# On linear functional equations in locally convex linear topological spaces

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In his famous memoir [1] F. Riesz has given what is essentially an extension to Banach spaces of Fredholm's theory of linear equations. This extension was supplemented by Hildebrandt [2] and Schauder [3]. A generalization of Riesz's theory for locally convex linear topological spaces has been given recently by Leray [4]. The proof of the "alternative of Fredholm" is based on a theorem on the invariance of the domain. This theorem was established by Schauder for Banach spaces and generalized by Leray for a wider class of spaces. However, the theorems of Schauder for the conjugate space were not obtained by Leray.

The purpose of this paper is to give a simple method of proof of the Riesz-Schauder theory in the case of locally convex linear topological spaces. This method is based on the above mentioned theory and permits us to generalize also other theorems concerning linear completely continuous transformations (see [5]). Finally, we observe that in order to prove the corresponding theorems it suffices to assume that the transformation in question is continuous in the sense of Heine.

**1.** Let $\mathfrak{X}$ be a *linear topological locally convex space*, i. e. a linear set on which a topology is imposed in such a fashion that the postulated operations of addition and multiplication by real numbers are continuous in the topology, moreover, for every neighbourhood $\mathfrak{U}_x$ of the element $x \in \mathfrak{X}$ there exists a convex neighbourhood $\mathfrak{V}_x$ such that $\mathfrak{V}_x \subset \mathfrak{U}_x$ (cf. [6] and [7]). It suffices to give the system $\mathfrak{W}$ of neighbourhoods of 0.

The system of neighbourhoods of an arbitrary element $x$ consists of the neighbourhoods of the form $\mathfrak{U}_x = x + \mathfrak{U}$ $(\mathfrak{U} \in \mathfrak{W})$.



A sequence $\{x_n\} \subset \mathfrak{X}$ is *convergent* if, for every neighbourhood $\mathfrak{U}$ of 0, there exists a number $N$ such that $m, n > N$ implies $x_n - x_m \in \mathfrak{U}$. A sequence $\{x_n\} \subset \mathfrak{X}$ is *convergent to* $x \in \mathfrak{X}$ if there exists an $N$ such that $n > N$ implies $x_n - x \in \mathfrak{U}$; we write $x_n \to x$, when $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

A transformation $y = U(x)$ of $\mathfrak{X}$ into $\mathfrak{X}$ is *continuous* if[1])

$$\lim_{n\to\infty} x_n = x$$

implies

$$\lim_{n\to\infty} U(x_n) = U(x), \qquad x \in \mathfrak{X}.$$

A transformation $y = U(x)$ is called *linear* if $U$ is continuous on $\mathfrak{X}$ and has the following property:

$$U(x+y) = U(x) + U(y) \quad \text{for arbitrary} \quad x, y \in \mathfrak{X};$$

hence $U(tx) = tU(x)$, where $t$ is a real number. A transformation $U$ is called *completely continuous* if there exists a neighbourhood $\mathfrak{U}$ of 0 such that the image $U(\mathfrak{U})$ is compact in this sense that every infinite subset has a limit point.

In a locally convex linear topological space $\mathfrak{X}$ the pseudonorm $|x|_{\mathfrak{V}}$, where $\mathfrak{V}$ is an arbitrary convex neighbourhood of 0, is defined for every $x \in \mathfrak{X}$ as follows[2]):

$$|x|_{\mathfrak{V}} = \text{g.l.b. of } h > 0 \quad \text{such that} \quad x/h \in \mathfrak{V}.$$

Then $|x|_{\mathfrak{V}}$ has the following properties:

1⁰ $|x|_{\mathfrak{V}} \geqslant 0$,

2⁰ $|x+y|_{\mathfrak{V}} \leqslant |x|_{\mathfrak{V}} + |y|_{\mathfrak{V}}$,

3⁰ $|tx|_{\mathfrak{V}} = |t|\,|x|_{\mathfrak{V}}$ for any real $t$.

The following classification of locally convex linear topological spaces is due to Mazur ([8], p. 199).

Let $\mathfrak{X}$ be a linear space and $\Phi$ an abstract set such that $\overline{\overline{\Phi}} = \aleph_\alpha$. Let us further assume that there exists a class of pseudonorms $|x|_\vartheta$, $\vartheta \in \Phi$, such that

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1) We consider continuity in the sense of Heine; continuity in the sense of Cauchy is stronger than sequential continuity.

2) We assume also, without loss of generality, the symmetry of $\mathfrak{V}$, $-\mathfrak{V} = \mathfrak{V}$, for an arbitrary neighbourhood of 0.

$$|x|_\vartheta = 0 \quad \text{for every} \quad \vartheta \in \Phi$$

is equivalent to $x=0$. Then $\mathfrak{X}$ is called a *$(B_a^*)$-space*. $\mathfrak{X}$ is a locally convex linear topological space, and the set of all elements $x \in \mathfrak{X}$ such that $|x-x_0|_{\vartheta_i} < \varepsilon$ $(i=1,2,\ldots,n)$, where $\varepsilon$ is a positive number, constitutes the neighbourhood of $x_0$.

