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S. A. PLAKSA and V. S. SHPAKIVSKYI

On limiting values of Cauchy type integral in a harmonic algebra with two-dimensional radical

Abstract. We consider a certain analog of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical. We establish sufficient conditions for an existence of limiting values of this integral on the curve of integration.

1. Introduction. Let Γ be a closed Jordan rectifiable curve in the complex plane $\mathbb{C}$. By D^+ and D^- we denote, respectively, the interior and the exterior domains bounded by the curve Γ.

N. Davydov [1] established sufficient conditions for an existence of limiting values of the Cauchy type integral

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)}{t-\xi}dt, \qquad \xi\in\mathbb{C}\setminus\Gamma, \tag{1}$$

on Γ from the domains D^+ and D^-. This result stimulated a development of the theory of Cauchy type integral on curves which are not piecewise-smooth.

In particular, using the mentioned result of the paper [1], the following result was proved: if the curve Γ satisfies the condition (see [2])

$$\theta(\varepsilon) := \sup_{\xi\in\Gamma}\ \theta_\xi(\varepsilon) = O(\varepsilon), \quad \varepsilon\to 0 \tag{2}$$

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(here $\theta_\xi(\varepsilon) := \operatorname{mes}\{t \in \Gamma : |t - \xi| \le \varepsilon\}$, where mes denotes the linear Lebesgue measure on Γ), and the modulus of continuity

$$\omega_g(\varepsilon) := \sup_{t_1,t_2\in\Gamma, |t_1-t_2|\le\varepsilon} |g(t_1) - g(t_2)|$$

of a function $g : \Gamma \to \mathbb{C}$ satisfies the Dini condition

$$\int_0^1 \frac{\omega_g(\eta)}{\eta} d\eta < \infty, \tag{3}$$

then the integral (1) has limiting values in every point of Γ from the domains D^+ and D^- (see [3]). The condition (2) means that the measure of a part of the curve Γ in every disk centered at a point of the curve is commensurable with the radius of the disk.

In this paper we consider a certain analogue of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical and study the question about an existence of its limiting values on the curve of integration.

2. A three-dimensional harmonic algebra with a two-dimensional radical. Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of algebra $\mathbb{A}_3$ with the multiplication table: $\rho_1\rho_2 = \rho_2^2 = 0$, $\rho_1^2 = \rho_2$.

$\mathbb{A}_3$ is a *harmonic* algebra, i.e. there exists a *harmonic* basis $\{e_1, e_2, e_3\} \subset \mathbb{A}_3$ satisfying the following conditions (see [5], [6], [7], [8], [9]):

$$e_1^2 + e_2^2 + e_3^2 = 0, \qquad e_j^2 \ne 0 \ \text{ for } j = 1, 2, 3. \tag{4}$$

P. Ketchum [5] discovered that every function $\Phi(\zeta)$ analytic with respect to the variable $\zeta := xe_1 + ye_2 + ze_3$ with real x, y, z satisfies the equalities

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi(\zeta) = \Phi''(\zeta)\,(e_1^2 + e_2^2 + e_3^2) = 0 \tag{5}$$

owing to the equality (4). I. Mel'nichenko [7] noticed that doubly differentiable in the sense of Gateaux functions form the largest class of functions Φ satisfying the equalities (5).

All harmonic bases in $\mathbb{A}_3$ are constructed by I. Mel'nichenko in [9].

Consider a harmonic basis

$$e_1 = 1, \qquad e_2 = i + \frac{1}{2}\,i\rho_2, \qquad e_3 = -\rho_1 - \frac{\sqrt{3}}{2}\,i\rho_2$$

in $\mathbb{A}_3$ and the linear envelope $E_3 := \{\zeta = x + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ over the field of real numbers $\mathbb{R}$, that is generated by the vectors $1, e_2, e_3$. Associate with a domain $\Omega \subset \mathbb{R}^3$ the domain $\Omega_\zeta := \{\zeta = x + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3.

