On limiting values of stochastic differential equations with small noise intensity tending to zero.

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The differential equation

We consider a differential equation in $\mathbb{R}^N$:

$$(ODE) \qquad x'(t) = f(x(t)),\ t \geq 0,\ x(0) = x$$

where $f : \mathbb{R}^N \mapsto \mathbb{R}^N$ is possibly discontinuous (but bounded and measurable)

Remark It is well known that for f only measurable, existence of solution to (ODE) may fail.

We consider the stochastic differential equation (with ε small) :

$$(SDE) \qquad dX_\varepsilon(t) = f(X_\varepsilon(t))dt + \varepsilon dW_t,\ t \geq 0,\ x(0) = x,$$

where $(W(t), t \geq 0)$ denotes an N-dimensional standard Brownian motion on some complete probability space $(\Omega, \mathcal{F}, P)$

Remark The SDE has always a weak solution.

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Remark The SDE has always a weak solution.

The differential inclusion

Our main aim is to compare the limits of solutions to SDE with Filippov's solutions to ODE which are solutions to the differential inclusion

$$(DI) \qquad x'(t) \in F(x(t)),\ t \geq 0,\ x(0) = x$$

where F is the smallest *set valued-map with closed graph and compact convex values* which contains f almost everywhere.

Existence for differential inclusion

Recall that if $F\mathbb{R}^n \mapsto \mathbb{R}^n$, is a (bounded) set valued-map with closed graph and compact convex values the differential inclusion (DI) has always at least one solution.

Scheme of the proof

- For any $\varepsilon > 0$ there exist a *Lipschitz* function f_ε with

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B.$$

- One can obtain a solution to the differential inclusion by taking any cluster point of solutions to the following differential equations

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Outline

- 1- Filippov's solutions to ordinary differential equations
- 2- Limits of stochastic differential equations
- 3- Examples and generalizations

Filippov's regularization

Definition To $f : \mathbb{R}^N \mapsto \mathbb{R}^N$, we associate the following set-valued map

$$F_f(x) := \bigcap_{\lambda(N)=0} \bigcap_{\delta>0} cof((x+\delta B)\backslash N);$$

the first intersection is taken over all sets $N \subset \mathbb{R}^N$, being neglectable with respect to the Lebesgue measure.

An absolutely continuous solution $t \mapsto x(t)$ is a Filippov's solution to ODE iff it is a solution of the following differential inclusion

$$x'(t) \in F_f(x(t)),\ t \geq 0,\ x(0) = x.$$

Remarks

If f is continuous, $F = f$, and so classical solutions are Filippov's solutions.

Consider $f(x) := 1_{\mathbb{R}\backslash\{0\}}$, then $x(t) = 0$ is a classical solution but not a Filippov's solution.

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Properties of the Filippov regularization

Let $f : \mathbb{R}^N \mapsto \mathbb{R}^n$ be a measurable and (locally) bounded function. Then

- **i)** There exists a set N_f neglectable such that,

$$\forall x \in \mathbb{R}^N, \; F_f(x) = \bigcap_{\delta > 0} cof((x + \delta B) \setminus N_f);$$

- **ii)** For almost all x, we have $f(x) \in F(x)$.
- **iii)** F_f is the smallest map with closed graph and closed convex values with $f(x) \in F(x)$ for almost all x
- **iv)** The map $x \mapsto F_f(x)$ is single-valued iff $\exists$ a continuous function g which coincide a. e. with f. Then $F_f(x) = \{g(x)\}$ for a. e. x.