Mazur ([8], p. 199) and v. Neumann (independently of each other) have shown that every locally convex linear topological space is isomorphic to a certain $(B_a^*)$-space. Thus, a *sequence* $\{x_n\} \subset \mathfrak{X}$ *is convergent* to an element $x \in \mathfrak{X}$ if

$$\lim_{n\to\infty} |x_n - x|_\vartheta = 0 \quad \text{for every} \quad \vartheta \in \Phi.$$

Let $U$ be a linear completely continuous transformation and $\mathfrak{U}$ the neighbourhood of 0 such that the image $U(\mathfrak{U})$ is compact. Then there exists a neighbourhood $\mathfrak{B}$ of 0 such that $\mathfrak{B} \subset \mathfrak{U}$, where $\mathfrak{B}$ is defined by the inequalities $|x|_{\vartheta_i} < \varepsilon$ $(i=1,2,\ldots,n)$. We define the pseudonorm $|x| = \sup_i |x|_{\vartheta_i}$. Then the image of any for this pseudonorm bounded set is compact. Hence we have the following

Lemma. *If $U$ is a linear completely continuous transformation, there exists for every $\vartheta \in \Phi$ a number $M_\vartheta$ such that $|U(x)|_\vartheta \leqslant M_\vartheta |x|$.*

Proof. The set $U(x/|x|)$ is compact. If $|x| \neq 0$ there exists a number $M_\vartheta$ such that $|U(x)|_\vartheta / |x| \leqslant M_\vartheta$; hence $|U(x)|_\vartheta \leqslant M_\vartheta |x|$. The condition $|x|=0$ implies that $|\lambda x| = 0$ for every $\lambda$. Since the set $\lambda U(x)$ is compact, we have $|\lambda| |U(x)|_\vartheta < +\infty$. Since $\lambda$ is arbitrary, it is necessary that $|U(x)|_\vartheta = 0$ for every $\vartheta \in \Phi$.

On the basis of lemma 1 we construct an auxiliary Banach space as follows:

Let $U$ be a linear completely continuous transformation having its domain and range in $\mathfrak{X}$ and $|x|$ the above chosen pseudonorm. We divide the space $\mathfrak{X}$ into classes and we say that $x_1$ and $x_2$ *belong to the same class* $\mathfrak{x}$, if $|x_1 - x_2| = 0$. The set $\Theta$ of all elements $x \in \mathfrak{X}$ such that $|x|=0$ constitutes the zero class. Thus we have obtained a $(B^*)$-space $\mathfrak{X}^*$ with the norm $|\mathfrak{x}| = |x|$, where $x \in \mathfrak{x} \subset \mathfrak{X}$. Denote by $\mathfrak{X}'$ the completion of $\mathfrak{X}^*$. $\mathfrak{X}'$ is a Banach space.

The transformation $y = U(x)$ defines in the space $\mathfrak{X}^*$ a transformation $\mathfrak{y} = \mathfrak{U}(\mathfrak{x})$, where $y \in \mathfrak{y}$, $x \in \mathfrak{x}$. We shall show that $\mathfrak{U}$ is a completely continuous transformation. If $\{\mathfrak{x}_n\}$ is an arbitrary sequence from $\mathfrak{X}^*$ such that $|\mathfrak{x}_n| < M$, where $M$ is a constant, then for $x_n \in \mathfrak{x}_n$ the sequence $\{U(x_n)\}$ is compact. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. Since the sequence $\{U(x_n)\}$ is compact, there exists an element $y_0 \in \mathfrak{X}$ such that each of the neighbourhoods defined by the condition $|x - y_0| < \varepsilon_n$ $(n=1,2,\ldots)$ contains infinitely many elements of the sequence $\{U(x_n)\}$, whence it follows that the sequence $\{\mathfrak{U}(\mathfrak{x}_n)\}$ contains a subsequence convergent to $\mathfrak{y}_0$, where $y_0 \in \mathfrak{y}_0$.



The transformation $\mathfrak{U}$ can be extended over the whole space $\mathfrak{X}'$ and its range is contained in $\mathfrak{X}^*$. In fact, if $\mathfrak{x}$ is an arbitrary element of $\mathfrak{X}'$, then there exists a sequence $\{\mathfrak{x}_n\} \subset \mathfrak{X}^*$ such that $\mathfrak{x}_n \to \mathfrak{x}$ for $n \to \infty$. Thus the sequence $\{\mathfrak{U}(\mathfrak{x}_n)\}$ is compact and convergent in the space $\mathfrak{X}^*$; hence there exists an element $\mathfrak{y} \in \mathfrak{X}^*$ such that

$$\lim_{n\to\infty} \mathfrak{U}(\mathfrak{x}_n) = \mathfrak{y},$$

and consequently $\mathfrak{y} = \mathfrak{U}(\mathfrak{x})$.

Theorem 1. *If $U$ is a linear completely continuous transformation, then the range of the transformation $T(x) = x - U(x)$ is closed.*

Proof. Consider the transformation $\mathfrak{T}(\mathfrak{x}) = \mathfrak{x} - \mathfrak{U}(\mathfrak{x})$ defined on $\mathfrak{X}'$. By a theorem of Riesz ([1], Satz 5) its range is closed. If $\mathfrak{y} \in \mathfrak{X}^*$ is a limit point of the range of the transformation $\mathfrak{T}$ then there exists an element $\mathfrak{x} \in \mathfrak{X}'$ such that $\mathfrak{T}(\mathfrak{x}) = \mathfrak{x} - \mathfrak{U}(\mathfrak{x}) = \mathfrak{y}$. Since $\mathfrak{U}(\mathfrak{x}) \in \mathfrak{X}^*$, also $\mathfrak{x} \in \mathfrak{X}^*$. Thus the range of the transformation $\mathfrak{T}$ with its domain restricted to $\mathfrak{X}^*$ is closed in $\mathfrak{X}^*$. The homomorphism $x \to \mathfrak{x}$, where $x \in \mathfrak{x} \subset \mathfrak{X}$, $\mathfrak{x} \in \mathfrak{X}^*$, is continuous; this implies that the range of $T$ is closed.