The algebra $\mathbb{A}_3$ have the unique maximal ideal $\{\lambda_1\rho_1+\lambda_2\rho_2 : \lambda_1,\lambda_2 \in \mathbb{C}\}$ which is also the radical of $\mathbb{A}_3$. Thus, it is obvious that the straight line $\{ze_3 : z \in \mathbb{R}\}$ is contained in the radical of algebra $\mathbb{A}_3$.

$\mathbb{A}_3$ is a Banach algebra with the Euclidean norm

$$\|a\| := \sqrt{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2}\,,$$

where $a = \xi_1 + \xi_2 e_2 + \xi_3 e_3$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

We say that a continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists $\Phi'(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\varepsilon\to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

For monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_3$ we established basic properties analogous to properties of analytic functions of the complex variable: the Cauchy integral theorem, the Cauchy integral formula, the Morera theorem, the Taylor expansion (see [11]).

3. On existence of limiting values of a hypercomplex analogue of the Cauchy type integral. In what follows, $t_1, t_2, x, y, z \in \mathbb{R}$ and the variables x, y, z with subscripts are real. For example, x_0 and x_1 are real, etc.

Let $\Gamma_\zeta := \{\tau = t_1 + t_2 e_2 : t_1 + it_2 \in \Gamma\}$ be the curve congruent to the curve $\Gamma \subset \mathbb{C}$. Consider the domain $\Pi^\pm_\zeta := \{\zeta = x + ye_2 + ze_3 : x + iy \in D^\pm, z \in \mathbb{R}\}$ in E_3. By Σ_ζ we denote the common boundary of domains Π^+_ζ and Π^-_ζ.

Consider the integral

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau \tag{6}$$

with a continuous density $\varphi : \Gamma_\zeta \to \mathbb{R}$. The function (6) is monogenic in the domains Π^+_ζ and Π^-_ζ, but the integral (6) is not defined for $\zeta \in \Sigma_\zeta$.

For the function $\varphi : \Gamma_\zeta \to \mathbb{R}$ consider the modulus of continuity

$$\omega_\varphi(\varepsilon) := \sup_{\tau_1,\tau_2\in\Gamma_\zeta, \|\tau_1-\tau_2\|\le\varepsilon} |\varphi(\tau_1) - \varphi(\tau_2)|,$$

and a singular integral

$$\int_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon\to 0} \int_{\Gamma_\zeta\setminus\Gamma^\varepsilon_\zeta(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1} d\tau,$$

where $\zeta_0 \in \Gamma_\zeta$ and $\Gamma^\varepsilon_\zeta(\zeta_0) := \{\tau \in \Gamma_\zeta : \|\tau - \zeta_0\| \le \varepsilon\}$.

Below, in Theorem 1 in the case where the curve Γ satisfies the condition (2) and the modulus of continuity of the function φ satisfies a condition of the type (3), we establish the existence of certain limiting values of the integral (6) in points $\zeta_0 \in \Gamma_\zeta$ when ζ tends to ζ_0 from Π^+_ζ or Π^-_ζ along a curve that is not tangential to the surface Σ_ζ outside of the plane of curve Γ_ζ.

For the Euclidean norm in $\mathbb{A}_3$ the following inequalities are fulfilled:

(7) $$\|ab\| \le 2\sqrt{14}\|a\|\|b\| \quad \forall a, b \in \mathbb{A}_3,$$

(8) $$\left\| \int\limits_{\Gamma'_\zeta} \psi(\tau)d\tau \right\| \le 9M \int\limits_{\Gamma'_\zeta} \|\psi(\tau)\|\|d\tau\|$$

with the constant $M := \max\{1, \|e_2^2\|, \|e_2e_3\|, \|e_3^2\|\}$ for any measurable set $\Gamma'_\zeta \subset \Gamma_\zeta$ and all continuous functions $\psi : \Gamma'_\zeta \to \mathbb{A}_3$.

