INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. 2002; **16**:435–453 (DOI: 10.1002/acs.703)

On-line almost-sure parameter estimation for partially observed discrete-time linear systems with known noise characteristics

Robert J. Elliott[1], Jason J. Ford[2,*,†] and John B. Moore[3]

[1] *Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada*
[2] *Weapons Systems Division, Defence Science Technology Organisation, P.O. Box 1500, Salisbury, SA, 5108, Australia*
[3] *Department of Systems Engineering & Cooperative Research Centre for Robust and Adaptive Systems, Research School of Information Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia*

SUMMARY

In this paper we discuss parameter estimators for fully and partially observed discrete-time linear stochastic systems (in state-space form) with known noise characteristics. We propose finite-dimensional parameter estimators that are based on estimates of summed functions of the state, rather than of the states themselves. We limit our investigation to estimation of the state transition matrix and the observation matrix. We establish almost-sure convergence results for our proposed parameter estimators using standard martingale convergence results, the Kronecker lemma and an ordinary differential equation approach. We also provide simulation studies which illustrate the performance of these estimators. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: adaptive estimation; parameter estimation; system identification

1. INTRODUCTION

The problem of identifying linear systems has received considerable attention in many scientific communities over many years, see References [1–7] for illustrative references. The origin of the identification problem can be traced back to the method of least squares and the work of Gauss in 1809. Since the 1950s, various recursive identification algorithms have become available including recursive least squares, recursive stochastic algorithms, instrumental variable methods, recursive maximum likelihood methods and general recursive prediction error approaches (see Reference [8] for general details about these algorithms). These recursive algorithms have been widely applied to many problems.

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*Correspondence to: Jason J. Ford, Weapons Systems Division, Defence Science Technology Organisation, P.O. Box 1500, Salisbury, SA, 5108, Australia.
†E-mail: Jason.Ford@dsto.defence.gov.au

Contract/grant sponsor: NSERC; contract/grant number: A7964

Published online 28 February 2002

Received 17 June 1998
Revised 17 May 2001
Accepted 16 August 2001

Identification of fully observed linear systems (where the output, inputs and states are all measured) can be readily understood in terms of the least squares problem. The linear relationship between input and output signals ensures efficient identification and global convergence of parameter estimates can be readily established in many problems, see References [6,7] for two examples. However, in partially observed linear systems (particularly if the input cannot be measured) similar convergence results are more difficult to obtain. In this situation, recursive approaches like Ljung's scheme lead to locally convergent algorithms, if convergence results can be established at all [8,9].

Although not considered here, sub-space methods appear to provide attractive approaches to identification of partially observed systems. In fact, strong convergence results have been established for sub-space algorithms [9] including global convergence results for the partially observed linear systems we consider here [10]. Two notable disadvantages of sub-space methods are their robustness problems and their computational requirements, see Reference [9].

Recently, in the related area of parameter estimation of hidden Markov models, almost-surely convergent parameter estimators algorithms have been proposed for hidden Markov models in discrete and continuous time [11,12]. Similarly, almost surely convergent algorithm for linear systems in continuous time have been developed [13]. Rather than work with a prediction error cost (in a parallel manner to Ljung's approach) these algorithms are based on estimates for summed functions of the state and these algorithms can be shown to be almost-surely convergent [11–13]. These three new algorithms suggest a new approach to the system identification problem for discrete-time linear stochastic systems.

This paper investigates the problem of recursively estimating a partially observed discrete-time linear stochastic system (in state-space form). We assume the input (or disturbance) is not measured but the statistics of the input and the measurement noise of the system are known. A typical application might be estimation of a noise model. In this paper we propose new recursive parameter estimators which are based on estimates of summed functions of the state rather than minimization of a cost function.

Convergence results for the algorithms are established using martingale convergence results, the Kronecker lemma and ODE methods. Firstly, preliminary almost-sure convergence results and convergence rates are established using martingale convergence results and the Kronecker lemma when the state is known or conditional mean estimates are available. Then, global convergence results are established using the ordinary differential equation (ODE) approach, which is introduced and discussed in References [8,14–17].

The paper is organized as follows: in Section 2, the partially observed discrete-time linear system model (in state-space form) is introduced and new recursive parameter estimators are proposed for the special case when the state is measured directly. Almost-sure convergence of these parameter estimators is established via martingale convergence results. In Section 3, parameter estimation is discussed when the state is not measured directly. We develop the relevant conditional mean filters, propose parameter estimators and establish almost-sure convergence results using an ODE approach. In Section 4 we present some simulation studies. Conclusions are given in Section 5.

2. DYNAMICS

Consider a probability space $(\Sigma, \mathscr{F}, P)$; suppose $\{\mathbf{x}_\ell\},\ \ell \in \mathscr{Z}^+$ is a discrete-time linear stochastic process, taking values in R^N, with dynamics given by

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{w}_{k+1}, \quad \mathbf{x}_0 \in R^{N\times 1} \tag{1}$$

Here $k \in \mathscr{Z}^+,\ \mathbf{A} \in R^{N\times N}$, $\mathbf{B} \in R^{N\times N}$ and $\{\mathbf{w}_\ell\} \in R^{N\times 1},\ \ell \in \mathscr{Z}^+$, is a sequence of vectors whose elements are independent and identically distributed $N(0,1)$ scalar random variables (with bounded 4th moment).

The state process $\{\mathbf{x}_\ell\}, \ell \in \mathscr{Z}^+$, is observed indirectly via the scalar observation process $\{y_\ell\},\ \ell \in \mathscr{Z}^+$, given by

$$y_k = \mathbf{C}\mathbf{x}_k + Dv_k \tag{2}$$

Here $k \in \mathscr{Z}^+,\ \mathbf{C} \in R^{1\times N}$, $D \in R$ and $\{v_\ell\},\ \ell \in \mathscr{Z}^+$, is a sequence of independent and identically distributed $N(0,1)$ scalar random variables. We assume that $\mathbf{x}_0$, $\{v_\ell\}$ and $\{\mathbf{w}_\ell\}$ are mutually independent. In this paper we assume scalar observations but this can be extended to vector observations.

Let $\mathscr{F}_k = \sigma\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ and $\mathscr{Y}_k = \sigma\{y_0, y_1, \ldots, y_k\}$ denote the filtrations generated by $\mathbf{x}$ and y, respectively. Also, let $\mathscr{G}_k = \sigma\{\mathbf{x}_0, \ldots, \mathbf{x}_k, y_0, \ldots, y_k\}$ denote the filtration generated by $\mathbf{x}$ and y.

Denote the model (1), (2) by

$$\lambda = \lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}, D, \mathbf{x}_0) \tag{3}$$

We are interested in estimation of $\mathbf{A}$ and $\mathbf{C}$.

2.1. *Parameter estimation—full observations*

In this subsection we assume that both $\{\mathbf{x}_k\}$ and $\{y_k\}$ are fully observed. The results in this section for the full observation case are presented as a stepping stone to the more interesting and general results of Section 3.

From (1), by post-multiplication by $\mathbf{x}'_k$ and summing, we obtain

$$\sum_{i=1}^{k} \mathbf{x}_i\mathbf{x}'_{i-1} = \mathbf{A}\sum_{i=1}^{k} \mathbf{x}_{i-1}\mathbf{x}'_{i-1} + \mathbf{B}\sum_{i=1}^{k} \mathbf{w}_i\mathbf{x}'_{i-1} \tag{4}$$

Here the prime $'$ denotes the transpose operation. Now consider the matrices

$$\mathbf{J}_k := \sum_{i=1}^{k} \mathbf{x}_i\mathbf{x}'_{i-1} \quad \mathbf{O}_k := \sum_{i=1}^{k} \mathbf{x}_{i-1}\mathbf{x}'_{i-1} \quad \mathbf{M}_k := \sum_{i=1}^{k} \mathbf{w}_i\mathbf{x}'_{i-1} \tag{5}$$

From (4) we see $\mathbf{J}_k = \mathbf{A}\mathbf{O}_k + \mathbf{B}\mathbf{M}_k$. Assuming that $\mathbf{x}_k$ is observed, a reasonable estimate for $\mathbf{A}$ is therefore

$$\hat{\mathbf{A}}_{k|x} = \mathbf{J}_k\mathbf{O}_k^{-1} \tag{6}$$

when $\mathbf{O}_k^{-1}$ exists. The error in this estimate is $\hat{\mathbf{A}}_{k|x} - \mathbf{A} = \mathbf{B}\mathbf{M}_k\mathbf{O}_k^{-1}$.