Theorem 2. *If $U$ is a linear completely continuous transformation, then the solutions of the equation $x - U(x) = 0$ form an Euclidean space.*

Proof. Consider the equation $\mathfrak{x} - \mathfrak{U}(\mathfrak{x}) = \theta$ corresponding to the equation $x - U(x) = 0$, where $x \in \mathfrak{x}$. By a theorem of Riesz ([1], Satz 1) the set of solutions constitutes a linear space of finite dimension contained in $\mathfrak{X}^*$. Let $x_1, x_2, \ldots, x_n$ be linearly independent solutions of the equation $x - U(x) = 0$. If $x_i \in \mathfrak{x}_i$, then $\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n$ are linearly independent solutions of the equation $\mathfrak{x} - \mathfrak{U}(\mathfrak{x}) = \theta$.

In fact, suppose that there exist numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1\mathfrak{x}_1 + \lambda_2\mathfrak{x}_2 + \ldots + \lambda_n\mathfrak{x}_n = \theta$. Then we have $|\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n| = 0$. By lemma 1 we obtain $U(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) = 0$. Since a linear combination of solutions is also a solution, we have $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Hence it follows also that the condition $|\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n| = 0$, where $\beta_1, \beta_2, \ldots, \beta_n$ are arbitrary numbers and $x_1, x_2, \ldots, x_n$ — arbitrary solutions of the equation $x - U(x) = 0$, implies $\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n = 0$.

Theorem 2'. *The solutions of the equation $T^n(x) = 0$ for any positive integer $n$ form an Euclidean space.*

Proof. The solutions of the equation $\mathfrak{T}^n(\mathfrak{x}) = \theta$ have the required property and the same is true for the equation $T^n(x) = 0$.

Denote by $G_n$ the set of solutions of the equation $T^n(x) = 0$ $(n = 1, 2, \ldots)$. We prove now

Theorem 3. *There exists a positive integer $\nu$ such that $G_n = G_\nu$ for $n > \nu$ and $G_n \neq G_\nu$ for $n < \nu$ (or, equivalently, $T^n(x) = 0$ implies $T^\nu(x) = 0$ for $n > \nu$, and for $n < \nu$ there exists an element $x$ such that $T^{n+1}(x) = 0$, but $T^n(x) \neq 0$).*

Proof. By a theorem of Riesz ([1], Satz 2) there exists a positive integer $\nu$ such that $\mathfrak{G}_\nu = \mathfrak{G}_n$ for $n > \nu$ and $\mathfrak{G}_\nu \neq \mathfrak{G}_n$ for $n < \nu$, where by $\mathfrak{G}_n$ we denote the set of all solutions of the equation $\mathfrak{T}^n(\mathfrak{x}) = \theta$. If $T^{\nu+1}(x) = 0$ then $\mathfrak{T}^{\nu+1}(\mathfrak{x}) = \theta$ and $\mathfrak{T}^\nu(\mathfrak{x}) = \theta$, where $x \in \mathfrak{x}$. Hence $|T^\nu(x)| = 0$ and $T^{\nu+1}(x) = T^\nu(x) - UT^\nu(x) = T^\nu(x) = 0$. On the other hand, if $T^{n+1}(x) = 0$ implies $T^n(x) = 0$ and if we assume that $\mathfrak{T}^{n+1}(\mathfrak{x}) = \theta$, then for $x \in \mathfrak{x}$ we have $|T^{n+1}(x)| = 0$. If $y = T^{n+1}(x)$, then $T^m(y) = y$ for an arbitrary $m$. We have $T^{n+1}(x) - y = 0$, hence $T^{n+1}(x - y) = 0$, where $x - y \in \mathfrak{x}$. This implies $T^n(x - y) = 0$ and $\mathfrak{T}^n(\mathfrak{x}) = \theta$.

Theorem 4. *If the equation $T(x) = y$ has a solution for any $y \in \mathfrak{X}$, then it has only one solution, i. e. the equation $x - U(x) = 0$ has the only solution $x = 0$.*

Proof. This follows from theorem 3.

We denote by $L_n$ the range of the transformation $T^n$. Since the range of the transformation $T^m$ is closed, so is $L_n$.

Theorem 5. *There exists a positive integer $\nu$ such that for $n > \nu$ $L_n = L_\nu$ and for $n < \nu$ $L_{n+1} \neq L_n$. The number $\nu$ coincides with that defined in theorem 3.*

Proof. Since $\mathfrak{T}^n = (I - \mathfrak{U})^n = I - \mathfrak{U}(nI - \ldots)$ and the range of the transformation $\mathfrak{U}$ is contained in $\mathfrak{X}^*$, $\mathfrak{T}^n(\mathfrak{x}) \in \mathfrak{X}^*$ implies $\mathfrak{x} \in \mathfrak{X}^*$. Denote by $\mathfrak{L}_n$ the range of $\mathfrak{T}^n$. Let $\nu$ be the number defined in theorem 3; from a theorem of Riesz ([1], Satz 6) it follows that $\mathfrak{L}_n = \mathfrak{L}_\nu$ for $n > \nu$. Let $x$ be an arbitrary element belonging to $\mathfrak{X}$ and $x \in \mathfrak{x} \in \mathfrak{X}^*$; there exists an element $\mathfrak{y} \in \mathfrak{X}'$ such that $\mathfrak{T}^\nu(\mathfrak{x}) = \mathfrak{T}^n(\mathfrak{y})$. $\mathfrak{T}^\nu(\mathfrak{x}) \in \mathfrak{X}^*$, hence $\mathfrak{y} \in \mathfrak{X}^*$. If $y \in \mathfrak{y}$, then $T^\nu(x) = T^n(y) + z$, where $|z| = 0$, and we obtain $T^n(y) + z = T^n(y + z)$. Since $x$ is an arbitrary element of $\mathfrak{X}$, we have $L_n = L_\nu$. On the other hand, if $L_{m+1} = L_m$, then for any $x$ there exists an element $y$ such that $T^m(x) = T^{m+1}(y)$. This implies $\mathfrak{T}^m(\mathfrak{x}) = \mathfrak{T}^{m+1}(\mathfrak{y})$, where $x \in \mathfrak{x}$, $y \in \mathfrak{y}$, for any $\mathfrak{x} \in \mathfrak{X}^*$. Since $\mathfrak{X}^*$ is dense in $\mathfrak{X}'$ and the range of $\mathfrak{T}^n$ is closed, we have $\mathfrak{L}_{m+1} = \mathfrak{L}_m$ and the theorem is proved.