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- **v)** If a function $\tilde{f}$ coincide almost everywhere with f then $F_f(x) = F_{\tilde{f}}(x)$ for all $x \in \mathbb{R}^n$.
- **vi)** $\exists \bar{f}$ which is equal a. e. to f and such that

$$F_f(x) = \bigcap_{\delta>0} co\bar{f}((x+\delta B)).$$

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$$F_f(x) := \bigcap_{\tilde{f}=f\ a.e.} \bigcap_{\delta>0} co\tilde{f}((x+\delta B)),$$

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Proofs of properties of the Filippov map

Define N_f the complement of set of points $x \in \mathbb{R}^N$ of approximate continuity of f, namely points x such that

$$\forall \varepsilon, \lim_{r \to 0^+} \frac{\lambda\{y \in (x + rB), |f(y) - f(x)| > \varepsilon\}}{\lambda(x + rB)} = 0.$$

Observe that

$$\varepsilon \frac{\lambda\{y \in (x + rB), |f(y) - f(x)| > \varepsilon\}}{\lambda(x + rB)} \leq \frac{1}{\lambda(x + rB)} \int_{x+rB} |f(y) - f(x)| dy.$$

The second term tends to 0 for all the Lebesgue points of f. So N_f is Lebesgue neglectable set.

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$$\bigcap_{\delta > 0} cof((x + \delta B) \backslash N_f) = \bigcap_{\delta > 0} cof((x + \delta B) \backslash (N_f \cup N)).$$

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$$\forall x \in \mathbb{R}^N \backslash N_f, \; f(x) \in F_f(x).$$

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Consider another set-valued map G upper semicontinuous with compact convex values such that for some N_G of measure 0 we have :

$$f(x) \in G(x), \ \forall x \in \mathbb{R}^N \backslash N_G.$$

Fix $y \in \mathbb{R}^N$. From the upper semicontinuity of G, there exists a sequence $\delta_n \downarrow 0^+$ with

$$G(y + \delta_n B) \subset G(y) + \frac{1}{n} B, \ \forall n \geq 1.$$

Clearly,

$$f((y + \delta_n B) \backslash (N_f \cup N_G)) \subset G(y + \delta_n B) \subset G(y) + \frac{1}{n} B, ,$$

and consequently, because $G(y)$ is a compact convex set of $\mathbb{R}^n$,

$$\bigcap_{n \geq 1} cof((y + \delta_n B) \backslash (N_f \cup N_G)) \subset G(y).$$

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Proofs of properties of the Filippov map

iv) Assume that $F_f(x) = \{g(x)\}$ for all $x \in R^N$. Because $x \mapsto \{g(x)\}$ is upper semicontinuous as a set-valued map, this yields that the function g is continuous. Furthermore, from **iii)** , $g(x) = f(x)$ for almost every x. Conversely, suppose that there exists some g continuous which coincide with f on the complement of some neglectable set N. We have for any x,

$$F_f(x) = \cap_{\delta>0} cof((x + \delta B)\backslash(N_f \cup N)) = \bigcap_{\delta>0} cog((x + \delta B)\backslash(N_f \cup N)).$$

The last term reduces to $\{g(x\}$ thanks to the continuity of g.

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v) Suppose that $\tilde{f}(x) = f(x)$ for any $x \in \mathbb{R}^N \setminus \tilde{N}$, where $\tilde{N}$ is a neglectable set. Then for any set N of measure 0 we have for any x

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By taking the intersection over all sets N of null measure, we obtain $F_f(x) = F_{\tilde{f}}(x)$, which proves our claim.
vi) Let us define $\bar{f}$ by setting

$$\bar{f}(x) = f(x) \text{ if } x \notin N_f$$

if $x \in N_f$ we choose for $\bar{f}(x)$ any element of $F_f(x)$.

Clearly $\bar{f}$ coincides with f on $\mathbb{R}^N \setminus N_f$. We can prove that $\bar{f}$ is suitable for our proof.

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v) Suppose that $\tilde{f}(x) = f(x)$ for any $x \in \mathbb{R}^N \backslash \tilde{N}$, where $\tilde{N}$ is a neglectable set. Then for any set N of measure 0 we have for any x

$$\bigcap_{\delta>0} co f((x+\delta B)\backslash(\tilde{N}\cup N)) = \bigcap_{\delta>0} co \tilde{f}((x+\delta B)\backslash(\tilde{N}\cup N)).$$

By taking the intersection over all sets N of null measure, we obtain $F_f(x) = F_{\tilde{f}}(x)$, which proves our claim.

vi) Let us define $\bar{f}$ by setting

$$\bar{f}(x) = f(x) \text{ if } x \notin N_f$$

if $x \in N_f$ we choose for $\bar{f}(x)$ any element of $F_f(x)$.