Similarly, from (2) by post-multiplication by $\mathbf{x}'_k$ and summing we obtain

$$\sum_{i=1}^{k} y_{i-1}\mathbf{x}'_{i-1} = \mathbf{C} \sum_{i=1}^{k} \mathbf{x}_{i-1}\mathbf{x}'_{i-1} + D \sum_{i=1}^{k} v_{i-1}\mathbf{x}'_{i-1} \tag{7}$$

Now consider the vectors

$$\mathbf{T}_k := \sum_{i=1}^{k} y_{i-1}\mathbf{x}'_{i-1} \quad \mathbf{V}_k := \sum_{i=1}^{k} v_{i-1}\mathbf{x}'_{i-1} \tag{8}$$

Then (7) can be written $\mathbf{T}_k = \mathbf{C}\mathbf{O}_k + D\mathbf{V}_k$. A reasonable estimate for $\mathbf{C}$ is therefore

$$\hat{\mathbf{C}}_{k|x} = \mathbf{T}_k\mathbf{O}_k^{-1} \tag{9}$$

when $\mathbf{O}_k^{-1}$ exists. The error in this estimate is $\hat{\mathbf{C}}_{k|x} - \mathbf{C} = D\mathbf{V}_k\mathbf{O}_k^{-1}$.

Remark

Although we assume in this paper that $\mathbf{B}$ and D are known, it is possible to estimate these quantities at the same time as estimating $\mathbf{A}$ and $\mathbf{C}$. Local convergence results for online estimation of $\mathbf{B}$ and D are given in Reference [18]. Convergence results for off-line estimators are given in Reference [19].

2.2. Almost-sure convergence

In this subsection we discuss convergence of these estimators. Before proceeding to our convergence results we first state stability and ergodicity results for linear systems.

A time-invariant system with state transition matrix A is *strictly stable* if the following condition holds

$$\sigma_{\max}(\mathbf{A}) < 1 \tag{10}$$

where $\sigma_{\max}(\mathbf{A})$ is the largest magnitude of the eigenvalue of the matrix $\mathbf{A}$.

Lemma 1

When the time-invariant system is strictly stable, i.e. (10) holds, the state sequence is ergodic, that is,

$$E[f(\mathbf{x}_k)] = \lim_{T\to\infty} \frac{1}{T} \sum_{\ell=1}^{T} f(\mathbf{x}_\ell) \quad \text{a.s. for any } k \tag{11}$$

Proof

This follows from ergodic theory [4, p. 34] because the system is uniformly stable and the elements of $\mathbf{w}_k$ have zero mean, finite variance and bounded fourth moments. □

Lemma 2

The system (1), (2) is persistently excited in that

$$\lim_{k\to\infty} \left(\frac{1}{k}\mathbf{O}_k\right)^{-1} \text{ is finite and } \left(\frac{1}{k}\mathbf{O}_k\right)^{-1} \text{ is bounded for all } k \tag{12}$$

Proof

See Reference [20]. □

Now consider

$$\mathbf{R}_k := \sum_{i=1}^{k} \rho(i)^{-1/2} \Delta \mathbf{M}_i$$

where $\Delta \mathbf{M}_i := \mathbf{M}_i - \mathbf{M}_{i-1} = \mathbf{w}_i \mathbf{x}'_{i-1}$ and here $\rho(k)$ is any function for which $\rho(k)$, $k \geqslant 0$, is positive, non-decreasing and such that $\lim_{k\to\infty} \sum_{i=0}^{k} \rho(i)^{-1} = B_\rho < \infty$. An example of $\rho(k)$ satisfying this requirement is $\rho(k) = \max(1, k \log_n(k)(\log_n(\log_n(k)))^\alpha)$, for any $\alpha > 1$.

Lemma 3

Suppose the system (1), (2) is strictly stable such that (10) holds. Then $\mathbf{R}_k$ is a matrix whose elements, $\mathbf{R}_k^{ij}$ for $i, j = 1, \ldots, N$, are square integrable martingales with respect to $\mathscr{F}_k$ so that $\lim_{k\to\infty} \mathbf{R}_k^{ij} = \xi^{ij}(\omega) < \infty$ exists a.s.

Proof

$\mathbf{R}_k^{ij}$ is a martingale because $E[\mathbf{R}_k^{ij} | \mathscr{F}_{k-1}] = E[\rho(k)^{-1/2} \Delta \mathbf{M}_k^{ij} + \mathbf{R}_{k-1}^{ij} | \mathscr{F}_{k-1}] = \mathbf{R}_{k-1}^{ij}$, where $\Delta \mathbf{M}_k^{ij}$ is the ijth element of $\Delta \mathbf{M}_k$. Also, the $\mathbf{R}_k^{ij}$ are bounded in L^2 because

$$\begin{aligned} E[(\mathbf{R}_k^{ij})^2] &= E\left[\sum_{\ell=1}^{k} \rho(\ell)^{-1} (\Delta \mathbf{M}_\ell^{ij})^2\right] \\ &= E\left[\sum_{\ell=1}^{k} \rho(\ell)^{-1} E[(\mathbf{w}_\ell^i)^2 (\mathbf{x}_{\ell-1}^j)^2 | \mathscr{F}_{\ell-1}]\right] \\ &= \sum_{\ell=1}^{k} \rho(\ell)^{-1} E[(\mathbf{x}_{\ell-1}^j)^2] \\ &< \infty \end{aligned}$$

We have used that $\mathbf{w}_\ell$ and $\mathbf{x}_{\ell-1}$ are uncorrelated, that $E[(\mathbf{w}_\ell^i)^2 | \mathscr{F}_{\ell-1}] = 1$ and that for strictly stable systems $E[(\mathbf{x}_k^j)^2] < B_\infty$ for all k, j for some $B_\infty < \infty$. By standard martingale results, [21,22],

$$\lim_{k\to\infty} \mathbf{R}_k^{ij} = \xi^{ij}(\omega) < \infty \quad \text{a.s. for } i, j = 1, \ldots, N$$ □

Lemma 4

Suppose the system (1), (2) is strictly stable, i.e. (10). Then

$$\lim_{k\to\infty} \rho(k)^{-1/2} \sum_{i=1}^{k} \Delta \mathbf{M}_i = \mathbf{0}_{N\times N} \quad \text{a.s.}$$

where $\mathbf{0}_{N\times N}$ is the $(N \times N)$ matrix of all zeros. That is,

$$\lim_{k\to\infty} \rho(k)^{-1/2} \mathbf{M}_k = \mathbf{0}_{N\times N} \quad \text{a.s.}$$

Proof
Follows from Lemma 3 and by applying the Kronecker Lemma to each element of $\mathbf{R}_k$, see also References [22,23]. □

Theorem 1
Consider the linear system (1), (2) and suppose the $\mathbf{x}_k$ and y_k are both observed. Suppose (10) holds. Then

$$\lim_{k\to\infty} \hat{\mathbf{A}}_{k|x}, \hat{\mathbf{C}}_{k|x}, = \mathbf{A}, \mathbf{C} \quad \text{a.s. and}$$

the almost sure convergence rate is at least $(k^{-2}\rho(k))^{1/2}$.

Proof
First consider the error in estimation of $\mathbf{A}$, that is $\hat{\mathbf{A}}_k - \mathbf{A} = \mathbf{B}((1/k)\mathbf{M}_k)((1/k)\mathbf{O}_k)^{-1}$. From Lemma 4 we have that

$$\lim_{k\to\infty} \frac{1}{k}\mathbf{M}_k = \mathbf{0}_{N\times N} \quad \text{a.s.} \tag{13}$$

at a rate of $(\rho(k)/k^2)^{1/2}$. Now $\lim_{k\to\infty}((1/k)O_k)^{-1}$ is finite, so

$$\lim_{k\to\infty} \tilde{\mathbf{A}}_k := \hat{\mathbf{A}}_{k|x} - \mathbf{A} = \mathbf{0}_{N\times N} \text{ a.s.}$$

Indeed $\tilde{\mathbf{A}}_k \to \mathbf{0}_{N\times N}$ a.s. at a rate $(\rho(k)/k^2)^{1/2}$. The result for C follows similarly using that $\mathbf{x}_{\ell-1}$ and $v_{\ell-1}$ are uncorrelated and that $E[v_k^2] = 1$. □

Remark
Similar convergence results for fully observed stochastic linear systems are common, see References [6,7,9,10] for examples.