$h$ is called a *regular value* of $U$ if the equation $x - hU(x) = 0$ has only the solution $x = 0$, otherwise $h$ is called a *proper* or *characteristic value* of $U$. The set of all proper values of $U$ forms the so-called *spectrum*. Obviously, if $h$ is a regular value of $U$, then $h$ is also a regular value of $\mathfrak{U}$: if $\mathfrak{x}_0$ is a solution of the equation $\mathfrak{x} - h\mathfrak{U}(\mathfrak{x}) = \theta$, then $\mathfrak{x}_0 \in \mathfrak{X}^*$ and there exists an $x_0 \in \mathfrak{x}_0$ such that $x_0 - hU(x_0) = 0$; hence $x_0 = 0$ and $\mathfrak{x}_0 = \theta$. Conversely, if $\mathfrak{x} - h\mathfrak{U}(\mathfrak{x}) = \theta$ implies $\mathfrak{x} = \theta$, then if there exists an element $x \in \mathfrak{X}$ such that $x - hU(x) = 0$, then $x \in \theta$; by lemma 1 $U(x) = 0$ and thus $x = 0$. The transformations $U$ and $\mathfrak{U}$ have the same spectrum.

Theorem 6. *If $h$ is a regular value of $U$, then the transformation $x - hU(x)$ is a one-to-one mapping of $\mathfrak{X}$ onto itself and its inverse is continuous, i. e. the mapping is a homeomorphism.*

Proof. $h$ is also a regular value of $\mathfrak{U}$. The mapping $\mathfrak{x} - h\mathfrak{U}(\mathfrak{x})$ is by a theorem of Riesz ([1], Satz 7) a homeomorphism of $\mathfrak{X}'$. If $y \in \mathfrak{y} \in \mathfrak{X}^*$ and $\mathfrak{x}_0$ is a solution of the equation $\mathfrak{x} - h\mathfrak{U}(\mathfrak{x}) = \mathfrak{y}$, then there exists an $x_0 \in \mathfrak{x}_0$ such that $x_0 - hU(x_0) = y$. Suppose $x_n - hU(x_n) \to 0$, when $n \to \infty$. Then $|x_n - hU(x_n)| \to 0$, hence also $|\mathfrak{x}_n - h\mathfrak{U}(\mathfrak{x}_n)| \to 0$. Since the transformation $\mathfrak{x} - h\mathfrak{U}(\mathfrak{x})$ has a continuous inverse, $|\mathfrak{x}_n| \to 0$ and also $|x_n| \to 0$; hence, by lemma 1, $U(x_n) \to 0$ and consequently $x_n \to 0$.

The element $x \in \mathfrak{X}$ is called a *null-element* if it is a solution of the equation $T^\nu(x) = 0$. The element $y$ of the form $y = T^\nu(x)$, where $x \in \mathfrak{X}$ is called a *kernel-element*.

Theorem 7. *Every element of $\mathfrak{X}$ can be represented as the sum of a null-element and a kernel-element in only one manner.*

Proof. This results from theorems 3-5 in exactly the same way as in the case of Banach spaces and it can be obtained directly from the corresponding theorem of Riesz ([1], Satz 8) as follows: Let $x$ be any element of $\mathfrak{X}$ and $x \epsilon \mathfrak{x}$, then $\mathfrak{x}=\mathfrak{x}'+\mathfrak{x}''$, where $\mathfrak{x}' \epsilon \mathfrak{G}_\nu$, $\mathfrak{x}'' \epsilon \mathfrak{L}_\nu$. Since $\mathfrak{T}^\nu(\mathfrak{x}')=\theta$ and $\mathfrak{x} \epsilon \mathfrak{X}^*$, we have $\mathfrak{x}', \mathfrak{x}'' \epsilon \mathfrak{X}^*$. If $x' \epsilon \mathfrak{x}'$ then $T^\nu(x')=y$ and $|y|=0$, $x=(x'-y)+(x-x'+y)$, $T^\nu(x'-y)=T^\nu(x')-T^\nu(y)=T^\nu(x')-y=0$, whence $x'-y \epsilon G_\nu$. Since $x-x'+y \epsilon \mathfrak{x}''$ and $\mathfrak{x}''=\mathfrak{T}^\nu(\mathfrak{w})$, where $\mathfrak{w} \epsilon \mathfrak{X}^*$, then if $w \epsilon \mathfrak{w}$ we have $T^\nu(w) \epsilon \mathfrak{x}''$, hence $x-x'+y=T^\nu(w)+\overline{w}$, where $|\overline{w}|=0$ and $x-x'+y=T^\nu(w+\overline{w}) \epsilon L_\nu$. The condition $\mathfrak{G}_\nu \mathfrak{L}_\nu=0$ implies $G_\nu L_\nu=0$.