Lemma 1. *Let Γ be a closed Jordan rectifiable curve satisfying the condition* (2) *and the modulus of continuity of a function $\varphi : \Gamma_\zeta \to \mathbb{R}$ satisfies the condition of the type* (3)*. If a point ζ tends to $\zeta_0 \in \Gamma_\zeta$ along a curve γ_ζ for which there exists a constant $m < 1$ such that the inequality*

(9) $$|z| \le m\|\zeta - \zeta_0\|$$

is fulfilled for all $\zeta = x + ye_2 + ze_3 \in \gamma_\zeta$, then

$$\lim_{\zeta\to\zeta_0,\zeta\in\gamma_\zeta} \int\limits_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta)^{-1}d\tau = \int\limits_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1}d\tau.$$

Proof. Let $\varepsilon := \|\zeta - \zeta_0\|$. Consider the difference

$$\int\limits_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta)^{-1}d\tau - \int\limits_{\Gamma_\zeta} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1}d\tau$$

$$= \int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta)^{-1}d\tau - \int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta_0)^{-1}d\tau$$

$$+ (\zeta - \zeta_0) \int\limits_{\Gamma_\zeta\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_0)\Big)(\tau - \zeta)^{-1}(\tau - \zeta_0)^{-1}d\tau =: I_1 - I_2 + I_3.$$

To estimate I_1 we choose a point $\zeta_1 = x_1 + y_1e_2$ on Γ_ζ such that $\|\zeta - \zeta_1\| = \min\limits_{\tau\in\Gamma_\zeta} \|\tau - \zeta\|$. Using the inequalities (7) and (8), we obtain

$$\|I_1\| = \left\| \int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)} \Big(\varphi(\tau) - \varphi(\zeta_1)\Big)(\tau - \zeta)^{-1}d\tau + \Big(\varphi(\zeta_1) - \varphi(\zeta_0)\Big) \int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)} (\tau - \zeta)^{-1}d\tau \right\|$$

$$\le 18\sqrt{14}M \int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)} |\varphi(\tau) - \varphi(\zeta_1)|\, \|(\tau - \zeta)^{-1}\|\, \|d\tau\|$$

$$+|\varphi(\zeta_1)-\varphi(\zeta_0)|\left\|\int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)}(\tau-\zeta)^{-1}d\tau\right\| =: I_1'+I_1''.$$

It follows from Lemma 1.1 [9] that

$$(\tau-\zeta)^{-1}=\frac{1}{t-\xi}-\frac{z}{(t-\xi)^2}\,\rho_1+\left(\frac{i}{2}\,\frac{y-t_2-\sqrt{3}z}{(t-\xi)^2}+\frac{z^2}{(t-\xi)^3}\right)\rho_2 \tag{10}$$

for all $\zeta=x+ye_2+ze_3\in\Pi_\zeta^{\pm}$ and $\tau=t_1+t_2e_2\in\Gamma_\zeta$, where $\xi:=x+iy$ and $t:=t_1+it_2$. The following inequality follows from the relations (9) and (10):

$$\|(\tau-\zeta)^{-1}\|\le c(m)\frac{1}{|t-\xi|}, \tag{11}$$

where the constant $c(m)$ depends only on m.

Using the inequality $|t-\xi|\ge|t-\xi_1|/2$ with $\xi_1:=x_1+iy_1$ and the inequality (11), we obtain:

$$\|I_1'\|\le 18\sqrt{14}\,Mc(m)\int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)}\frac{|\varphi(\tau)-\varphi(\zeta_1)|}{|t-\xi|}\|d\tau\|$$

$$\le 36\sqrt{14}\,Mc(m)\int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)}\frac{|\varphi(\tau)-\varphi(\zeta_1)|}{|t-\xi_1|}\|d\tau\|$$

$$\le 36\sqrt{14}\,Mc(m)\int\limits_{[0,4\varepsilon]}\frac{\omega_\varphi(\eta)}{\eta}d\theta_{\xi_1}(\eta),$$

where the last integral is understood as a Lebesgue–Stieltjes integral.

To estimate the last integral we use Proposition 1 [10] (see also the proof of Theorem 1 [4]) and the condition (2). So, we have

$$\int\limits_{[0,4\varepsilon]}\frac{\omega_\varphi(\eta)}{\eta}d\theta_{\xi_1}(\eta)\le\int\limits_0^{8\varepsilon}\frac{\theta_{\xi_1}(\eta)\omega_\varphi(\eta)}{\eta^2}d\eta\le c\int\limits_0^{8\varepsilon}\frac{\omega_\varphi(\eta)}{\eta}d\eta\ \to 0,\quad \varepsilon\to 0,$$

where the constant c does not depend on ε.