Next we propose parameter estimators for partially observed linear systems based on the conditional mean estimates of $\mathbf{J}_k$, $\mathbf{O}_k$ and $\mathbf{T}_k$. Convergence results are also presented.

3. CONDITIONAL MEAN ESTIMATES

In this section we consider parameter estimators based on conditional mean estimates in lieu of the states. Consider again the system (1), (2). We define a model set, Λ, of allowable model estimates $\hat{\lambda}_k$ and assume that the 'correct' model $\lambda \in \Lambda$. The model set considered in this paper is $\Lambda = \{\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}, D, \mathbf{x}_0) : N\text{th order model}\}$. In this paper we consider estimation of $\mathbf{A}$ and $\mathbf{C}$ only and assume that $\mathbf{B}$, D and $\mathbf{x}_0$ are known (even if x_0 is not known its influence on estimation will be forgotten as k increases). That is, $\hat{\lambda}_k = \lambda(\bar{\mathbf{A}}_{k-1}, \mathbf{B}, \bar{\mathbf{C}}_{k-1}, D, \mathbf{x}_0)$ where $\bar{\mathbf{A}}_{k-1}$ and $\bar{\mathbf{C}}_{k-1}$ are estimates for $\mathbf{A}$ and $\mathbf{C}$ based on measurements up until time $k-1$ (we will give the estimators later).

Let us denote the associated conditional mean estimates based on the signal generating system λ as in (3), also termed the 'correct' model as

$$\hat{\mathbf{J}}_{k|k,\lambda} = E[\mathbf{J}_k|\mathcal{Y}_k, \lambda], \quad \hat{\mathbf{O}}_{k|k,\lambda} = E[\mathbf{O}_k|\mathcal{Y}_k, \lambda], \quad \hat{\mathbf{T}}_{k|k,\lambda} = E[\mathbf{T}_k|\mathcal{Y}_k, \lambda] \tag{14}$$

For the purposes of the next definition we allow the system (1), (2) to be possibly time-varying. Let λ_k denote the system at time k, then denote the associated conditional mean estimates based on a possibly time varying estimate $\hat{\Lambda}_k = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_k\}$, also termed the 'incorrect' model, as

$$\hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k} = E[\mathbf{J}_k|\mathcal{Y}_k, \hat{\Lambda}_k], \quad \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} = E[\mathbf{O}_k|\mathcal{Y}_k, \hat{\Lambda}_k], \quad \hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k} = E[\mathbf{T}_k|\mathcal{Y}_k, \hat{\Lambda}_k] \tag{15}$$

where $E[J_k|\mathcal{Y}_k, \hat{\Lambda}_k] = E[J_k|\mathcal{Y}_k, \lambda_k = \hat{\lambda}_k, \lambda_{k-1} = \hat{\lambda}_{k-1}, \ldots]$ etc. In the rest of the paper we will use both conditional means estimates based on one model (if the subscript has one model) and conditional mean estimates based on model sequences (if the subscript has a model sequence).

Recursions for (14) and (15) are given in Reference [19]. We repeat them below (for (14) set $\hat{\lambda}_k = \lambda$ for all k):

$$\hat{\mathbf{J}}^{ij}_{k|k,\hat{\Lambda}_k} = \mathbf{a}(J)^{ij}_k + \mathbf{b}(J)^{ij'}_k \mu_k + \sum_{p=1}^{N} \sum_{q=1}^{N} \mathbf{d}(J)^{ij}_k(p,q)\mathbf{R}_k(p,q) + \mu'_k \mathbf{d}(J)^{ij}_k \mu_k \tag{16}$$

$$\hat{\mathbf{O}}^{ij}_{k|k,\hat{\Lambda}_k} = \mathbf{a}(O)^{ij}_k + \mathbf{b}(O)^{ij'}_k \mu_k + \sum_{p=1}^{N} \sum_{q=1}^{N} \mathbf{d}(O)^{ij}_k(p,q)\mathbf{R}_k(p,q) + \mu'_k \mathbf{d}(O)^{ij}_k \mu_k \tag{17}$$

$$\hat{\mathbf{T}}^{i}_{k|k,\hat{\Lambda}_k} = \mathbf{a}(T)^{i}_k + \mathbf{b}(T)^{i'}_k \mu_k \tag{18}$$

for $1 \leqslant i, j \leqslant N$ where $\mathbf{d}(J)^{ij}_k(p,q)$ and $\mathbf{R}_k(p,q)$ are the pqth elements of $\mathbf{d}(J)^{ij}_k$ and $\mathbf{R}_k$, respectively. Here,

$$\begin{aligned}\mathbf{a}(J)^{ij}_{k+1} &= \mathbf{a}(J)^{ij}_k + \mathbf{b}(J)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k + \mathrm{Tr}[\mathbf{d}(J)^{ij}_k \sigma^{-1}_{k+1}] \\ &+ \mu'_k \mathbf{R}^{-1}_k \sigma^{-1}_{k+1} \mathbf{d}(J)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k\end{aligned} \tag{19}$$

$$\mathbf{b}(J)^{ij}_{k+1} = \mathbf{B}^{-2} \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} (\mathbf{b}(J)^{ij}_k + 2\mathbf{d}(J)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k) + \mathbf{e}_i \mathbf{e}'_j \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k \tag{20}$$

$$\mathbf{d}(J)^{ij}_{k+1} = \mathbf{B}^{-2} \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} \mathbf{d}(J)^{ij}_k \sigma^{-1}_{k+1} \bar{\mathbf{A}}_k \mathbf{B}^{-2} + \tfrac{1}{2}(\mathbf{e}_i \mathbf{e}'_j \sigma^{-1}_{k+1} \bar{\mathbf{A}}_k \mathbf{B}^2 + \mathbf{B}^2 \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} \mathbf{e}_j \mathbf{e}'_i) \tag{21}$$

$$\begin{aligned}\mathbf{a}(O)^{ij}_{k+1} &= \mathbf{a}(O)^{ij}_k + \mathbf{b}(O)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k + \mathrm{Tr}[\mathbf{d}(O)^{ij}_k \sigma^{-1}_{k+1}] \\ &+ \mu'_k \mathbf{R}^{-1}_k \sigma^{-1}_{k+1} \mathbf{d}(O)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k\end{aligned} \tag{22}$$

$$\mathbf{b}(O)^{ij}_{k+1} = \mathbf{B}^{-2} \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} (\mathbf{b}(O)^{ij}_k + 2\mathbf{d}(O)^{ij}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k) \tag{23}$$

$$\mathbf{d}(O)^{ij}_{k+1} = \mathbf{B}^{-2} \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} \mathbf{d}(O)^{ij}_k \sigma^{-1}_{k+1} \bar{\mathbf{A}}_k \mathbf{B}^{-2} + \tfrac{1}{2}(\mathbf{e}_i \mathbf{e}'_j + \mathbf{e}_j \mathbf{e}'_i) \tag{24}$$

$$\mathbf{a}(T)^{i}_{k+1} = \mathbf{a}(T)^{i}_k + \mathbf{b}(T)^{i'}_k \sigma^{-1}_{k+1} \mathbf{R}^{-1}_k \mu_k \tag{25}$$

$$\mathbf{b}(T)^{i}_{k+1} = \mathbf{B}^{-2} \bar{\mathbf{A}}_k \sigma^{-1}_{k+1} \mathbf{b}(T)^{i}_k + \mathbf{e}_i y_{k+1} \tag{26}$$

with $\mathbf{a}(\ell)^{ij}_0 = 0$, $\mathbf{b}(\ell)^{ij}_0 = \mathbf{0}_{N\times 1}$ and $\mathbf{d}(\ell)^{ij}_0 = \mathbf{0}_{N\times N}$ for $\ell = J$ and $\ell = O$, $\mathbf{a}(T)^{i}_0 = 0$ and $\mathbf{b}(T)^{i}_0 = \mathbf{e}_i y_0$. Here Tr(.) denotes the trace of a matrix, $\mathbf{B}^{-2}$ is shorthand for $(\mathbf{B}^2)^{-1}$, $\mathbf{e}_i$ is a vector of zeros except the ith element which is 1, $\sigma_k = \bar{\mathbf{A}}_{k-1} \mathbf{B}^{-2} \bar{\mathbf{A}}_{k-1} + \mathbf{R}^{-1}_{k-1}$, and μ_k and $\mathbf{R}_k$ are the mean and variance given by standard Kalman filter equations, that is,

$$\mu_k = \mathbf{R}_k \mathbf{B}^{-2} \bar{\mathbf{A}}_{k-1} \sigma_k^{-1} \mathbf{R}_{k-1}^{-1} \mu_{k-1} + \mathbf{R}_k \bar{\mathbf{C}}'_{k-1} D^{-2} y_k \tag{27}$$

$$\mathbf{R}_k = [(\bar{\mathbf{A}}_{k-1} \mathbf{R}_{k-1} \bar{\mathbf{A}}'_{k-1} + \mathbf{B}^2)^{-1} + \bar{\mathbf{C}}'_{k-1} D^{-2} \bar{\mathbf{C}}_{k-1}]^{-1} \tag{28}$$

Remark

Although the system we are estimating is time-invariant, our filters for $\mathbf{J}_k$ etc. are based on a possibly time-varying system (i.e. on time-varying estimates $\bar{\mathbf{A}}_k$ and $\bar{\mathbf{C}}_k$).