Theorem 8. *There exists a unique linear transformation $T^{(0)}$, which maps every kernel-element into itself and every null-element into 0. $T^{(0)}$ maps every element of $\mathfrak{X}$ into a kernel-element and $I-T^{(0)}$ maps every element into a null-element. Moreover, $T^{(0)^2}=T^{(0)}$ and $UT^{(0)}=T^{(0)}U$.*

Proof. For every element $x \epsilon \mathfrak{X}$ we have, by theorem 7, $x=x'+(x-x')$, where $x'$ is a kernel-element and $x-x'$ is a null-element.

Denote by $T^{(0)}$ the mapping $x \to x'$. It is obvious that $T^{(0)}(x')=x'$ if $x'$ is a kernel-element, $T^{(0)}(x)''=0$ if $x''$ is a null-element, and for every $x \epsilon \mathfrak{X}$, $T^{(0)}(x)$ is a kernel-element and $I(x)-T^{(0)}(x)$ is a null-element. Since $G_\nu$ is of finite dimension and $G_\nu \cdot L_\nu=0$, for $x_n \epsilon \mathfrak{X}$, we can write $x_n=\lambda_1^n x_1^0+\lambda_2^n x_2^0+\ldots+\lambda_p^n x_p^0+x_n'$ where the elements $x_1^0, x_2^0, \ldots, x_p^0$ form a base in $G_\nu$ and $x_n' \epsilon L_\nu$. Suppose that $x_n \to 0$, when $n \to \infty$, and put $r_n=|\lambda_1^n|+|\lambda_2^n|+\ldots+|\lambda_p^n|$. We shall show that $r_n \to 0$. Suppose that there exists a number $\beta$ and a subsequence $\{n_k\}$ such that $r_{n_k}>\beta$; then

$$\frac{1}{r_{n_k}}(\lambda_1^{n_k}x_1^0+\lambda_2^{n_k}x_2^0+\ldots+\lambda_p^{n_k}x_p^0+x_{n_k}') \to 0.$$

It follows that $\lambda_i^{n_k}/r_{n_k} \to 0$, when $k \to \infty$, for $i=1,2,\ldots,p$; otherwise there would exist a number $\beta'$ and an index $i_0$ such that $\lambda_{i_0}^{n_k}/r_{n_k}>\beta'$ for infinitely many elements of the sequence $\{n_k\}$ and we could choose a subsequence $\{\overline{n}_k\}$ of indices such that the sequences $\lambda_i^{\overline{n}_k}/r_{\overline{n}_k}$ would be convergent and $\lambda_{i_0}^{\overline{n}_k}/r_{\overline{n}_k} \to \lambda_{i_0} \neq 0$, which gives

$$\frac{1}{r_{\overline{n}_k}}(\lambda_1^{\overline{n}_k}x_1^0+\lambda_2^{\overline{n}_k}x_2^0+\ldots+\lambda_p^{\overline{n}_k}x_p^0) \to x_0 \neq 0,$$

where $x_0 \epsilon G_\nu$.

This would imply $x'_{\overline{n}_k}/r_{\overline{n}_k} \to -x_0$ and $x_0 \epsilon L_\nu$, which is impossible. We obtain $\lambda_i^{n_k}/r_{n_k} \to 0$ when $k \to \infty$, but then



$$\frac{|\lambda_1^{n_k}|+|\lambda_2^{n_k}|+\ldots+|\lambda_p^{n_k}|}{r_{n_k}} \to 0.$$

The obtained contradiction shows that $r_n \to 0$, hence $\lambda_1^n x_1^0+\lambda_2^n x_2^0+\ldots+\lambda_p^n x_p^0 \to 0$ and also $x_n' \to 0$. Thus the continuity of the transformation $T^{(0)}$ is proved.

Arguing as in the case of Banach spaces we can prove the other statements of the theorem.

Theorem 9. *The transformation $U$ can be decomposed in one and only one way into two components: $U=U_1+U_2$, where*

$1^0$ *$U_1$ is a linear transformation, which maps all null-elements into 0 and $U_2$ maps all kernel-elements into 0;*

$2^0$ *$U_1$ coincides with $U$ for kernel-elements and $U_2$ coincides with $U$ for null-elements;*

$3^0$ *For any $x$, $U_1(x)$ is a kernel-element and $U_2(x)$ a null-element; $U_1$ and $U_2$ are ortogonal, i. e. $U_1U_2=U_2U_1=0$; $U_1$ and $U_2$ are completely continuous.*

Proof. Putting $U_1=T^{(0)}U=UT^{(0)}$ and $U_2=(I-T^{(0)})U=U(I-T^{(0)})$ we obviously have $U_1+U_2=U$. We shall show that the transformations $U_1$ and $U_2$ are completely continuous; everything else can be proved in exactly the same way as in the case of Banach space ([1], Satz 10).

By theorem 8 we have $T^{(0)}(x)=x' \epsilon L_\nu$ and $(I-T^{(0)})(x)=x'' \epsilon G_\nu$, hence $\mathfrak{x}=\mathfrak{x}'+\mathfrak{x}''$, where $x' \epsilon \mathfrak{x}'$, $x'' \epsilon \mathfrak{x}''$, $x \epsilon \mathfrak{x}$, $\mathfrak{x}' \epsilon \mathfrak{L}_\nu$, $\mathfrak{x} \epsilon'' \mathfrak{G}_\nu$. By a lemma of Riesz ([1], Hilfssatz 6) there exist numbers $M_1$, $M_2$ such that $|\mathfrak{x}'| \leqslant M_1|\mathfrak{x}|$ and $|\mathfrak{x}|'' \leqslant M_2|\mathfrak{x}|$; hence follows $|x'| \leqslant M_1|x|$ and $|x|'' \leqslant M_2|x|$. Thus the transformations $U_1=UT^{(0)}$ and $U_2=U(I-T^{(0)})$ map the neighbourhood of 0, defined by the condition $|x|<\varepsilon$, into compact sets.