To estimate I_1'' we introduce the domain $D_\zeta^{2\varepsilon}(\zeta_0):=\{\tau=t_1+t_2e_2 : t_1+it_2\in D^+,\ \|\tau-\zeta_0\|\le 2\varepsilon\}$ and its boundary $\partial D_\zeta^{2\varepsilon}(\zeta_0)$. Using the inequalities (8) and (11), we obtain:

$$\|I_1''\| \le \omega_\varphi\big(\|\zeta_1-\zeta_0\|\big)\Bigg\|\int\limits_{\Gamma_\zeta^{2\varepsilon}(\zeta_0)}(\tau-\zeta)^{-1}d\tau\Bigg\|$$

$$=\omega_\varphi\big(\|\zeta_1-\zeta_0\|\big)\Bigg\|\int\limits_{\partial D_\zeta^{2\varepsilon}(\zeta_0)}(\tau-\zeta)^{-1}d\tau-\int\limits_{\partial D_\zeta^{2\varepsilon}(\zeta_0)\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)}(\tau-\zeta)^{-1}d\tau\Bigg\|$$

$$\le \omega_\varphi\big(\|\zeta_1-\zeta_0\|\big)\left(2\pi+9Mc(m)\int\limits_{\partial D_\zeta^{2\varepsilon}(\zeta_0)\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)}\frac{\|d\tau\|}{|t-\xi|}\right)$$

$$\le \omega_\varphi(2\varepsilon)\left(2\pi+9Mc(m)\frac{1}{\varepsilon}2\pi2\varepsilon\right)\to 0,\quad \varepsilon\to 0.$$

Estimating I_2, by analogy with the estimation of I_1', we obtain:

$$\|I_2\|\le c\int\limits_0^{4\varepsilon}\frac{\omega_\varphi(\eta)}{\eta}d\eta\ \to 0,\quad \varepsilon\to 0,$$

where the constant c does not depend on ε.

Using the inequality $|t-\xi|\ge|t-\xi_0|/2$, where the point $\xi_0 := x_0+iy_0$ corresponds to the point $\zeta_0=x_0+y_0e_2$, and using the relations (7), (8), (11) and (2), by analogy with the estimation of I_1', we obtain:

$$\|I_3\|\le 9M(2\sqrt{14})^2\,\varepsilon\int\limits_{\Gamma_\zeta\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)}|\varphi(\tau)-\varphi(\zeta_0)|\,\|(\tau-\zeta)^{-1}\|\,\|(\tau-\zeta_0)^{-1}\|\,\|d\tau\|$$

$$\le c\,\varepsilon\int\limits_{\Gamma_\zeta\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)}\frac{|\varphi(\tau)-\varphi(\zeta_0)|}{|t-\xi||t-\xi_0|}\|d\tau\|\le c\,\varepsilon\int\limits_{\Gamma_\zeta\setminus\Gamma_\zeta^{2\varepsilon}(\zeta_0)}\frac{|\varphi(\tau)-\varphi(\zeta_0)|}{|t-\xi_0|^2}\|d\tau\|$$

$$\le c\,\varepsilon\int\limits_{[2\varepsilon,d]}\frac{\omega_\varphi(\eta)}{\eta^2}d\theta_{\xi_0}(\eta)\le c\,\varepsilon\int\limits_{2\varepsilon}^{2d}\frac{\theta_{\xi_0}(\eta)\omega_\varphi(\eta)}{\eta^3}d\eta$$

$$\le c\,\varepsilon\int\limits_{2\varepsilon}^{2d}\frac{\omega_\varphi(\eta)}{\eta^2}d\eta\to 0,\quad \varepsilon\to 0,$$

where $d := \max_{\xi_1,\xi_2\in\Gamma}|\xi_1-\xi_2|$ is the diameter of Γ and c denotes different constants which do not depend on ε. The lemma is proved. □

Let $\widehat{\Phi}^{\pm}(\zeta_0)$ be the boundary value of function (6) when ζ tends to $\zeta_0\in\Gamma_\zeta$ along a curve γ_ζ for which there exists a constant $m<1$ such that the inequality (9) is fulfilled for all $\zeta=x+ye_2+ze_3\in\gamma_\zeta$.