The following lemma holds.

Lemma 5

Consider the linear system (1), (2) and a sequence of model estimates $\bar{\mathbf{A}}_k$, $\bar{\mathbf{C}}_k$. Assume that $[\bar{\mathbf{A}}_k, \mathbf{B}]$ is completely stabilisable and $[\bar{\mathbf{A}}_k, \bar{\mathbf{C}}_k]$ is completely detectable (see References [24,25] for more details). Then $\Delta\mathbf{J}_k$, $\Delta\mathbf{O}_k$ and $\Delta\mathbf{T}_k$ are derived from exponentially stable systems, or equivalently are exponentially forgetting of initial conditions, where

$$\Delta\mathbf{J}_k := \hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{J}}_{k-1|k-1,\hat{\Lambda}_{k-1}}, \ \Delta\mathbf{O}_k := \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{O}}_{k-1|k-1,\hat{\Lambda}_{k-1}}, \quad \text{and}$$

$$\Delta\mathbf{T}_k := \hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{T}}_{k-1|k-1,\hat{\Lambda}_{k-1}}$$

Proof

We first establish the result for $\Delta\mathbf{J}_k$. Firstly, note that it is shown in Reference [24] and elsewhere that under the lemma conditions the Kalman filter for a time-varying system is exponentially stable and that $\mathbf{R}_k^{-1}$ is bounded. Hence, we need only examine the stability of the recursions (19)–(21).

To show exponential stability for $\Delta\mathbf{J}_k$ we exploit the fact that the Kalman filter is exponentially stable. We rewrite (27) and left multiply by $\mathbf{R}_k^{-1}$ giving a recursion for $\mathbf{R}_k^{-1}\mu_k$ which is exponentially stable under the lemma conditions (because the Kalman filter is exponentially stable)

$$\mathbf{R}_k^{-1}\mu_k = \mathbf{B}^{-2} \bar{\mathbf{A}}_{k-1} \sigma_k^{-1} \mathbf{R}_{k-1}^{-1} \mu_{k-1} + \bar{\mathbf{C}}'_{k-1} D^{-2} y_k \tag{29}$$

We first consider the $\mathbf{d}(J)_k^{ij}$ recursion (21) because it is uncoupled from the $\mathbf{a}(J)_k^{ij}$ and $\mathbf{b}(J)_k^{ij}$ recursions. The transition term in (29) appears twice in (21) and hence the recursion for $\mathbf{d}(J)_k^{ij}$ is exponentially stable.

Likewise, for the $\mathbf{b}(J)_k^{ij}$ recursion (20) we note that μ_k, $\mathbf{R}_k^{-1}$ and $\mathbf{d}(J)_k^{ij}$ are exponentially stable and hence we need only consider the term involving $\mathbf{b}(J)_k^{ij}$. This term has the same transition term as the (29) recursion and hence (20) is exponentially stable.

Now, from (16) we write

$$\Delta \mathbf{J}_k^{ij} = \mathbf{a}(J)_k^{ij} + \mathbf{b}(J)_k^{ij'} \mu_k + \sum_{p=1}^{N} \sum_{q=1}^{N} \mathbf{d}(J)_k^{ij}(p,q)\mathbf{R}_k(p,q) + \mu_k' \mathbf{d}_k(J)^{ij} \mu_k$$
$$- \Bigg[\mathbf{a}(J)_{k-1}^{ij} + \mathbf{b}(J)_{k-1}^{ij'} \mu_k + \sum_{p=1}^{N} \sum_{q=1}^{N} \mathbf{d}(J)_{k-1}^{ij}(p,q)\mathbf{R}_k(p,q)$$
$$+ \mu_k' \mathbf{d}_{k-1}(J)^{ij} \mu_k \Bigg]$$

Substitution of the recursion for $\mathbf{a}(J)_k^{ij}$ in this equation shows that $\Delta \mathbf{J}_k^{ij}$ depends only on terms in $\mathbf{b}(J)_k^{ij}$, $\mathbf{d}(J)_k^{ij}$, $\mathbf{R}_k$ and μ_k, which are all exponentially stable. Hence, $\Delta \mathbf{J}_k$ is exponentially stable.

The results for $\Delta \mathbf{O}_k$ and $\Delta \mathbf{T}_k$ follow in the same way. □

Remark

To ensure that $[\bar{\mathbf{A}}_k, \mathbf{B}]$ is completely stabilisable and $[\bar{\mathbf{A}}_k, \bar{\mathbf{C}}_k]$ is completely detectable a projection into a stability domain for the filters may be required, see Reference [26] for details.

3.1. *Estimation using conditional mean estimates*

Initially, we consider the (somewhat artificial) special case when it is assumed that the conditional mean estimates based on the correct model are available. We consider the following parameter estimates:

$$\hat{\mathbf{A}}_{k|k,\lambda} = \hat{\mathbf{J}}_{k|k,\lambda} \hat{\mathbf{O}}_{k|k,\lambda}^{-1}, \quad \hat{\mathbf{C}}_{k|k,\lambda} = \hat{\mathbf{T}}_{k|k,\lambda} \hat{\mathbf{O}}_{k|k,\lambda}^{-1} \tag{30}$$

Following this we will consider estimation using conditional mean estimates based on adaptive model estimates. We assume that $\mathbf{B}$ and D are known. We consider the following parameter estimates:

$$\hat{\mathbf{A}}_{k|k,\hat{\Lambda}_k} = \mathrm{Proj}\{\hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k} \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}^{-1}\}, \quad \hat{\mathbf{C}}_{k|k,\hat{\Lambda}_k} = \mathrm{Proj}\{\hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k} \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}^{-1}\} \tag{31}$$

or in recursive form (but not algebraically equivalent form)

$$\hat{\mathbf{A}}_k = \mathrm{Proj}\left\{ \hat{\mathbf{A}}_{k-1} + \frac{1}{k}(\Delta \mathbf{J}_{k|\hat{\Lambda}_k} - \hat{\mathbf{A}}_{k-1} \Delta \mathbf{O}_{k|\hat{\Lambda}_k}) \left(\frac{1}{k} \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} \right)^{-1} \right\} \tag{32}$$

$$\hat{\mathbf{C}}_k = \mathrm{Proj}\left\{ \hat{\mathbf{C}}_{k-1} + \frac{1}{k}\left(\Delta \mathbf{T}_{k|\hat{\Lambda}_k} - \hat{\mathbf{C}}_{k-1} \Delta \mathbf{O}_{k|\hat{\Lambda}_k}\right) \left(\frac{1}{k} \hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} \right)^{-1} \right\} \tag{33}$$

where $\hat{\mathbf{A}}_k$ etc. denote the recursive form of the estimates and define $\Delta \mathbf{J}_{k|\hat{\Lambda}_k} := \hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{J}}_{k-1|k-1,\hat{\Lambda}_{k-1}}$ etc.

The model estimate is adaptively updated as follows:

$$\hat{\lambda}_{k+1} = \lambda(\hat{\mathbf{A}}_k, \mathbf{B}, \hat{\mathbf{C}}_k, D, \mathbf{x}_0) \quad \text{and} \quad \hat{\Lambda}_{k+1} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{k+1}\} \tag{34}$$

Here Proj{.} is a projection operation described as follows. Let P_c denote a compact set of strictly stable models, that are completely stabilisable and completely detectable, in canonical form. The set of strictly stable models is not a compact set and hence we limited the projection to an appropriate compact set of strictly stable models, for example bandlimited models in canonical form. The projection operation is projection onto the set P_c.

