \mathbf{C}.$$

We thus obtain

$$\overline{\lim_{r\to\infty}} \frac{T(r,f)}{r} \le \frac{2\pi}{c_j}.$$

Letting $c_j \to \infty$ we get

$$\lim_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

By Lemma 3, we deduce a contradiction.

(B) Suppose there are infinitely many $n_j \le 0$ such that (8) holds. As in (A), we can draw the ray L'_j and the ray L''_j; hence (10) follows. Using (10) to estimate the integrand, we can deduce

$$|f^{(n_j)}(z)| = O(|z|^{2+n_j}), \qquad z \in L'_j \text{ or } L''_j.$$

It follows from this equality, (8) and the Phragmén–Lindelöf principle that

$$\overline{\lim_{r\to\infty}} \frac{T(r,f^{(n_j)})}{r} \le \frac{2\pi}{c_j}. \tag{11}$$

On the other hand, by Lemma 4,

$$T(r,f^{(n_j+1)}) \le T(r,f^{(n_j)}) + m\left(r, \frac{(f^{(n_j)})'}{f^{(n_j)}}\right) \le \big(1+o(1)\big)T(r,f^{(n_j)}) + k_j \log r$$

for sufficiently large r, $r \notin E_j^1$, $\operatorname{mes} E_j^1 < \infty$. Here k_j is a positive constant. Note that $n_j \le 0$. Repeating the above estimation, we obtain

$$T(r,f) \le \big(1+o(1)\big)T(r,f^{(n_j)}) + K_j \log r \tag{12}$$

for $r \notin E_j^1$, $\operatorname{mes} E_j^1 < \infty$. Here K_j is a positive constant. By (11) and (12), we have

$$\underline{\lim_{r\to\infty}} \frac{T(r,f)}{r} \le \frac{2\pi}{c_j}.$$

Furthermore, by the same method as used in (A), we can deduce a contradiction. The proof of Theorem 2 is complete.

The proof of Theorem 3. Assume there exist a horizontal strip region S and an integer n such that $J(f^{(n)}) \cap S$ is bounded. Denote the width of S by c. Choose a point $z_0 \in S$, and draw the ray $L : z = z_0 + t$, $t > 0$. By Lemma 2,

$$(13) \qquad |f^{(n)}(z)| = O\left(\exp \frac{2\pi}{c}|z|\right), \qquad z \in L.$$

Since all limiting directions of $J(f)$ belong to $(-\pi/\varrho, \pi/\varrho)$, we can draw two rays: $L' : z = z_0 + te^{i\theta}$, $t > 0$, and $L'' : z = z_0 + te^{-i\theta}$, $t > 0$, such that $(-\theta, \theta) \subset (-\pi/\varrho, \pi/\varrho)$ and all limiting directions of $J(f)$ belong to $(-\theta, \theta)$. By Lemma 1,

$$(14) \qquad |f(z)| = O(|z|^k), \qquad z \in L' \text{ or } L'';$$

here k is a positive constant. Put $m = \min(n, 0)$. Using (13) and (14) to estimate the integrand, we can obtain

$$(15) \qquad |f^{(m)}(z)| = O\left(|z|^{k_1} \exp \frac{2\pi}{c}|z|\right), \qquad z \in L,$$

$$(16) \qquad |f^{(m)}(z)| = O(|z|^{k_2}), \qquad z \in L' \text{ or } L'';$$

here k_1, k_2 are two positive constants. It follows from (15), (16) and the Phragmén–Lindelöf principle that

$$(17) \qquad |f^{(m)}(z)| = O\left(|z|^{k_1} \exp \frac{2\pi}{c}|z|\right), \qquad -\theta \le \arg(z - z_0) \le \theta.$$

Since all limiting directions of $J(f)$ belong to $(-\pi/\varrho, \pi/\varrho)$, by Lemma 1, there exists a positive constant k such that (14) holds for $\theta \le \arg(z - z_0) \le 2\pi - \theta$. This and (17) imply that $f^{(m)}$ is of order ≤ 1. This is a contradiction. The proof of Theorem 3 is complete.

The proof of Theorem 4. Assume there exist a strip region S which parallels $\theta \in (-\pi/2\lambda, \pi/2\lambda)$, and some $J(f^{(n)})$ such that $J(f^{(n)}) \cap S$ is bounded. Denote the width of S by c. Choose a point $z_0 \in S$, by Lemma 2,

$$|f^{(n)}(z)| = O\left(\exp \frac{2\pi}{c}|z|\right), \qquad z \in L: \ z = z_0 + te^{i\theta}, \ t > 0.$$

Obviously, we can draw two rays

$$L' : z = z_0 + te^{i\theta_1}, \quad t > 0, \qquad L'' : z = z_0 + te^{i\theta_2}, \quad t > 0,$$

such that $(\theta_1, \theta_2) \supset [-\pi/2\lambda, \pi/2\lambda]$, $\theta_2 - \theta < \pi/\lambda$ and $\theta - \theta_1 < \pi/\lambda$. Using the same method as in the proof of Theorem 3, we can deduce

$$|f^{(n)}(z)| = O\left(\exp \frac{2\pi}{c}|z|\right), \qquad \theta_1 \le \arg(z - z_0) \le \theta_2,$$

$$|f^{(n)}(z)| = O(|z|^2), \qquad \theta_2 \le \arg(z - z_0) \le \theta_1 + 2\pi.$$

It follows that f is of order ≤ 1. This is a contradiction. The proof of Theorem 4 is thus complete.

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