Theorem 1. *Let Γ be a closed Jordan rectifiable curve satisfying the condition* (2) *and the modulus of continuity of a function $\varphi : \Gamma_\zeta \to \mathbb{R}$ satisfies the condition of the type* (3). *Then the integral* (6) *has boundary values $\widehat{\Phi}^{\pm}(\zeta_0)$ for all $\zeta_0 \in \Gamma_\zeta$ that are expressed by the formulas:*

$$\widehat{\Phi}^{+}(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1} d\tau + \varphi(\zeta_0)$$

$$\widehat{\Phi}^{-}(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1} d\tau.$$

The theorem follows from the Lemma 1 and the equalities

$$\frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} \varphi(\tau)(\tau-\zeta)^{-1} d\tau = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} (\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta)^{-1} d\tau + \varphi(\zeta_0) \quad \forall\, \zeta \in \Pi_\zeta^{+},$$

$$\frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta)^{-1} d\tau \quad \forall\, \zeta \in \Pi_\zeta^{-}.$$

In comparison with Theorem 1, note that additional assumptions about the function φ are required for an existence of limiting values of the function (6) from Π_ζ^+ or Π_ζ^- on the boundary Σ_ζ. We are going to state these results in next papers.

References

[1] Davydov, N. A., *The continuity of an integral of Cauchy type in a closed domain*, Dokl. Akad. Nauk SSSR **64**, no. 6 (1949), 759–762 (Russian).

[2] Salaev, V. V., *Direct and inverse estimates for a singular Cauchy integral along a closed curve*, Mat. Zametki **19**, no. 3 (1976), 365–380 (Russian).

[3] Gerus, O. F., *Finite-dimensional smoothness of Cauchy-type integrals*, Ukrainian Math. J. **29**, no. 5 (1977), 490–493.

[4] Gerus, O. F., *Some estimates of moduli of smoothness of integrals of the Cauchy type*, Ukrainian Math. J. **30**, no. 5 (1978), 594–601.

[5] Ketchum, P. W., *Analytic functions of hypercomplex variables*, Trans. Amer. Math. Soc. **30** (1928), 641–667.

[6] Kunz, K. S., *Application of an algebraic technique to the solution of Laplace's equation in three dimensions*, SIAM J. Appl. Math. **21**, no. 3 (1971), 425–441.

[7] Mel'nichenko, I. P., *The representation of harmonic mappings by monogenic functions*, Ukrainian Math. J. **27**, no. 5 (1975), 499–505.

[8] Mel'nichenko, I. P., *Algebras of functionally invariant solutions of the three-dimensional Laplace equation*, Ukrainian Math. J. **55**, no. 9 (2003), 1551–1559.

[9] Mel'nichenko, I. P., Plaksa, S. A., *Commutative algebras and spatial potential fields*, Inst. Math. NAS Ukraine, Kiev, 2008 (Russian).

[10] Plaksa, S. A., *Riemann boundary-value problem with infinite index of logarithmic order on a spiral contour. I*, Ukrainian Math. J. **42**, no. 11 (1990), 1509–1517.

[11] Shpakivskyi, V. S., Plaksa, S. A., *Integral theorems and a Cauchy formula in a commutative three-dimensional harmonic algebra*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **60** (2010), 47–54.

S. A. Plaksa
Department of Complex Analysis and Potential Theory
Institute of Mathematics of the National Academy of Sciences of Ukraine
Tereshchenkivska St. 3
01601 Kiev-4
Ukraine
e-mail: plaksa@imath.kiev.ua

V. S. Shpakivskyi
Department of Complex Analysis and Potential Theory
Institute of Mathematics of the National Academy of Sciences of Ukraine
Tereshchenkivska St. 3
01601 Kiev-4
Ukraine
e-mail: shpakivskyi@mail.ru

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