Now suppose the persistently excitation condition associated with the signal generating system λ and its estimate $\hat{\Lambda}_k$ holds, so that:

$$\lim_{k\to\infty}\left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1} = \mathbf{O}^L(\omega) \text{ is finite a.s. and } \left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1} \text{ is bounded in } L_2 \text{ a.s.} \quad (35)$$

Remark

When (12) holds, this additional persistently excitation condition (35) is essentially a condition on the sequence of model estimates, see Reference [20] for details on persistence of excitation conditions.

3.2. Preliminary convergence result

In this subsection, we established convergence results for the artificial case when the conditional mean estimates based on the true model are available. In most realistic applications these quantities will not be available but the results in this section are needed to establish the more general convergence result that follows.

Theorem 2

Consider the linear system (1), (2) denoted by λ. Suppose (10) holds. Also assume that conditional mean estimates based on the 'correct model' are available. Assume the true model $\lambda \in P_c \in \Lambda$. Then

$$\lim_{k\to\infty} \hat{\mathbf{A}}_{k|k,\lambda}, \hat{\mathbf{C}}_{k|k,\lambda} = \mathbf{A}, \mathbf{C} \quad \text{a.s.} \quad (36)$$

the almost sure convergence rate is at least $(k^{-2}\rho(k))^{1/2}$.

Proof

We first proceed with the lemma result for $\hat{\mathbf{A}}_{k|k,\lambda}$. Simple manipulations of (4), (14) and (30) give the error term, as:

$$\tilde{\mathbf{A}}_{k|k,\lambda} = \hat{\mathbf{A}}_{k|k,\lambda} - \mathbf{A} = E[k^{-1}\mathbf{B}\mathbf{M}_k|\mathcal{Y}_k,\lambda](E[k^{-1}\mathbf{O}_k|\mathcal{Y}_k,\lambda])^{-1} \quad (37)$$

From Lemma 4 write that $\phi_k := \rho(k)^{-1/2}\mathbf{M}_k = \rho(k)^{-1/2}\sum_{i=1}^{k}\Delta\mathbf{M}_i$, so that $\lim_{k\to\infty}\phi_k = \mathbf{0}_{N\times N}$

a.s. Now, elements of ϕ_k are bounded in L_2 since

$$
\begin{aligned}
E[(\phi_k^{ij})^2] &= E\left[\sum_{\ell=1}^{k}\sum_{t=1}^{k} \rho(k)^{-1/2}\mathbf{w}_{\ell+1}^i \mathbf{x}_\ell^j \mathbf{x}_t^j \mathbf{w}_{t+1}^i \rho(t)^{-1/2}\right] \\
&= E\left[\sum_{\ell=1}^{k} \rho(\ell)^{-1} E[(\mathbf{w}_{\ell+1}^i)^2|\mathscr{F}_\ell] E[(\mathbf{x}_\ell^j)^2|\mathscr{F}_\ell]\right] \\
&= \sum_{\ell=1}^{k} \rho(\ell)^{-1} E[(\mathbf{x}_\ell^j)^2] \\
&< \infty
\end{aligned}
$$

Here we have used that for stable systems $E[(\mathbf{x}_k^j)^2] < B_\infty$ for all k, j. Also that $E[w_\ell w_t|\mathscr{F}_{\min(\ell,t)}] = 0$ for $\ell \neq t$ and $E[(w_{k+1}^i)^2|\mathscr{F}_k] = 1$.

The elements of ϕ_k bounded in L_2 is a uniform integrability condition, which together with the property $\lim_{k\to\infty} \phi_k^{ij} = 0$ a.s. for all $i, j = 1, \ldots, N$ ensures convergence in conditional mean:

$$
E[\phi_k^{ij}|\mathscr{Y}_k, \lambda] \to 0 \quad \text{a.s. for all } i, j = 1, \ldots, N \quad \text{or} \quad E[\phi_k|\mathscr{Y}_k, \lambda] \to \mathbf{0}_{N\times N} \quad \text{a.s.}
$$

(Also $E[\phi_k] \to \mathbf{0}_{N\times N}$.) Hence, $E[k^{-1}M_k|\mathscr{Y}_k, \lambda] \to \mathbf{0}_{N\times N}$ a.s. at a rate $(k^{-2}\rho(k))^{1/2}$. This gives the convergence (and rate) result (36) for $\hat{\mathbf{A}}_{k|k,\lambda}$ under the excitation condition (35) (with $\hat{\lambda}_k = \lambda$ for all k).

Similarly, the lemma convergence result holds for $\hat{\mathbf{C}}_{k|k,\lambda}$. □

Remark

Optimal finite-dimensional filters for $\mathbf{O}_k$, $\mathbf{J}_k$ and $\mathbf{T}_k$ were given earlier and require $O(N^5)$ calculations per time instant [19]. Approximate filters can be implemented for $\mathbf{O}_k$, $\mathbf{J}_k$ and $\mathbf{T}_k$ from Kalman filter state estimates. For example, $\hat{\mathbf{O}}_{k|k,\lambda}^{\text{sub}} = \sum_{\ell=1}^{k} \hat{\mathbf{x}}_{\ell-1}' \hat{\mathbf{x}}_{\ell-1}$. Convergence results when these approximate estimates are used are neither included nor excluded by convergence results of this paper.

We proceed in the next section to consider the more realistic case when conditional mean estimates based on an adaptive model estimate are used.

3.3. Global convergence result

Let $(\mathbf{A}^c, \mathbf{B}^c, \mathbf{C}^c, D^c)$ denote the companion canonical form of the linear system (1), (2).

Theorem 3

Consider the linear system (1), (2) denoted by λ. Suppose (10). Assume $\lambda \in P_c \in \Lambda$. Consider a sequence of estimated models $\hat{\Lambda}_k$ adaptively updated by previous parameter estimates (given by (32) and (33)) so that $\hat{\lambda}_{k+1} = \lambda(\hat{\mathbf{A}}_k, \mathbf{B}, \hat{\mathbf{C}}_k, D, \mathbf{x}_0)$ and $\hat{\Lambda}_{k+1} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{k+1}\}$. We suppose that $\hat{\lambda}_k$ is persistently exciting, along with λ in that (35) holds. Then

$$
\lim_{k\to\infty} \hat{\mathbf{A}}_k, \hat{\mathbf{C}}_k = \mathbf{A}^c, \mathbf{C}^c \quad \text{a.s.} \tag{38}
$$

or to a boundary point of P_c.

Proof
See Appendix A. The proof is based on the results of Kushner [27].

Remark
Here we have considered only adaptive estimation of **A** and **C**.

Summary of algorithm.
We provide here a summary of the on-line estimation algorithm.

1. Choose initial guess, $\hat{\lambda}_1 = (\hat{\mathbf{A}}_0, \mathbf{B}, \hat{\mathbf{C}}_0, D, \mathbf{x}_0)$ and set $\hat{\Lambda}_1 = \{\hat{\lambda}_1\}$. Set $k = 1$.
2. Calculate $\hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k}$, $\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}$ and $\hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k}$ using (16)–(18) and $\hat{\Lambda}_k$.
3. Estimate $\hat{\mathbf{A}}_k$ and $\hat{\mathbf{C}}_k$ using $\hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k}$, $\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}$ and $\hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k}$ by (32) and (33).
4. Update model estimate $\hat{\lambda}_{k+1} = (\hat{\mathbf{A}}_k, \mathbf{B}, \hat{\mathbf{C}}_k, D, \mathbf{x}_0)$ and Λ_{k+1} according to (34).
5. $k = k + 1$. Return to Step 2 (continue until some convergence criteria met).

Remark
In Step 3 above, it is often useful to use Polyak averaging [28] the forms of the estimators (32) and (33), to accelerate convergence.

4. SIMULATIONS

In this section, simulation studies are presented. The companion form used in these simulations is the companion canonical form used in Matlab$^{\text{TM}}$.

Example 1. Adaptive estimation
A 10 000 point, 2-state linear system (1), (2) is generated with parameter values $\mathbf{A} = \begin{bmatrix} 0 & -0.8 \\ 1 & -0.1 \end{bmatrix}$, $\mathbf{B} = [1, 0]'$, $\mathbf{C} = [1, 0.2]'$, $D = 0.01$. This system is in Matlab's companion canonical form and has eigenvalues of $-0.0500 \pm 0.8930i$. In this simulation we assumed that **B** and D are known and **A** and **C** are to be estimated using (16)–(18), (32) and (33). Our initial guess for the model is, $\hat{\mathbf{A}}_0 = \begin{bmatrix} 0.0579 & 0.8132 \\ 0.3529 & 0.0099 \end{bmatrix}$ which has eigenvalues of 0.5701 and -0.5023 and $\hat{\mathbf{C}}_0 = [0.8, 0]'$. Note that this initial guess is not in companion canonical form but will be projected into companion form by the algorithm. Figure 1 shows the evolution of **C** estimates. The final estimated system was $\hat{\mathbf{A}}_{10000} = \begin{bmatrix} 0.0000 & -0.7897 \\ 1.0000 & -0.0621 \end{bmatrix}$ (which has eigenvalues of $-0.0311 \pm 0.8881i$) and $\hat{\mathbf{C}}_{10000} = [0.9638, 0.1824]'$. This estimated system compares well with the generating system. Note that the parameter estimators are not turned on until after 1000 points to allow $\mathbf{J}_k$, $\mathbf{O}_k$ and $\mathbf{T}_k$ filters to forget initial conditions.