Theorem 10. *The transformation $T_1=I-U_1$ has an inverse, i. e. there exists a transformation $T_1^{-1}$ such that $T_1T_1^{-1}=T_1^{-1}T_1=I$. The equations $T^m(x)=0$ and $T_2^m(x)=0$, where $T_2=I-U_2$ have the same solutions. The equations $T^m(x)=y$ and $T_2^m(x)=y$ with the same right-hand sides either both have solutions or both have no solutions.*

Proof. The theorem results from theorems 6 and 7 in the same manner as in the case of Banach spaces ([1], Satz 11).

Theorem 11. *The proper values of $U$ have no finite limit points.*

Proof. This follows from a theorem of Riesz ([1], Satz 12) for the linear completely continuous transformation $\mathfrak{U}$ defined in the Banach space $\mathfrak{X}'$ and from the fact that $U$ and $\mathfrak{U}$ have the same spectrum.

Theorem 12. *All the values of the parameter $h\neq 1$ are regular values of the transformation $U_2$ defined in theorem* 9.

Proof. In accordance with the decomposition $x=x'+x''$, where $x'\in L_\nu$, $x''\in G_\nu$ we obtain the following decomposition for the space $\mathfrak{X}'$: if $x\in\mathfrak{x}$, $x'\in\mathfrak{x}'$ then $\mathfrak{x}'\in\mathfrak{L}_\nu$, $\mathfrak{x}''\in\mathfrak{G}_\nu$ and we set $\mathfrak{x}=\mathfrak{x}'+\mathfrak{x}''$. The transformation $\mathfrak{T}^{(0)}(\mathfrak{x})=\mathfrak{x}'$, corresponding to the transformation $T^{(0)}(x)=x'$, is defined by the homomorphism $x\to\mathfrak{x}$, where $x\in\mathfrak{x}$. In an analogous way we obtain the transformations $\mathfrak{T}^{(0)}\mathfrak{U}=\mathfrak{U}_1$ and $(I-\mathfrak{T}^{(0)})\mathfrak{U}=\mathfrak{U}_2$, corresponding to the transformations $T^{(0)}U=U_1$ and $(I-T^{(0)})U=U_2$ and identical with those defined in theorems 8 and 9 for the Banach space $\mathfrak{X}'$ on the basis of the corresponding theorems of Riesz.

Suppose now that $\mathfrak{x}-h\mathfrak{U}_2(\mathfrak{x})=\theta$ implies $\mathfrak{x}=\theta$. Hence it follows that if there exists an element $x\in\mathfrak{X}$ such that $x-hU_2(x)=0$, then we have $x\in\theta$ and also $x=0$, since $U_2(x)=0$.

On the other hand, if $x-hU_2(x)=0$ implies $x=0$ and if $\mathfrak{x}\in\mathfrak{X}'$ is an element such that $\mathfrak{x}-h\mathfrak{U}_2(\mathfrak{x})=\theta$, then there exists an element $x\in\mathfrak{x}$ such that $x-hU_2(x)=0$; hence $x=0$ and $\mathfrak{x}=\theta$. Thus we have established that the transformations $U_2$ and $\mathfrak{U}_2$ have the same regular values. But by a theorem of Riesz ([1], Satz 13) the assertion is true for $\mathfrak{U}_2$ and therefore also for $U_2$.

**2.** We shall denote by $\bar{\mathfrak{X}}$ the set of all linear functionals $X$ defined on $\mathfrak{X}$.

A sequence of linear functionals $\{X_n\}$ is said to be *strongly convergent to a linear functional* $X\in\bar{\mathfrak{X}}$ if it is uniformly convergent in a certain neighbourhood $\mathfrak{O}$ of 0, i. e.

$$\sup_{x\in\mathfrak{O}}|X_n(x)-X(x)|\to 0 \quad \text{when} \quad n\to\infty.$$

A set $\mathfrak{M}$ of linear functionals is said to be *bounded* if there exists a neighbourhood $\mathfrak{O}$ of 0 and a constant $M$ such that $|X(x)|<M$ for all $X\in\mathfrak{M}$, $x\in\mathfrak{O}$.

If $U$ is a linear transformation defined on $\bar{\mathfrak{X}}$, then the transformation $X=\bar{U}(Y)$ defined by the formula $X(x)=YU(x)$ is said to be the *adjoint* of $U$.



A transformation defined on $\bar{\mathfrak{X}}$ is said to be *completely continuous* if it maps any bounded set into a compact one.

A set $\mathfrak{M}\subset\bar{\mathfrak{X}}$ is said to be *compact* if any infinite part of $\mathfrak{M}$ contains a sequence strongly convergent to some element of $\bar{\mathfrak{X}}$.

Theorem 13. *If $y=U(x)$ is a linear completely continuous transformation then so is the adjoint $X=\bar{U}(Y)$ and its range belongs to the conjugate space of $\mathfrak{X}'$.*

Proof. Let $\{Y_n\}$ be a bounded sequence of linear functionals, i. e.