Clearly $\bar{f}$ coincides with f on $\mathbb{R}^N \backslash N_f$. We can prove that $\bar{f}$ is suitable for our proof.

Outline

- 1- Filippov's solutions to ordinary differential equations
- 2- Limits of stochastic differential equations
- 3- Examples and generalizations

Limit of solutions to SDES

THEOREM Suppose that f is bounded (by M) and Lebesgue measurable.
For any ε, let X_ε be a solution to (SDE).
Then, along a subsequence, X_ε converges in law, as $\varepsilon \to 0$, to some X which belongs almost surely to the set of Filippov's solutions to ODE.

Proof of Main Theorem

Consider a weak solution $(X_\varepsilon, W_\varepsilon)$ to

$$X_\varepsilon(t) = x + \int_0^t f(X_\varepsilon(s))ds + \varepsilon W_\varepsilon(t),\ t \in [0, T].$$

Note that $(X_\varepsilon, W_\varepsilon)$ is still solution to the same equation with f replaced by $\bar{f}$. So we assume that

$$\forall x \in \mathbb{R}^N,\ F_f(x) = \bigcap_{\delta>0} cof(x + \delta B).$$

The laws $\{P_\varepsilon(X_\varepsilon, W_\varepsilon)^{-1},\ \varepsilon > 0\}$ are tight. Hence $\exists \varepsilon_n \to 0^+$ with

$$P_{\varepsilon_n}(X_{\varepsilon_n}, W_{\varepsilon_n})^{-1} \to P(X, W)^{-1} \text{ in } \mathcal{D}, \text{ as } n \to +\infty.$$

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Proof of Main Theorem

Set $Y_\varepsilon(t) := X_\varepsilon(t) - \varepsilon W_\varepsilon(t)$ which satisfies

$$Y'_\varepsilon(t) = f(X_\varepsilon(t)),\ t \geq 0,\ Y_\varepsilon(0) = x.$$

Using Skohorod's Theorem we can find a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and stochastic processes $\widetilde{X}_{\varepsilon_n}, \widetilde{W}_{\varepsilon_n}, \widetilde{X}, \widetilde{W}$ defined over $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, such that

- $\widetilde{P}(\widetilde{X}_{\varepsilon_n}, \widetilde{W}_{\varepsilon_n})^{-1} = P(X_{\varepsilon_n}, W_{\varepsilon_n})^{-1}$, $n \geq 1$, and $\widetilde{P}(\widetilde{X}, \widetilde{W})^{-1} = P(X, W)^{-1}$,
- in the topology of the uniform convergence on compacts, $\widetilde{X}_{\varepsilon_n} \to \widetilde{X}$, $\widetilde{W}_{\varepsilon_n} \to \widetilde{W}$, $\widetilde{P}$-a.s.

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Proof of Main Theorem

Hence, for $T > 0$,

$$\widetilde{X}_{\varepsilon_n} \to \widetilde{X},\ \widetilde{W}_{\varepsilon_n} \to \widetilde{W} \text{ in } C([0,T),\mathbb{R}^n),\ \widetilde{P}\text{- a.s.},$$

from where we easily get that

$$\widetilde{Y}_{\varepsilon_n} := \widetilde{X}_{\varepsilon_n} - \varepsilon_n \widetilde{W}_{\varepsilon_n} \to \widetilde{X},\ \text{in } C([0,T),\mathbb{R}^n),\ \widetilde{P}\text{- a.s.}.$$

Because $|\widetilde{Y}'_{\varepsilon_n}(t)| \le M$, so outside a $\widetilde{P}$-null set, $\widetilde{Y}_{\varepsilon_n}$ converges P. a. s. to $\widetilde{X}$ in $W^{1,\infty}_{weak}([0,T],\mathbb{R}^n)$, and hence also in $W^{1,\infty}_{weak}([0,T],L^2(\widetilde{\Omega},\mathbb{R}^n))$.