Example 2. More noise
The same process is generated except that $D = 1$ (more measurement noise) and $T = 60\,000$. Again we assumed that **B** and D are known and **A** and **C** are to be estimated by (16)–(18), (32)

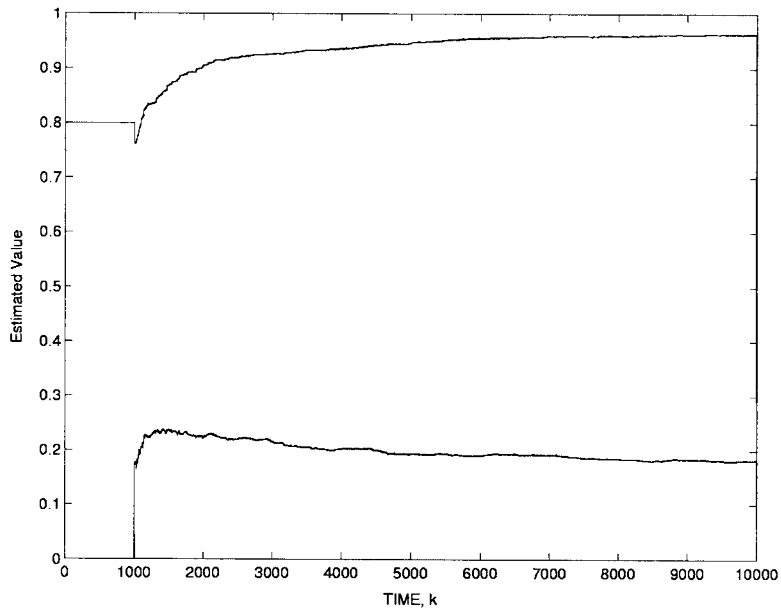

Figure 1. Estimation of C in low noise.

and (33). We use the same initial guess. The estimated values after 60 000 points are $\hat{\mathbf{A}}_{60000} = \begin{bmatrix} 0.0000 & -0.8079 \\ 1.0000 & -0.0755 \end{bmatrix}$ and $\hat{\mathbf{C}}_{60000} = [1.0094, 0.1712]'$. Convergence is slower in this example because of the increased measurement noise level.

Example 3. Real data

To examine the performance of the algorithm with more realistic data we used torque measurement (input) to arm acceleration (output) data for a mechanical robot system. The data is provided by the DAISY project [29]. The input signal is roughly zero mean Gaussian.

We first estimated the system using both input and output measurements and Matlab's n4sid (n4sid requires that both the input and output signals are measured). To illustrate the performance of our algorithm we assume that only the output measurements are available and ignore the input signal. We assume that the n4sid algorithm has estimated the true system and now see if our algorithm can estimate the same system using only the output signal.

For simplicity, we consider only estimation of $\mathbf{A}$. We assume that we have the true $\mathbf{B}$, $\mathbf{C}$ and D (actually we use the $\mathbf{B}$, $\mathbf{C}$ and D estimated by n4sid). Our algorithm was able to estimate the $\mathbf{A}_{\text{n4sid}}$ from a variety of initializations without knowledge of the input signal (for our purposes here it is considered a unknown white noise input). We tried this for a variety of model orders and estimation was possible in each case. For example, when the model order is 2, n4sid estimated $\mathbf{A}_{\text{n4sid}} = \begin{bmatrix} 0 & -0.9788 \\ 1.0000 & 1.4007 \end{bmatrix}$ and $\mathbf{C}_{\text{n4sid}} = [-0.1821, -0.2484]'$, our estimate of $\mathbf{A}$ is

$\hat{\mathbf{A}} = \begin{bmatrix} 0 & -0.9852 \\ 1.0000 & 1.4162 \end{bmatrix}$. This data set was relatively short (1024 data points) and several passes through the data are required.

5. CONCLUSIONS

Global convergence results have been developed for new finite dimensional adaptive schemes for estimation of the parameters of a partially observed discrete-time linear stochastic system. The convergence results are developed using standard martingale properties and convergence results, the Kronecker lemma and an ordinary differential equation approach. We emphasize that, for stable linear systems driven by white noise, there is consistent estimation and, with the model estimate used in adaptive state estimation, this is asymptotically optimal.

ACKNOWLEDGEMENTS

The authors acknowledge the significant contribution of several anonymous reviewers. In particular, the reviewers identified several errors in the original manuscript and their comments led to many improvements to the paper's style and content.

The support of NSERC grant A7964 is gratefully acknowledged.

APPENDIX A: PROOF OF THEOREM 3

Proof

This proof follows the results of Kushner [27]. The proof can also be established using the results of Ljung [8,14]. Related results can be found in References [15–17].

First consider estimation of $\mathbf{A}^{\mathbf{c}}$ only. As given earlier the estimator of A is

$$\hat{\mathbf{A}}_k = \text{Proj}\left\{ \hat{\mathbf{A}}_{k-1} + \frac{1}{k}(\Delta\mathbf{J}_{k|\hat{\Lambda}_k} - \hat{\mathbf{A}}_{k-1}\Delta\mathbf{O}_{k|\hat{\Lambda}_k})\left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1} \right\} \tag{A1}$$

where we defined $\Delta\mathbf{J}_{k|\hat{\Lambda}_k} := \hat{\mathbf{J}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{J}}_{k-1|k-1,\hat{\Lambda}_{k-1}}$ (for a sequence of models) and $\Delta\mathbf{J}_{k|\bar{A}} := \hat{\mathbf{J}}_{k|k,\bar{A}} - \hat{\mathbf{J}}_{k-1|k-1,\bar{A}}$ (for a single model). Define $\Delta\mathbf{O}_{k|\hat{\Lambda}_k}$ and $\Delta\mathbf{O}_{k|\bar{A}}$ similarly.

Convergence of recursion (A1) can be shown by considering an associated ordinary differential equation(ODE). We introduce the following ODE and then show how it is associated with (A1),

$$\frac{\mathrm{d}\bar{\mathbf{A}}^{cv}(\tau)}{\mathrm{d}\tau} = \mathbf{R}_A^{-1}(\bar{\mathbf{A}}(\tau))\mathbf{f}^A(\bar{\mathbf{A}}(\tau)) + \mathbf{z}^A(\bar{\mathbf{A}}(\tau)) \tag{A2}$$

where $\mathbf{z}^A(\bar{\mathbf{A}}(\tau))$ is the projection required to keep $\bar{\mathbf{A}}(\tau)$ in P_c, and $\bar{\mathbf{A}}^{cv}(\tau) := \text{col vec}(\bar{\mathbf{A}}(\tau))$. For an arbitrary matrix $\mathbf{A}$ we define col vec$(\mathbf{A}) := [\mathbf{A}_{11}, \ldots, \mathbf{A}_{N1}, \mathbf{A}_{12}, \ldots, \mathbf{A}_{N2}, \ldots, \mathbf{A}_{NN}]'$.

Let us abbreviate $\bar{\mathbf{A}}(\tau)$ as $\bar{\mathbf{A}}$ and we define $\mathbf{f}^A(\bar{\mathbf{A}})$ and $\mathbf{R}_A(\bar{\mathbf{A}})$ as follows (the superscript and subscript here denote that these functions related to recursion (A2)):

$$\mathbf{f}^A(\bar{\mathbf{A}}) := \text{col vec}(E[\Delta\hat{\mathbf{J}}_{\ell|\bar{\mathbf{A}}} - \bar{\mathbf{A}}\Delta\hat{\mathbf{O}}_{\ell|\bar{\mathbf{A}}}|\bar{\mathbf{A}}]) \quad \text{and}$$
$$\mathbf{R}_A(\bar{\mathbf{A}}) := E\left[\left(\frac{1}{\ell}\hat{\mathbf{O}}_{\ell|\ell,\bar{\mathbf{A}}}\right) \otimes \mathbf{I}_N \,|\bar{\mathbf{A}}\right] \quad \text{for any } \ell \tag{A3}$$

where $\mathbf{I}_N$ is the identity matrix of size $N \times N$ and $\otimes$ is the Kronecker product.