$$\sup_{y\in\mathfrak{O}}|Y_n(y)|<M,$$

where $\mathfrak{O}$ is a neighbourhood of 0. There exists a neighbourhood $\mathfrak{O}_1$ of 0 defined by the inequalities $|y|_{\theta_1'}<\varepsilon_0$, $|y|_{\theta_2'}<\varepsilon_0$, $\dots$, $|y|_{\theta_k'}<\varepsilon_0$ such that $\mathfrak{O}_1\subset\mathfrak{O}$. We define the pseudonorm:

$$|y|_1=\sup_{1\leqslant i\leqslant k}|y|_{\theta_i'}.$$

The set of all elements $y$ such that $|y|_1<\varepsilon_0$ belongs to $\mathfrak{O}$, and hereby

$$\sup_{|y|_1<1}|Y_n(y)|<\frac{M}{\varepsilon_0}=M'.$$

We have

$$|X_n(x)|=|Y_nU(x)|\leqslant\sup_{|y|_1\leqslant 1}|Y_n(y)|\,|U(x)|_1.$$

By lemma 1 there exists a constant $M_1$ such that $|U(x)|_1\leqslant M_1|x|$. Finally we obtain

$$\sup_{|x|\leqslant 1}|X_n(x)|\leqslant\sup_{|y|_1\leqslant 1}|Y_n(y)|\cdot M_1. \tag{1}$$

From (1) follows the linearity of the adjoint transformation. By a well known theorem, for any linear functional the inequality

$$|Y(y)|\leqslant N\sup(|y|_{\beta_1},\dots,|y|_{\beta_n}),$$

where $N$ is a constant and $n$ a positive integer, is true. From this fact and from (1) it follows that the range of the adjoint transformation $X=\bar{U}(Y)$ belongs to the conjugate space of $\mathfrak{X}'$. If we consider the range of the transformation $y=U(x)$ as a space with the pseudonorm $|y|_1$, then we see that the transformation $U$ defined on $\mathfrak{X}$ with the pseudonorm $|x|$ is completely continuous; hence it follows that its range is separable for the pseudonorm $|y|_1$. Thus by use of the diagonal method we can choose a subsequence $\{Y_{n_i}\}$ convergent for every $y$ of the range of $U$. We have

$$\lim_{i\to\infty} Y_{n_i} U(x) = \lim_{i\to\infty} X_{n_i}(x).$$

From (1) we infer that

$$\lim_{i\to\infty} X_{n_i}(x) = X(x)$$

is a linear functional for the pseudonorm $|x|$ and hence also on $\mathfrak{X}'$. Let $\{x_i\}\subset\mathfrak{X}$ be a sequence such that

$$|x_i|=1, \qquad |X(x_i)-X_{n_i}(x_i)|\geqslant \frac{1}{2}\sup_{|x|\leqslant 1}|X(x)-X_{n_i}(x)|.$$

Suppose that there exists a positive number $\lambda$ such that

$$\sup_{|x|\leqslant 1}|X(x)-X_{n_i}(x)|>\lambda \tag{2}$$

for every $i$, then we have

$$|\lim_{i\to\infty} Y_{n_i} U(x_i) - Y_{n_i} U(x_i)|\geqslant \frac{1}{2}\lambda.$$

Since $|x_i|=1$, there exists an element $y_0$ and a sequence of indices $\{i_k\}$ such that

$$\lim_{k\to\infty}|U(x_{i_k})-y_0|_1=0.$$

Now let $\varepsilon$ be an arbitrary positive number; there exists an index $k_0$ such that for $k>k_0$ we have

$$|y_0-U(x_{i_k})|_1<\varepsilon$$

and

$$|Y_{i_k}(y_0)-\lim_{k\to\infty} Y_{i_k}(y_0)|<\varepsilon.$$

For $k>k_0$ we have

$$|Y_{i_k}U(x_{i_k})-\lim_{k\to\infty} Y_{i_k}U(x_{i_k})|\leqslant M'\varepsilon+\varepsilon+M'\varepsilon,$$

which is incompatible with condition (2), since $\varepsilon$ is arbitrary. Thus the sequence $\{X_{n_i}\}$ is uniformly convergent in the neighbourhood $\mathfrak{D}_0$ defined by the inequality $|x|<1$ to the linear functional $X$ and this completes the proof.



Theorem 14. *If the equation $Y=X-\bar{U}(X)$, where $U$ is a linear completely continuous transformation having its domain and range in $\mathfrak{X}$, has a solution for any $Y\epsilon\mathfrak{X}$, then the equation $X-\bar{U}(X)=0$ has only the solution $X=0$.*

Proof. Since, by theorem 13, $\bar{U}(X)\epsilon\bar{\mathfrak{X}}'$ for every $X\epsilon\bar{\mathfrak{X}}$ and $X\mathfrak{U}(\mathfrak{x})=XU(x)$ for $x\epsilon\mathfrak{x}$, $X\epsilon\bar{\mathfrak{X}}'$, we have $\bar{U}(X)=\bar{\mathfrak{U}}(X)$ for $X\epsilon\bar{\mathfrak{X}}'$. Further, the equation $Y=X-\bar{\mathfrak{U}}(X)$ has for any $Y\epsilon\bar{\mathfrak{X}}'$ a solution belonging also to $\bar{\mathfrak{X}}'$. By a theorem of Schauder [3] $\bar{\mathfrak{U}}$ is a completely continuous transformation, hence by a theorem of Riesz ([1], Satz 3) the equation $X-\bar{\mathfrak{U}}(X)=0$ has only the solution $X=0$; therefore $X-\bar{U}(X)=0$ implies $X=0$.

Theorem 15. *If the equation $X-\bar{U}(X)=0$, where $U$ is a linear completely continuous transformation having its domain and range in $\mathfrak{X}$ has only the solution $X=0$, then the equation $Y=X-\bar{U}(X)$ has a solution for any $Y\epsilon\bar{\mathfrak{X}}$.*

Proof. As follows from the condition given in the theorem, the equation $X-\bar{\mathfrak{U}}(X)=0$, where $X\epsilon\bar{\mathfrak{X}}'$, has only the solution $X=0$.