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Proof of Main Theorem

Fix $\delta > 0$. Then, since

$$\eta_{\varepsilon_n} = \sup_{k \geq n,\ t \in [0,T]} |\widetilde{X}_{\varepsilon_k}(t) - \widetilde{X}(t)| \geq 0 \text{ converge to } 0\ \widetilde{P} \text{ a.s.}$$

$\widetilde{Y}'_{\varepsilon_n}(t) = f(\widetilde{X}_{\varepsilon_n}(t)) \in f(\widetilde{X}(t) + \eta_{\varepsilon_n} B) \subset cof(\widetilde{X}(t) + \delta B)$, $\forall n$ large enough.
Passing to the limit when $n \to \infty$,

$$\widetilde{X}'(t) \in cof(\widetilde{X}(t) + \delta B),$$

$\widetilde{P}$-a.s.. Hence, δ being arbitrary,

$$\widetilde{X}'(t) \in \bigcap_{\delta > 0} cof(\widetilde{X}(t) + \delta B) = F_f(\widetilde{X}(t)) \text{ a.e } t \geq 0.$$

Observe that

$$1 = \widetilde{P}[\widetilde{X}'(t) \in F_f(\widetilde{X}(t)) \text{ for a.e } t \geq 0.] = P[X'(t) \in F_f(X(t)) \text{ for a.e } t \geq 0.]$$

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- 1- Filippov's solutions to ordinary differential equations
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Examples in dimension 1

[Bafico Baldi] Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous around $x \in \mathbb{R}$ an isolated zero of f and for some $r > 0$,

$$\int_x^{x+r} \frac{1}{f(y)} < +\infty, \int_x^{x-r} \frac{1}{f(y)} = +\infty,$$

and for some $\delta > 0$ the functions

$$h(x) = \min_{[x,x+\delta]} f, \; g(x) = \max_{[x-\delta,x]} f$$

satisfies

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Examples in dimension 1

[Gradinaru, Herrmann, Roynette] Fix $0 < \gamma < 1$. The differential equation

$$x'(t) = \mathrm{sgn}x(t)|x(t)|^{\gamma},\ x(0) = 0$$

has two extremal solutions

$$x_1(t) = (t(1-\gamma))^{\frac{1}{1-\gamma}},\ x_2(t) = -(t(1-\gamma))^{\frac{1}{1-\gamma}}$$

Let p_t^{ε} the density of the law of the solution X_t^{ε} of SDE.

If $|x| > x_1(t)$ then there exists a positive function k_t with

$$\lim_{\varepsilon \to 0^+} \varepsilon^2 p_t^{\varepsilon}(x) = -k_t(|x|)$$

If $x_1(t) > |x| > x_2(t)$ then there exists a function l_t with

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Applications to existence of solution to some ODE

Consider f measurable bounded and suppose that there is some $C > 0$ with

$$< f(x) - f(y), x - y > \leq C|x - y|^2, \ \forall x, y \in \mathbb{R}^n$$

Then the ODE has a unique Filippov solution which is the limit of solutions to SDEs

- The solutions of the SDEs converge to some process X supported by Filippov's solutions
- The process X is deterministic.
- There is at most one Filippov's solution.

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Further generalizations

Consider σ Lipschitz and b measurable bounded

$$dX(t) = b(X(t))dt + \sigma(X(t))dW_t, \; t \geq 0, \; x(0) = x,$$

$$\text{(SSDE)} \qquad dX(t) = b(X(t))dt + \sigma(X(t))dW_t + \varepsilon dB_t, \; t \geq 0, \; x(0) = x,$$

where B_t is independent with W_t**Result** The solutions to (SSDE) converge weakly to a solution to

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