To show how (A2) is related to (A1) we appeal to the Theorem 2.3 in Chapter 5 of Reference [27]. Let

$$\begin{aligned}\Delta_k^A &:= \text{col vec}\left((\Delta\mathbf{J}_{k|\hat{\Lambda}_k} - \hat{\mathbf{A}}_{k-1}\Delta\mathbf{O}_{k|\hat{\Lambda}_k})\left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1}\right)\\ &= \left(\left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1} \otimes \mathbf{I}_N\right)\text{col vec}(\Delta\mathbf{J}_{k|\hat{\Lambda}_k} - \hat{\mathbf{A}}_{k-1}\Delta\mathbf{O}_{k|\hat{\Lambda}_k})\end{aligned}$$

and $\mathbf{g}^A(A) := \mathbf{R}_A^{-1}(A)\mathbf{f}^A(A)$.

The following conditions are satisfied:

1. (A2.1) Under (35), $((1/k)\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k})^{-1}$ is L_2 bounded and hence so is it's expectation. It then follows that $E[|\Delta_k^A|^2] < \infty$.
2. (A2.3) From definition, $\mathbf{g}^A(.)$ is a continuous function of $\mathbf{A}$.
3. (A2.4) The gain sequence, $\varepsilon_\ell = 1/\ell$, satisfies $\sum_\ell \varepsilon_\ell^2 < \infty$.
4. (A2.8) Define $\beta_k^A := E[\Delta_k^A|\hat{\mathbf{A}}_0, \Delta_i^A, i<k] - \mathbf{g}^A(\hat{\mathbf{A}}_{k-1})$. Here $\mathbf{g}^{\mathbf{A}}(\mathbf{A})$ is continuous in k. Also, see Lemma 6 to show $\beta_k^A \to 0$ with probability 1.

Theorem 2.3 in Chapter 5 of Reference [27] now applies and hence the recursion (A1) converges to any locally asymptotically stable points (in the sense of Lyapunov) of (A2) or to a boundary point of P_c.

To examine the Lyapunov stability of (A2) consider the following candidate Lyapunov function:

$$\begin{aligned}W^A(\bar{\mathbf{A}}) &= \frac{1}{2}E\left[E\left[\sum_{\ell=1}^{n} \|\mathbf{x}_\ell - \bar{\mathbf{A}}\mathbf{x}_{\ell-1}\|^2\middle|\bar{\mathbf{A}}, \mathcal{Y}_n\right] - E\left[\sum_{\ell=1}^{n-1} \|\mathbf{x}_\ell - \bar{\mathbf{A}}\mathbf{x}_{\ell-1}\|^2\middle|\bar{\mathbf{A}}, \mathcal{Y}_{n-1}\right]\middle|\bar{\mathbf{A}}\right]\\ &= \frac{1}{2}E[\|x_n - \bar{A}x_{n-1}\|^2|\bar{A}] \geqslant 0 \quad \text{for any } n\end{aligned} \tag{A4}$$

The last line follows from classical expectation results, including that $E[E[X|A_2]|A_1] = E[X|A_1]$ when $A_1 \subset A_2$. Under asymptotic ergodicity and certain smoothness conditions the differentiation w.r.t. $\bar{\mathbf{A}}$ and the expectation operations can be interchanged. Hence,

$$\frac{\mathrm{d}W^A(\bar{\mathbf{A}}))}{\mathrm{d}\bar{\mathbf{A}}^{cv}} = -E[(\hat{J}_{\ell|\ell,\bar{A}} - \bar{\mathbf{A}}\hat{O}_{\ell|\ell,\bar{A}}) - (\hat{J}_{\ell-1|\ell-1,\bar{A}} - \bar{\mathbf{A}}\hat{O}_{\ell-1|\ell-1,\bar{A}})|\bar{\mathbf{A}}] = -\mathbf{f}^A(\bar{\mathbf{A}})' \tag{A5}$$

for any ℓ and it then follows that

$$\frac{\mathrm{d}W^A(\bar{\mathbf{A}})}{\mathrm{d}\tau} = \frac{\mathrm{d}W^A(\bar{\mathbf{A}})\mathrm{d}\bar{\mathbf{A}}^{cv}}{\mathrm{d}\bar{\mathbf{A}}^{cv}\quad \mathrm{d}\tau} = -\mathbf{f}^A(\bar{\mathbf{A}})'\mathbf{R}_A(\bar{\mathbf{A}})^{-1}\mathbf{f}^A(\bar{\mathbf{A}}) \tag{A6}$$

Under ergodicity and (35), $\mathbf{R}_A(\bar{\mathbf{A}})^{-1}$ is positive definite and hence $\mathrm{d}W^A(\bar{\mathbf{A}})/\mathrm{d}\tau < 0$. It then follows from Lyapunov's direct method, Ljung [8] and Equation (A5) that $\bar{\mathbf{A}}(\tau)$ converges to the set $\{\bar{\mathbf{A}}|\lim_{\tau\to\infty} \mathbf{f}(\bar{\mathbf{A}}) = 0\} \in P_c$ or to a boundary point of P_c a.s.

Convergence occurs to solutions of the N^2 simultaneous equations $\mathrm{d}W^A(\bar{\mathbf{A}})/\mathrm{d}\bar{\mathbf{A}}_{ij} = E[\mathbf{x}^j_{\ell-1}(\mathbf{x}^i_\ell - \sum_{n=1}^N \bar{\mathbf{A}}_{in}\mathbf{x}^n_{\ell-1})|\bar{\mathbf{A}}] = E[\mathbf{x}^j_{\ell-1}\mathbf{x}^i_\ell|\bar{\mathbf{A}}] - \sum_{n=1}^N \bar{\mathbf{A}}_{in}E[\mathbf{x}^j_{\ell-1}\mathbf{x}^n_{\ell-1}|\bar{\mathbf{A}}] = 0$ for $i,j = 1,\dots,N$ (actually only N distinct linear equations). In companion canonical form A has only N free variables and hence there will be only one solution to this homogeneous system of linear equations (if there is one). The persistence of excitation conditions imply that there is a solution (see Theorem 2), hence $\bar{\mathbf{A}}$ converges uniquely to $\mathbf{A}^\mathbf{c}$ (or to a boundary point of P_c). It then follows from Theorem 2.3 in Chapter 5 of Reference [27] that (A1) converges a.s. to $\mathbf{A}^\mathbf{c}$ as required (or to a boundary point of P_c).

We proceed to prove convergence when simultaneously estimating **A** and **C**.

As given earlier the estimator of C is

$$\hat{\mathbf{C}}_k = \mathrm{Proj}\left\{\hat{\mathbf{C}}_{k-1} + \frac{1}{k}\left(\Delta\mathbf{T}_{k|\hat{\Lambda}_k} - \hat{\mathbf{C}}_{k-1}\Delta\mathbf{O}_{k|\hat{\Lambda}_k}\right)\left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k}\right)^{-1}\right\} \tag{A7}$$

where $\Delta\mathbf{T}_{k|\hat{\Lambda}_k} := \hat{\mathbf{T}}_{k|k,\hat{\Lambda}_k} - \hat{\mathbf{T}}_{k-1|k-1,\hat{\Lambda}_{k-1}}$. Let $\hat{\theta}_k := \begin{bmatrix}\text{col vec}(\hat{\mathbf{A}}_k)\\ \text{col vec}(\hat{\mathbf{C}}_k)\end{bmatrix}$. Now consider the ODE associate with the **A** and **C** recursions

$$\frac{\mathrm{d}\bar{\theta}(\tau)}{\mathrm{d}\tau} = \mathbf{R}^{-1}(\bar{\theta}(\tau))\mathbf{f}(\bar{\theta}(\tau)) + \mathbf{z}(\bar{\theta}(\tau)) \tag{A8}$$

where $\bar{\theta}(\tau) := \begin{bmatrix}\text{col vec}(\bar{\mathbf{A}}(\tau))\\ \text{col vec}(\bar{\mathbf{C}}(\tau))\end{bmatrix}$ and $\mathbf{z}(\bar{\theta}(\tau))$ is the projection required to keep $\bar{\theta}(\tau)$ in P_c.