Using the well known theorem for the Banach space $\mathfrak{X}'$ we find that the equation $\mathfrak{x}-\mathfrak{U}(\mathfrak{x})=\theta$ has only the solution $\mathfrak{x}=\theta$. This implies that the equation $x-U(x)=0$ also has only the solution $x=0$. According to Theorem 6 the transformation $y=x-U(x)$ is a one-to-one maping of the space $\mathfrak{X}$ onto itself having a continuous inverse.

Theorem 16. *If $T$ is a linear completely continous transformation having its domain and range in $\mathfrak{X}$, then the equations $x+U(x)=0$ and $X-\bar{U}(X)=0$ have the same number of linear independent solutions.*

Proof. It is easy to see that the equations $x-\bar{U}(x)=0$ and $\mathfrak{x}-\mathfrak{U}(\mathfrak{x})=\theta$ have the same number of linearly independent solutions. The equations $X-\bar{U}(X)=0$ and $X-\bar{\mathfrak{U}}(X)=0$ have the same solutions. By a theorem of Schauder [3] the equations $\mathfrak{x}-\mathfrak{U}(\mathfrak{x})=\theta$ and $X-\bar{\mathfrak{U}}(X)=0$ have the same number of linearly independent solutions and hence follows the above assertion.

Theorem 17. *The linear completely continuous transformation $U$ and its adjoint $\bar{U}$ have the same spectrum.*

Proof. The transformations $U$ and $\mathfrak{U}$ have the same spectrum and this holds also for $\bar{U}$ and $\bar{\mathfrak{U}}$.

Thus, given a system of equations $x-hU(x)=y$ and $X-h\bar{U}(X)=Y$, we can judge one of them from the behaviour of the other.

Theorem 18. *The set of all regular values of $U$ is an open set.*

Proof. This follows from the fact that $U$ and $\mathfrak{U}$ have the same spectrum.

Theorem 19. *If $x-hU(x)=0$ and $X-h'\bar{U}(X)=0$ $(h\neq h')$, then $X(x)=0$.*

Proof. $X(x)=hXU(x)=h\bar{U}(X)(x)$. Hence $X(x)=hX(x)/h'$ and we have $X(x)=0$.

If $U$ is a linear completely continuous transformation, then we can state for the equations

$$\text{(I)} \qquad x-hU(x)=0, \qquad X-h\bar{U}(X)=0$$

the generalization of Fredholms theorems for integral equations as follows:

Theorem 20. (a) *The equations* (I) *have the same number $d(h)$ of linearly independent solutions.*

(b) *If $d(h)=0$, then the assertion of Theorem* 6 *holds.*

(c) *If $d(h)>0$ and $\{x_i\}$, $\{X_i\}$ $(i=1,\ldots,d(h))$ are linearly independent solutions of the equations* (I), *then the equation $x-hU(x)=y$ has a solution for any $y$ satisfying the condition $X_i(y)=0$ $(i=1,2\ldots d(h))$, and the equation $X-h\bar{U}(X)=Y$ has a solution for any $Y$ satysfying the condition $Y(x_i)=0$ $(i=1,2,\ldots,d(h))$.*

Proof. (c) Suppose that $Y\epsilon\bar{\mathfrak{X}}$ satisfies the condition $Y(x_i)=0$ $(i=1,\ldots,d(h))$.

There exist pseudonorms $|x|_{\vartheta_k}$ $(k=1,\ldots,n)$ and a number $N$ such that

$$|Y(x)|\leqslant N\sup_{1\leqslant k\leqslant n}|x|_{\vartheta_k}.$$

We define the pseudonorm

$$|x|_1=\sup(|x|,|x|_{\vartheta_1},\ldots,|x|_{\vartheta_n})$$

for which we can repeat the same reasoning as for the pseudonorm $|x|$.

Denote by $\mathfrak{X}_1^*$ the corresponding quotient space and by $\mathfrak{X}_1'$ its completion. We define the transformation $\mathfrak{y}=\mathfrak{U}'(\mathfrak{x})$ corresponding to the transformation $y=U(x)$, where $x\epsilon\mathfrak{x}\epsilon\mathfrak{X}_1^*$, $y\epsilon\mathfrak{y}\epsilon\mathfrak{X}_1^*$. Let $\mathfrak{O}_1$ be the neighbourhood defined by the inequality $|x|_1<\varepsilon$. Since the condition $|x|_1<\varepsilon$ implies $|x|<\varepsilon$, then $U(\mathfrak{O}_1)$ is a compact set. By lemma 1, established for the pseudonorm $|x|_1$, the transformation $\mathfrak{U}'$ is completely continuous and can be extended over the whole $\mathfrak{X}_1'$. We have $Y\epsilon\bar{\mathfrak{X}}_1'$ and $Y(x_i)=Y(\mathfrak{x}_i)$, $x_i\epsilon\mathfrak{x}_i$ $(i=1,2,\ldots,d(h))$. Denote by $\mathfrak{G}'$ the set of all solutions of the equation $\mathfrak{x}-h\mathfrak{U}'(\mathfrak{x})=\theta_1$; then it is easy to see that $\{\mathfrak{x}_i\}$ $(i=1,2,\ldots,d(h))$ is a base in $\mathfrak{G}'$, whence by a theorem of Schauder [3] there exists a linear functional $X\epsilon\bar{\mathfrak{X}}_1'$ such that $Y=X-\bar{\mathfrak{U}}'(X)$. Obviously, $X\epsilon\bar{\mathfrak{X}}$, and for $x\epsilon\mathfrak{x}\epsilon\mathfrak{X}_1'$ we have $X\mathfrak{U}'(\mathfrak{x})=XU(x)=\bar{U}(X)(x)$, hence $Y=X-\bar{U}(X)$ and the theorem is proved.

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