With $\bar{\theta}(\tau)$ abbreviated as $\bar{\theta}$ we define $\mathbf{f}(\bar{\theta})$ and $\mathbf{R}(\bar{\theta})$ as follows

$$\mathbf{f}(\bar{\theta}) := \begin{bmatrix}\text{col vec}(E[\Delta\hat{\mathbf{J}}_{\ell|\bar{\theta}} - \bar{\mathbf{A}}\Delta\hat{\mathbf{O}}_{\ell|\bar{\theta}}|\bar{\theta}])\\ \text{col vec}(E[\Delta\hat{\mathbf{T}}_{\ell|\bar{\theta}} - \bar{\mathbf{C}}\Delta\hat{\mathbf{O}}_{\ell|\bar{\theta}}|\bar{\theta}])\end{bmatrix} \quad \text{and}$$

$$\mathbf{R}(\bar{\theta}) := E\left[\left(\frac{1}{\ell}\hat{\mathbf{O}}_{\ell|\ell,\bar{\theta}}\right) \otimes \mathbf{I}_{N+1}|\bar{\theta}\right] \tag{A9}$$

To show how (A8) is related to recursions (A1), (A7) we again appeal to the Theorem 2.3 in

Reference [27]. Let

$$\Delta_k := \begin{bmatrix} \text{col vec}\left((\Delta \mathbf{J}_{k|\hat{\Lambda}_k} - \hat{\mathbf{A}}_{k-1}\Delta \mathbf{O}_{k|\hat{\Lambda}_k}) \left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} \right)^{-1} \right) \\ \text{col vec}\left((\Delta \mathbf{T}_{k|\hat{\Lambda}_k} - \hat{\mathbf{C}}_{k-1}\Delta \mathbf{O}_{k|\hat{\Lambda}_k}) \left(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k} \right)^{-1} \right) \end{bmatrix} \quad \text{and}$$

$$\mathbf{g}(\theta) := \mathbf{R}^{-1}(\theta)\mathbf{f}(\theta)$$

The following conditions are satisfied:

1. (A2.1) Under (3.22), $(\frac{1}{k}\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k})^{-1}$ is L_2 bounded and hence so is it's expectation. $E[|\Delta_k|^2] < \infty$.
2. (A2.3) From definition, $\mathbf{g}(.)$ is a continuous function of θ.
3. (A2.4) The gain sequence, $\varepsilon_\ell = 1/\ell$, satisfies $\sum_\ell \varepsilon_\ell^2 < \infty$.
4. (A2.2), (A2.5) Define $\beta_k := E[\Delta_k|\hat{\theta}_0, \Delta_i, i<k] - \mathbf{g}(\hat{\theta}_{k-1})$. Here $\mathbf{g}(\boldsymbol{\theta})$ is continuous in k. Also, using the same techniques as used in Lemma 6 it can be shown that $\beta_k \to 0$ with probability 1.

Theorem 2.3 of Reference [27] now applies and hence the recursions (A1), (A7) converge to any locally asymptotically stable points (in the sense of Lyapunov) of (A8) or to a boundary point of P_c.

To examine the Lyapunov stability of (A8) consider the following candidate Lyapunov function:

$$\begin{aligned} \bar{W}(\bar{\theta}) = \frac{1}{2}E\Bigg[& E\left[\sum_{\ell=1}^{n} \|\mathbf{x}_\ell - \bar{\mathbf{A}}\mathbf{x}_{\ell-1}\|^2 \middle| \bar{\theta}, \mathcal{Y}_n \right] - E\left[\sum_{\ell=1}^{n-1} \|\mathbf{x}_\ell - \bar{\mathbf{A}}\mathbf{x}_{\ell-1}\|^2 \middle| \bar{\theta}, \mathcal{Y}_{n-1} \right] \\ & + E\left[\sum_{\ell=1}^{n} \|y_\ell - \bar{\mathbf{C}}\mathbf{x}_\ell\|^2 \middle| \bar{\theta}, \mathcal{Y}_n \right] - E\left[\sum_{\ell=1}^{n-1} \|y_\ell - \bar{\mathbf{C}}\mathbf{x}_\ell\|^2 \middle| \bar{\theta}, \mathcal{Y}_{n-1} \right] \Bigg| \bar{\theta} \Bigg] \end{aligned} \tag{A10}$$

for any n.

This is a Lyapunov function because $\bar{W}(\bar{\theta}) \geq 0$ and in a similar manner to the above case for estimation of A it can be shown that $\mathrm{d}\bar{W}(\bar{\theta})/\mathrm{d}\bar{\theta} = -\mathbf{f}(\bar{\theta})'$ and hence $\mathrm{d}\bar{W}(\bar{\theta})/\mathrm{d}\tau < 0$. It follows from Lyapunov's direct method that A and C estimates converge to the set $\{\bar{\mathbf{A}}, \bar{\mathbf{C}} | \lim_{\tau\to\infty} \mathrm{d}\bar{W}(\bar{\theta}(\tau))/\mathrm{d}\tau = 0\}$ (or a boundary point of P_c).

Convergence occurs to solutions of the $N^2 + N$ simultaneous equations $\mathrm{d}W^A(\bar{\theta})/\mathrm{d}\boldsymbol{\theta}_i = 0$ for $i = 1, \ldots, (N^2 + N)$. Which are the N^2 equations $E[\mathbf{x}_{\ell-1}^j(\mathbf{x}_\ell^i - \sum_{n=1}^N \bar{\mathbf{A}}_{in}\mathbf{x}_{\ell-1}^n)|\bar{\theta}] = 0$ for $i,j = 1, \ldots, N$ (actually only N distinct linear equations) and the N distinct linear equations $E[\mathbf{x}_\ell^i \times (y_\ell - \sum_{n=1}^N \bar{\mathbf{C}}_n\mathbf{x}_\ell^n)|\bar{\theta}] = 0$ for $i = 1, \ldots, N$.

The companion canonical form has only $2N$ free variables and hence there will be only one solution to this homogeneous system of linear equations (if there is one). The persistence of excitation conditions imply that there is a solution (see Theorem 2), hence $\bar{\mathbf{A}}$, $\bar{\mathbf{C}}$ converge uniquely to $\mathbf{A}^c$, $\mathbf{C}^c$ (or to a boundary point of P_c). It then follows from Theorem 2.3 of

Reference [27] that (A1) (A7) converge a.s. to $\mathbf{A}^{\mathbf{c}}$, $\mathbf{C}^{\mathbf{c}}$ as required (or to a boundary point of P_{c}). □

Lemma 6

Define $\beta_k^A := E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_0, \mathbf{\Delta}_i^A, i<k] - \mathbf{g}^A(\hat{\mathbf{A}}_{k-1})$ then $\beta_\ell^A \to 0$ *w.p.* 1.

Proof

This can be established by expressing β_k^A as follows:

$$\beta_k^A = E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_i, i<k] - E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_k] \tag{A11}$$

which follows from the definition of $\mathbf{\Delta}_k^A$ and $\hat{\mathbf{A}}_k$.

First note that Lemma 5 established that the terms $\Delta J_{k|\hat{\Lambda}_k}$ and $\Delta O_{k|\hat{\Lambda}_k}$ are exponentially forgetting (because elements of P_{c} are completely stabilisable and detectable). Also, it follows from (35) that $((1/k)\hat{\mathbf{O}}_{k|k,\hat{\Lambda}_k})^{-1}$ is asymptotically stable to $\mathbf{O}^L$. Now, the product of two asymptotically stable terms is also asymptotically stable (or asymptotically approaching exponential stability in this case), hence $\mathbf{\Delta}_k^A$ and $E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_i, i<k]$ both asymptotically forget $\hat{\mathbf{A}}_l$ as $k-l$ grows.

Then from (A1) and (35) we have that $|\hat{\mathbf{A}}_k - \hat{\mathbf{A}}_{k-1}| < (1/k)B_A$ a.s. for large k and some non-negative random variable B_A.

Hence, asymptotically stability (approaching exponential stability for large k) of $E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_i, i<k]$ with respect to $\hat{\mathbf{A}}_l$ together with $|\hat{\mathbf{A}}_k - \hat{\mathbf{A}}_{k-1}| < (1/k)B_A$ gives that $\beta_k^A \to 0$ as k grows. That is, with increasing k, changes in $\hat{\mathbf{A}}_k$ become smaller, and influence of these decreasing changes is asymptotically forgotten over time so that $E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_0, \hat{\mathbf{A}}_i, i<k]$ approaches $E[\mathbf{\Delta}_k^A|\hat{\mathbf{A}}_k]$ as k grows. □

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